UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXXVIII
2000

HOMOTOPY PROPERTIES OF CLOSED SYMPLECTIC MANIFOLDS

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Abstract. In this paper we discuss directions of research and some recent results in the area of symplectic topology, with an emphasis on methods of homotopy theory applied to closed symplectic manifolds. We describe basic open problems and possible ways of attacking them, in the framework of rational homotopy theory, symplectic fibrations, Lie group actions and Lefschetz fibrations. Also we prove some new results about symplectic blow-ups and symplectic G-manifolds with hamiltonian action of G.

1. INTRODUCTION

This paper is based on a mini-course given by the author during the conference "Geometry and Topology of Manifolds" held in Krynica Górska in April, 1999. The article is intended to serve the dual role of an exposition and a research paper. Our objective is to give a picture of an area of symplectic topology which is primarily concerned with the homotopy properties of closed symplectic manifolds.

In the last decades a new and intriguing subject of symplectic topology has emerged as a beautiful mixture of geometry, topology and analysis. The (non) existence of symplectic structures is a global property of manifolds and, therefore, the methods of algebraic topology play an essential role in this area.

In this paper, we stress the role of homotopy theory in the whole subject. Our main concern is a general picture rather than a detailed exposition of particular results. Also, we pay a special attention to *open problems*. For these reasons as well as for the sake of brevity we don't discuss examples referring to other sources. We assume that the reader is aware of the basics of symplectic geometry [**24**] and algebraic topology and, in particular, of rational homotopy theory [**12, 15, 22, 29, 32**].

The paper is organized as follows. In Section 2 we recall several topological notions which we use in the article. In Section 3 the basic problems of 'symplectic homotopy' are described. In Section 4 we discuss Lefschetz fibrations and possible applications of Donaldson's results to symplectic homotopy. Section 5 is devoted to the rational homotopy properties of symplectic blow-ups. We treat this topic in detail and give complete proofs. We prove that, roughly speaking, the non-formality is preserved under the symplectic blow-up construction. This fact has applications to symplectic topology. Section 6 deals with various aspects of homotopy properties of closed symplectic manifolds endowed with symplectic (hamiltonian) actions of compact Lie groups.

Acknowledgment. The author expesses his sincere thanks to the Organizers of the Conference, Jan Kubarski and Robert Wolak for their kind invitation to give this course and for a nice and creative atmosphere during the meeting.

2. Preliminaries on some topological invariants

In the paper, we consider (closed) symplectic manifolds, i.e. pairs (M^{2n}, ω), where M^{2n} is a $2n$-dimensional manifold and ω a non-degenerate closed 2-form. One can easily check using the Stokes formula that the de Rham cohomology class $[\omega]$ of ω is not zero up to the n-th power: $[\omega]^n \neq 0$. Hence, we get a first *homotopic* property of closed symplectic manifolds: their de Rham cohomology algebras are *c-symplectic.* By definition, a graded commutative finite-dimensional algebra $H = \oplus_{i=0}^{2n} H^i$ is c-symplectic, if there exists $\omega \in H^2$ such that $\omega^n \neq 0$. Until now, it is not known if there are other strictly homotopy properties dependent on the existence of symplectic structures. A lot of efforts has been put in looking for such properties in the framework of *rational homotopy theory.* The present paper also deals with rational homotopy properties, therefore, we give a very short account of some notions from this theory.

Recall that to every topological space X there is associated a differential graded commutative algebra $(\mathcal{M}_X, d)$ which is a *homotopy invariant* called the *minimal model of* X. We don't write the formal definition of $\mathcal{M}_X$ referring to [**12, 13, 29, 32**]. In the sequel, we will call $\mathcal{M}_X$ *the Sullivan minimal model,* since there is another approach to rational homotopy theory via the theory of differential graded Lie algebras developed by Quillen [**2, 29**]. This approach is relatively less known to mathematicians working in symplectic geometry. However, we believe that it will play an important role in future (see Section 3). Again, we omit detailed explanations referring to [**2, 20, 29**]. We only mention that to each *simply connected* topological space X we can assign in an invariant way a *free differential graded Lie algebra* $(\mathbb{L}(X), \partial)$ as follows. Consider the co-algebra structure on the *homology* $H_*(X)$ and generate a free graded Lie algebra $L(s^{-1}H_*(X))$ over the vector space $s^{-1}H_*(X)$, where s^{-1} denotes the desuspension. Then the structure of the differential algebra is given by the formula

$$\partial(s^{-1}c) = -\sum_{i<j}(-1)^{|c_i|}[s^{-1}c_i, s^{-1}c_j] - \frac{1}{2}\sum_i [s^{-1}c_i, s^{-1}c_j],$$

if the co-algebra structure on $H_*(X)$ is given by

$$\Delta(c) = c\otimes 1 + 1\otimes c + \sum_{i<j}\{c_i\otimes c_j + (-1)^{|c_i||c_j|}c_j\otimes c_i\} + \sum_i c_i\otimes c_i.$$

In the sequel we will call $\mathbb{L}(X)$ the *Quillen minimal model of* X.

The next important rational homotopy property is *formality.* By definition, a space X is formal, if there exists a quasi-isomorphism (i.e. a morphism inducing isomorphism on the cohomology level)

$$\rho : \mathcal{M}_X \to (H^*(\mathcal{M}_X), 0)$$

that is, we consider $H^*(\mathcal{M}_X)$ as a differential graded Lie algebra with zero differential. Clearly, it is a homotopy property of X. It is shared by all *Kähler manifolds.* Section 5 of the present paper deals with this property in case of symplectic manifolds. Note that formality can be reformulated in the language of Quillen models (see [**20**]).

Finally, we introduce the notion of the *hard Lefschetz property.* Let M be any c-symplectic manifold. Then, the de Rham cohomology algebra is written as $H^*(M) = \oplus_{i=0}^{2n}H^i$ and there exists $\omega \in H^2$ such that $\omega^n \neq 0$. We say that M satisfies the *hard Lefschetz property* if the linear maps

$$L_{\omega^k} : H^{n-k} \to H^{n+k},\;\; L_{\omega^k}(x) = \omega^k x$$

are isomorphisms for all $0 \leq k \leq n$. This property is shared by all Kähler manifolds.

3. Basic problems in symplectic homotopy

One of the first applications of homotopy theory to symplectic geometry is due to Thurston [**31**], who solved the following problem (attributed to Weinstein): *Are there symplectic non-Kähler closed manifolds?* The solution given in [**31**] was based on the following homotopic property of Kähler manifolds: they have *even* odd-dimensional Betti numbers. It was shown in [**31**] that there exists a manifold (which is called now the *Kodaira-Thurston manifold*) which is symplectic and has the first Betti number $b_1 = 3$. As a consequence, such manifold cannot admit Kähler structures for homotopic reasons. This observation resulted in a large number of publications exploring the subject 'symplectic versus Kähler'. The status of this area of research is summarized in [**32**].

Motivated by research in this area as well as in other directions of symplectic topology, many authors asked the following question: *Are there specific homotopic properties (invariants) of closed symplectic manifolds?* This problem is still open and seems to be very difficult to solve. However, it seems that the recent breakthrough made by Donaldson [**6**] as well as the efforts of many mathematicians in analyzing topological properties of closed symplectic manifolds will lead to essential progress in the nearest future.

Obviously, any closed symplectic manifold has the property that the top power of the cohomology class of the symplectic form is non-zero. Hence manifolds violating this property have no symplectic structures. The most general conjecture in 'symplectic homotopy' is the *Thurston conjecture*: *Any c-symplectic finite dimensional graded commutative algebra* $H = \oplus_{i=0}^{2n} H^i$ *with the Poincaré duality is a cohomology algebra of some closed symplectic manifold*:

$$H^*(M, \mathbb{R}) \cong H.$$

Note that the real difficulty in the proof (or disproof) of the conjecture lies in the fact that there is no method of verifying the non-degeneracy of a differential 2-form on a closed manifold corresponding to $[\omega] \in H$. Hence, it looks more promising to analyze various homotopic properties of closed symplectic manifolds obtained by some 'canonical' procedures from the known symplectic manifolds, e.g. 'symplectic surgery' [**7, 10**]. This is not very easy, however, the first attempts have been made recently by David Gay [**7**] and Paul Biran [**3**].

The conjecture of Thurston was relaxed by Sullivan [**28**] : *Are there topological properties or invariants of closed symplectic manifolds which ensure the existence of symplectic structures?* In particular, Sullivan asked *if the existence of symplectic structure can be read off the minimal model of the given manifold?*

The problems mentioned above are very general and difficult to solve, however, they have been stimulating a large number of research papers devoted to topology (homotopy) of symplectic manifolds. We would describe the known results in the area as results of the period of 'collecting information', the breakthrough ahead.

Keeping in mind our main conjectures and questions we can formulate several 'restricted' problems whose solution will hopefully help to settle our basic problems. To start with, note that Kähler manifolds have many homotopic properties which distinguish them from merely symplectic. We mention here *formality, the hard Lefschetz property, even odd-dimensional Betti numbers,* dd^c*-lemma* [**13**], *vanishing Massey products.* We know already (see

[**1, 5, 27, 32**] and Section 4) that *all* these properties are in general violated by symplectic manifolds (although in the simply-connected case this was proved very recently [**1, 27**]). However, it is important to learn: *what are the relations between the above mentioned properties in case of closed symplectic manifolds?* For example, there are two (contradicting) conjectures.

(1) (Merkulov [**25**]) *Any closed symplectic manifold with the hard Lefschetz property is formal,*
(2) *There are closed symplectic manifolds with the hard Lefschetz property but non-formal.*

Note that Greg Lupton [**20**] constructed an example of c-symplectic closed manifold which is not formal but which satisfies the hard Lefschetz property. This suggests the second conjecture, although his method does not allow to prove that his manifold is true symplectic. It is worth mentioning that in Merkulov's paper the multiplication structure of the de Rham algebra is not analyzed, so the 'formality' considered by the author of [**25**] is not the standard formality introduced in Section 2.

In [**6**] Donaldson has proved that any closed symplectic manifold after a blow-up along certain symplectic submanifold admits a structure of the *Lefschetz fibration.* Hence, in view of the questions, posed above, we can formulate the following problem

Describe homotopy properties of Lefschetz fibrations, and in particular, their Quillen (or Sullivan) models.

Finally, we should mention an intriguing area of research related to symplectic group actions. This subject has its origin in a beautiful theorem of Ginzburg and Kirwan [**8, 17**] that any symplectic closed manifold endowed with a *hamiltonian* action of a compact Lie group has the property that the Leray-Serre spectral sequence of the Borel fibration

$$M \to EG \times_G M \to BG$$

degenerates. Clearly, this is a *homotopic* characterization of the *hamiltonian* group action. It is natural to ask if there are other homotopy properties characterizing symplectic or hamiltonian group actions.

4. Lefschetz fibrations and symplectic surgery

Let M be a smooth manifold of dimension $2n+2, n \geq 0$ and let $f : M \to S^2$ be a smooth surjective mapping with a finite number of critical points. We will identify S^2 with the extended complex plane $\mathbb{C} \cup \{\infty\}$. If $z \in S^2$ is a regular value of f, then $f^{-1}(z)$ is called the regular fiber of f.

Definition. The smooth mapping $f : M \to S^2$ is called a *Lefschetz fibration* if each critical point p of f admits a coordinate neighbourhood with complex valued coordinates $(w_1, ..., w_{n+1})$ consistent with the given orientation of M, and if $f(p)$ has a complex coordinate z, consistent with the given orientation of S^2, then, locally, f has the form

$$f(w) = z_0 + w_1^2 + ... + w_{n+1}^2.$$

The most important for us information about Lefschetz fibrations is that they have a handlebody decomposition which is not very difficult to describe. Namely, we know that any Morse function $F : M \to \mathbb{R}$ determines a handlebody decomposition of M [**18**]. For any Lefschetz fibration $f : M \to S^2$ we assume that the regular fiber is X and has dimension $2n$ and that 0 and ∞ are regular values of f. Then we define $F : M \to \mathbb{R} \cup \infty$ by $F(p) = |f(p)|^2$. One can check that outside of $f^{-1}(0) \cup f^{-1}(\infty)$, F has only non-degenerate critical points, each of index $n+1$. Hence, if one knows the handlebody decomposition of X, one can describe the handlebody decomposition of M (see [**16**] for details).

The recent result of Donaldson relates Lefschetz fibrations and symplectic manifolds as follows.

THEOREM 4.1. [**6**] *If (M, ω) is a symplectic manifold with $[\omega]$ integral, then there exists a codimension 2 symplectic submanifold $B \subset M$ such that the symplectic blow-up $\tilde{M}$ of M along B admits a structure of the Lefschetz fibration.*

REMARK. Note that we don't introduce the defintion of the symplectic blow-up here, since we treat it in detail in the next section.

Now, let us speculate a bit about possible relations of this result with the main problems of symplectic homotopy. Since the Lefschetz fibrations have a 'controllable' handlebody decomposition, there is a hope to describe the rational homotopy invariants of such decompositions using *Quillen models* [**29**]. The advantage of using the Quillen models comes from the fact that it is possible to use them to calculate models of *homotopy pushouts* [**2**]. Since the handlebody decomposition can be considered from the homotopic point of view as a pushout, there is a hope to get new homotopic information about the Lefschetz fibrations. Combining this with the information about the blow-ups (see the next section) one may hope to obtain new homotopic results about symplectic manifolds. However, we stress that this is a mere speculation and we are not precise here.

In relation with the handlebody decompositions of symplectic manifolds, one can go in the 'reversed' direction and try to build symplectic manifolds using the procedure of attaching handles. The main problem here is: *can one do it symplectically?* In dimension 4, there is an answer to this question obtained recently by David Gay in his Ph. D. Thesis [**7**].

5. Blow-ups

The purpose of this section is to analyze the formality of a symplectic blow-up. Let $\tilde{X}$ be the result of the symplectic blow-up of a symplectic compact manifold X along a symplectic submanifold M. We are interested in describing relations between rational homotopy properties of $\tilde{X}$ and those of X and M. In general it seems to be rather difficult to handle this problem, however, the 'first step' obstructions to formality, namely, the *triple* Massey products appear to behave 'well' enough. The main results of the paper show that the triple Massey products in both $H^*(X)$ and $H^*(M)$ survive under the blow-up construction, and this gives many of examples of compact symplectic non-formal (and, hence, non-Kähler) manifolds. There are many examples of *non-simply connected* symplectic compact non-Kähler manifolds [**5, 32**], but until recently there were no examples in the simply-connected world (cf. [**32**]). The first example of this type was obtained recently by Babenko and Taimanov [**1**]. The present section contains results which essentially give 'qualitative' description of the whole problem. The proofs of our results appear to be a modification of Gitler's description of the multiplication structure of the cohomology of a complex blow-up [**9**].

The proofs of Theorems 5.1 and 5.2 below are based on the use of *Thom spaces*. The basic references to this subject are [**4, 26**]. If $k : M \to X$ is an embedding with the normal bundle N, then the tubular neighbourhood theorem yields inclusions

$$M \hookrightarrow N \hookrightarrow X$$

Let $T(N)$ denotes the Thom space of the vector bundle N. Define the *Browder-Novikov map*

$$t : X \to T(N)$$

as follows. Shrink $X - N$ to a point and get a map t to the one-point compactification of N, i.e. to the Thom space $T(N)$.

Results on formality of symplectic blow-ups.

THEOREM 5.1. *Let X be any compact symplectic manifold whose minimal model has non-vanishing triple Massey products. Then, its blow-up along any symplectic compact submanifold M is non-formal.*

THEOREM 5.2. *Let $\tilde{X}$ be the symplectic blow-up of X along a symplectic submanifold M of codimension $2k$, $k > 3$. If the minimal model of M has non-vanishing triple Massey products, $\tilde{X}$ is non-formal.*

REMARK. In this section we give only the proofs of the theorems. Using these results and the Gromov-Tischler embedding theorem [**14, 30**], it is possible to construct a lot of examples of simply connected closed symplectic non-formal manifolds (cf. [**1, 27**]).

Massey products and formality. In our paper we use only one obstruction to formality, namely, *triple Massey products.* Since this obstruction is essential in our considerations, we give details of it. Thus, let (A, d) be any differential graded algebra. Let $[u], [v], [w] \in H^*(A)$ be cohomology classes satisfying the equalities

$$[u][v] = [v][w] = 0.$$

Consider the corresponding conditions on the cochain level:

$$dx = uv, \quad dy = vw.$$

One can check that the cochain

$$uy + \bar{v}x, \quad \bar{v} = (-1)^{deg(v)}v$$

is in fact a cocycle.

DEFINITION. The set of all cohomology classes which can be constructed in this way is called the *triple Massey product* of $[u], [v], [w]$. This set is denoted by

$$\langle [u], [v], [w] \rangle.$$

Therefore,

$$\langle [u], [v], [w] \rangle = \{[uy + \bar{v}x], \text{ where } u, v, x, y \text{ vary over the choices } \}$$

This means that u, v vary over the choices of cocycles representing the same cohomology classes and x, y over the choices which satisfy $dx = uv, dy = vw$.

Note again that the triple Massey product is a *set* and there is an indeterminacy in the choice of the corresponding cohomology classes. This caution will be important in our considerations. However, this indeterminacy can be completely described. Namely, when defined, $\langle [u], [v], [w] \rangle$ is an element of the cohomology ring

$$H^*(A)/([a], [c])$$

where $([u], [w]) \subset H^*(A)$ denotes the ideal in $H^*(A)$ generated by $[u]$ and $[w]$. This can be proved very easily by calculating the difference between cohomology classes representing the triple Massey product and determined by different choices of u, v, w, x, y. Also, one may consult [**32**] and [**22**, p. 289]. The most important observation for us is that *any two cohomology classes representing the same Massey product differ by an element from* $([u], [w]) \subset H^*(A)$.

Triple Massey products give an obstruction to formality: *if* $(\mathcal{M}_X, H^*(X))$ *has non-trivial Massey products, X is non-formal.* The proof of this can be found in [**32**].

Symplectic blow-up. Let us begin with a symplectic submanifold $M^{2(n-k)} \subset X^{2n}$ with normal bundle $\nu : E \to M$. We choose ν to be ω-orthogonal to $TM \subset TX$. Therefore, we assume that the fiber of ν is $\mathbb{C}^k$ and the structure group of the associated principal frame bundle is $U(k)$, i.e.

$$U(k) \to P \to M.$$

Consider also the *canonical line bundle*

$$L \to \mathbb{C}P^{k-1}, \quad L = \{(z,l) \in \mathbb{C}^k \times \mathbb{C}P^{k-1} | \quad z \in l\}.$$

Note that $U(k)$ acts on $\mathbb{C}^k$ and $\mathbb{C}P^{k-1}$ by the same rule $z \to Az, [z] \to [Az]$. Hence, we may form the bundles

$$L \to \tilde{E} \to M, \quad \mathbb{C}P^{k-1} \to \tilde{M} \to M$$

with the total spaces

$$\tilde{E} = P \times_{U(k)} L, \quad \tilde{M} = P \times_{U(k)} \mathbb{C}P^{k-1},$$

respectively.
Also $\tilde{E}$ becomes a line bundle over $\tilde{M}$:

$$\tilde{E} = P \times_{U(k)} L \to \tilde{M} = P \times_{U(k)} \mathbb{C}P^{k-1}, \quad [p,(z,l)] \to [p,l]$$

where the square brackets stand for equivalence classes with respect to the $U(k)$-action, $p \in P, (z,l) \in L, l \in \mathbb{C}P^{k-1}$. Note also that $\tilde{M}$ is by definition the projectivized normal bundle associated to ν.

Define the map

$$\phi : \tilde{E} \to E, \quad \phi([p,(z,l)]) = [p,z]$$

and consider the sets $\tilde{E}_0 = \tilde{E} - \{\text{the zero section}\}$ and $E_0 = E - \{\text{the zero section}\}$. Also, M and $\tilde{M}$ can be embedded into $\tilde{E}$ and E as the zero sections of the vector bundles

$$E \to M, \quad \tilde{E} \to \tilde{M},$$

respectively.
These embedings can be described explicitly:

$$\pi(p) \to [p,0], \quad [p,l] \to [p,(0,l)]$$

for $\pi : E \to M$ (one checks immediately that everything is well-defined). Now, recall the construction of the symplectic blow-up of a symplectic manifold X along a symplectic submanifold M. Take the normal bundle ν and the corresponding disc bundle

$$V \to M.$$

According to the tubular neighbourhood theorem,

$$V \cong W \subset X$$

for tubular neighbourhood $W \supset M$ of M in X. Set

$$\tilde{V} = \phi^{-1}(V) \subset \tilde{E}$$

and note that ϕ is a diffeomorphism on $\tilde{E}_0$, i.e.

$$\phi|_{\tilde{E}_0} : \tilde{E}_0 \xrightarrow{\cong} E_0.$$

The latter can be seen from the representations

$$\tilde{E}_0 = \{[p,(z,l)], \quad z \neq 0, z \in l\}, \quad E_0 = \{[p,z], \quad z \neq 0\}$$

and the definition of ϕ. In particular, $\tilde{V} \cap \tilde{E}_0 \cong V \cap E_0$, and since $V \cap E_0$ does not contain M realized as the zero section, we get

$$\partial\tilde{V} \cong \partial V.$$

DEFINITION. The *blow-up* $\tilde{X}$ of X along M is the smooth manifold

$$\tilde{X} = \overline{(X - W)} \cup_{\partial\tilde{V}} \tilde{V},$$

where the identification of points is given by

$$\partial\tilde{V} \cong \partial V \cong \partial W.$$

The role of this construction is that it yields new symplectic manifolds: *any symplectic blow-up again carries a symplectic structure* [**24**].

In the sequel we will need one additional observation that

$$\tilde{V} = \{[p,(z,l)], \quad |z| \leq 1\} = P \times_{U(k)} \tilde{D}$$

where $\tilde{D}$ is the usual complex blow-up of a disc. Now, one can define a projection $\tilde{X} \to X$ separately on $\tilde{V}$ as a restriction of ϕ and on $X - W$ as identity. This projection will be also denoted by ϕ.

One can see that there is a natural map

$$\tilde{k} : \tilde{M} \to \tilde{X} = \overline{(X - W)} \cup_{\partial\tilde{V}} \tilde{V}$$

defined by

$$\tilde{k}([p,l]) = [p,(0,l)] \in \tilde{V} \subset \tilde{X}$$

We claim that the diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{k}} & \tilde{X} \\ {\scriptstyle\phi}\downarrow & & {\scriptstyle\phi}\downarrow \\ M & \xrightarrow{k} & X \end{array}$$

is commutative (note that we interpret M as an image of the zero section of E). Indeed, the point $[p,l] \in \tilde{M}$ is embedded into $\tilde{X}$ as $[p,(0,l)]$. Therefore, $\phi([p,(0,l)]) = [p,0]$. The latter can be identified with a point in the zero section of E, i.e., with a point in M.

Finally, we have come to

PROPOSITION 5.3. *There exists a commutative diagram*

$$\begin{array}{ccccccc} \tilde{M} & \longrightarrow & \tilde{N} & \longrightarrow & \tilde{X} & \xrightarrow{\tilde{t}} & T(\tilde{N}) \\ {\scriptstyle\phi}\downarrow & & \downarrow & & {\scriptstyle\phi}\downarrow & & {\scriptstyle\hat{\phi}}\downarrow \\ M & \longrightarrow & N & \longrightarrow & X & \xrightarrow{t} & T(N) \end{array}$$

PROOF. We have just proved the commutativity of the first two squares. The remaining squares are commutative because of the general property of Thom spaces: we consider the normal bundles $\tilde{N}$ and N of embeddings

$$\tilde{M} \hookrightarrow \tilde{X}, \quad M \hookrightarrow X.$$

and embed M and $\tilde{M}$ as zero sections, N and $\tilde{N}$ according to the tubular neighbourhood theorem, then the map between the normal bundles is induced by ϕ and hence $\hat{\phi}$ is the corresponding map of one-point compactifications. Note that we have more, namely, N is identified with E (it is by definition) and $\tilde{N}$ is identified with $\tilde{E}$ considered as a line bundle over $\tilde{M}$. Indeed, one can write an obvious inclusion $\tilde{M} \to \tilde{V}$ and recall the representation of $\tilde{V} = \{[p,(z,l)], \quad |z| \leq 1\} \subset \tilde{X}$ diffeomorphic to the tubular neighbourhood of $\tilde{M}$ in $\tilde{X}$. Since $\tilde{V}$ (open) is obviously diffeomorphic to $\tilde{E}$, the result follows. Note that we did not change notation for the 'open' bundle $\{[p,(z,l)], |z| < 1\}$ corresponding to $\tilde{V}$. □

The following results are direct corollaries of [**9**] and Proposition 5.3.

THEOREM 5.4. *We have a short exact sequence*

$$0 \longrightarrow H^q(T(N)) \xrightarrow{\alpha} H^q(X) \oplus H^q(T(\tilde{N})) \xrightarrow{\beta} H^q(\tilde{X}) \longrightarrow 0$$

$$\alpha(y) = (t^*y, \hat{\phi}^*y), \quad \beta(x,z) = \phi^*x - \tilde{t}^*z.$$

The sequence is not multiplicative.

In the sequel we denote the Thom class of the vector bundle $\tilde{N}$ by U. Note that $U \in H^2(T(\tilde{N}))$. It is known that $\tilde{t}^*(U) = \zeta$, where ζ is identified with the Euler class of the vector bundle $\tilde{N}$ (considered now as a real 2-bundle over M). Consider the usual cup-product $H^*(T(\tilde{N})) \otimes H^*(M) \to H^*(T(\tilde{N}))$, and the Thom isomorphism determined by it, i.e. $H^p(M) \to H^{p+2}(T(\tilde{N})), v \to U \cup v$.

THEOREM 5.5. *The multiplication rule in the cohomology of the symplectic blow-up $\tilde{X}$ of X along M is given as follows. In the direct sum of vector spaces*

$$H^*(\tilde{X}) \cong \phi^*H^*(X) \oplus_{v.s.} \tilde{H}^*(M)[\zeta]/(\zeta^k - c_1\zeta^{k-1} + \ldots + (-1)^{k-1}c_{k-1}\zeta + (-1)^k c_k)$$

one multiplies elements as indicated below:

(i) *in $H^*(X)$, in accordance with the multiplication rule in $H^*(X)$,*

(ii) *in* $\tilde{H}^*(M)[\zeta]/(\zeta^k - c_1\zeta^{k-1} + ... + (-1)^{k-1}c_{k-1}\zeta) = \tilde{H}^*(M)[\zeta]/I$ *by*

$$[u\zeta^p] \cdot [v\zeta^q] = \tilde{t}^*((U \cup \zeta^{p-1}u) \cdot (U \cup \zeta^{q-1}v))$$

(iii) *for* $\phi^* u \in H^*(X)$ *and* $[v\zeta^p] \in \tilde{H}^*(M)[\zeta]/I$, *by the formula*

$$\phi^* u \cdot [v\zeta^p] = [k^* uv\zeta^p]$$

where $k^* : H^*(X) \to H^*(M)$ *is induced by the embedding* k.

PROOF OF THEOREMS 5.4 AND 5.5. Consider, first, the manifolds

$$X = \overline{X - W} \cup_{\partial W} W \quad \text{and} \quad \tilde{X} = \overline{X - W} \cup_{\partial \tilde{V}} \tilde{V}$$

together with the accompanying homology ladder of Mayer-Vietoris sequences (with integral coefficients)

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & H_i(\overline{X - W}) \oplus H_i(\tilde{V}) & \longrightarrow & H_i(\tilde{X}) & \longrightarrow & H_{i-1}(\partial\tilde{V}) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\dots & \longrightarrow & H_i(\overline{X - W}) \oplus H_i(W) & \longrightarrow & H_i(X) & \longrightarrow & H_{i-1}(\partial W) & \longrightarrow
\end{array}$$

$$\begin{array}{ccc}
H_{i-1}(\overline{X - W}) \oplus H_{i-1}(\tilde{V}) & \longrightarrow & \dots \\
\downarrow & & \downarrow \\
H_{i-1}(\overline{X - W}) \oplus H_{i-1}(W) & \longrightarrow & \dots
\end{array}$$

Recall that, if Z is an oriented m-manifold with boundary ∂Z, then there is a natural isomorphism

$$H_m(Z, \partial Z) \cong H_{m-1}(\partial Z) \cong \mathbb{Z}.$$

This isomorphism implies that the inclusion $\partial Z \to Z$ induces the *zero* map $H_{m-1}(\partial Z) \to H_{m-1}(Z)$. Thus, in the Mayer-Vietoris ladder above we have

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_{2n}(\tilde{X}) & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \cong\downarrow & & \downarrow \\
0 & \longrightarrow & H_{2n}(X) & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

Therefore, we have proved that the blow-up projection ϕ is an isomorphism in $2n$-homology over $\mathbb{Z}$:

$$\phi_* : H_{2n}(\tilde{X}) \xrightarrow{\cong} H_{2n}(X)$$

Of course, this is also true for cohomology and this has the effect that the induced map on cohomology $\phi^* : H^*(X) \to H^*(\tilde{X})$ is *injective*. To see this, suppose that $\phi^*(\alpha) = 0$ for $\alpha \in H^j(X)$. By Poincaré duality, there exists a

$\beta \in H^{2n-j}(X)$ with $\alpha \cup \beta$ the (cohomology) fundamental class $\nu \in H^{2n}(X)$. But then, the isomorphism above gives a contradiction

$$\begin{aligned} 0 &\neq \phi^*(\nu) \\ &= \phi^*(\alpha) \cup \phi^*(\beta) \\ &= 0 \cup \phi^*(\beta) \\ &= 0. \end{aligned}$$

Hence we have proved the following three facts:

(i) ϕ^* is injective in cohomology and preserves orientation;
(ii) $\phi : \tilde{X} \to X$ is a *degree one map* (by definition);
(iii) the long exact cohomology sequence associated to the inclusion of X into the mapping cylinder of $\phi : \tilde{X} \to X$ yields a short exact sequence of graded vector spaces

$$0 \longrightarrow H^*(X) \xrightarrow{\phi^*} H^*(\tilde{X}) \longrightarrow H^*(X, \tilde{X}) \longrightarrow 0 \qquad (**)$$

The last group denotes the cohomology of the pair $(M_\phi, \tilde{X})$, where M_ϕ is the mapping cylinder of ϕ.

Use the excision property

$$H^*(Y - U, B - U) \cong H^*(Y, B)$$

(recall that U is an open set in Y whose closure $\overline{U}$ is contained in the interior of B). Take

$$Y = X, \quad B = \tilde{X}, \quad U = X - M = \tilde{X} - \tilde{M}.$$

Then

$$H^*(X, \tilde{X}) = H^*(X - (X - M), \tilde{X} - (\tilde{X} - \tilde{M})) = H^*(M, \tilde{M}) \qquad (***)$$

Having all this in mind we can proceed as follows. By (i), ϕ is a degree one map and, therefore, one can define the *Gysin map*

$$f! : H^*(\tilde{X}) \to H^*(X), \quad [X] \cap f!\tilde{x} = f_*([\tilde{X}] \cap \tilde{x})$$

(where $\cap$ denotes the cap-product and $[X]$ and $[\tilde{X}]$ denote the fundamental homology classes. One can easily check that $f!$ is a left invertible of f^*: $f!f^* = \mathrm{id}_{H^*(X)}$ and hence

$$H^*(\tilde{X}) = f^*H^*(X) \oplus (\mathrm{Ker}\, f!).$$

This check-up is made, e.g., in [**9**]. Indeed, one can notice that

$$[X] \cap f!f^*x = f_*([\tilde{X}] \cap f^*x) = f_*[\tilde{X}] \cap x = [X] \cap x$$

as required. In the same vein, one can prove the following relation. □

LEMMA 5.6. *For any* $y \in (Ker f!)$ *and* $x \in H^*(X)$

$$y \cup f^* x \cdot \in Ker f!.$$

PROOF. See [**9**]. □

Now, to proceed further, we need the diagram of spaces given by Proposition 3.5. Consider the decompositions

$$\tilde{X} = (\tilde{X} - \tilde{M}) \cup \tilde{N}, \quad (\tilde{X} - \tilde{M}) \cap \tilde{N} = \tilde{N} - \tilde{M}$$

$$X = (X - M) \cup N, \quad (X - M) \cap N = N - Y$$

They satisfy the excision property and therefore one can write the Mayer-Vietoris sequences

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & H^{q-1}(\tilde{N} - \tilde{M}) & \xrightarrow{\tilde{\alpha}} & H^q(\tilde{X}) & \xrightarrow{\tilde{j}^*} & H^q(\tilde{N}) \oplus H^q(\tilde{X} - \tilde{M}) & \xrightarrow{\tilde{\beta}} & \\
\uparrow & & \cong \uparrow f_2^* & & \phi^* \uparrow & & \bar{f}^* \oplus f_1^* \uparrow & & \\
\ldots & \longrightarrow & H^{q-1}(N - M) & \xrightarrow{\alpha} & H^q(X) & \xrightarrow{j^*} & H^q(N) \oplus H^q(X - M) & \xrightarrow{\beta} &
\end{array}$$

$$\begin{array}{ccc}
H^q(\tilde{N} - \tilde{M}) & \longrightarrow & \ldots \\
f_2^* \uparrow & & \\
H^q(N - M) & \longrightarrow & \ldots
\end{array}$$

where f_1^* and f_2^* are isomorphisms, ϕ^* and $\bar{f}^*$ are monomorphisms. Indeed, f_1^* and f_2^* are induced by diffeomorphisms $\tilde{X} - \tilde{M} \to X - M$ and $\tilde{N} - \tilde{M} \to N - M$. Note that the latter manifolds are diffeomorphic, since $\tilde{E}_0$ and E_0 are diffeomorphic. Then it follows that

$$\tilde{X} - \tilde{M} = \overline{(X - W)} \cup_{\partial \tilde{V}} (\tilde{V} - \tilde{M}) \cong \overline{(X - W)} \cup_{\partial V} (V - M) = X - M$$

and the diffeomorphism follows from the fact that $\tilde{V} - \tilde{M} \cong V - M$. As fas as $\bar{f}^*$ is concerned, one can notice that $\tilde{M}$ is a projectivized normal bundle over M and hence

$$H^*(\tilde{M}) \cong H^*(M)[\zeta]/I, \quad I = (\zeta^k - c_1\zeta^{k-1} + \ldots(-1)^k c_k)$$

where $c_1, \ldots, c_k$ denote the Chern classes of N. This is known and we refer to [**32**]. Also, $\bar{f}^*$ is the same as injection

$$g^* : H^*(M) \to H^*(\tilde{M}) = H^*(M)[\zeta]/I.$$

Also, as a map in the Mayer-Vietoris sequence, $\tilde{j}^* = (\tilde{j}_1^*, \tilde{j}_2^*)$, where $\tilde{j}_1^* : H^*(\tilde{X}) \to H^*(\tilde{N}) \cong H^*(\tilde{M})$ is induced by inclusion $\tilde{N} \to \tilde{X}$ and $\tilde{J}_2^* : H^*(\tilde{X}) \to H^*(\tilde{X} - \tilde{M})$ is induced by $\tilde{X} - \tilde{M} \to \tilde{X}$.

LEMMA 5.7. *The map* $\tilde{j}^*$ *in the Mayer-Vietoris diagram above induces an isomorphism*

$$Ker\,\phi! \cong H^*(\tilde{M})/g^*H^*(M).$$

PROOF. The proof follows from diagram chasing. Indeed, consider the following algebraic situation. Let

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\alpha} & C & \xrightarrow{\beta} & D & \longrightarrow & \cdots \\
\uparrow & & \cong\uparrow\theta & & i_B\uparrow & & i_C\uparrow & & \cong\uparrow\eta & & \uparrow \\
\cdots & \longrightarrow & A & \xrightarrow{\mu_1} & B_1 & \xrightarrow{\alpha_1} & C & \xrightarrow{\beta_1} & D & \longrightarrow & \cdots
\end{array}$$

be a commutative diagram in which horizontal rows are exact. Assume that θ and η are isomorphisms and that i_B and i_C are monomorphisms. Then $B/i_B(B_1) \cong C/i_C(C_1)$. The proof is accomplished as follows. By exactness, $\mathrm{Im}\mu = \mathrm{Ker}\,\alpha$ and $\mathrm{Im}\,\mu_1 = \mathrm{Ker}\,\alpha_1$. Obviously, $i_B(\mathrm{Im}\,\mu_1) \subset \mathrm{Im}\,\mu$. Also, from the commutativity of the first square and injectivity of i_B the converse inclusion follows, which gives

$$\mathrm{Im}\,\mu = i_B(\mathrm{Im}\,\mu_1) \implies \mathrm{Ker}\,\alpha = i_B(\mathrm{Ker}\,\alpha_1).$$

The latter implies also the inclusion

$$\mathrm{Ker}\,\alpha \subset i_B(B_1).$$

Consider the decomposition $B = i_B(B_1) \oplus W$. Note that $\alpha|_W$ is injective. From the commutativity of the middle square we get $i_C(\mathrm{Im}\,\alpha_1) = \alpha(i_B(B_1))$ which implies $\alpha(i_B(B_1)) \subset i_C(C_1)$. The latter yields a well-defined map

$$\bar{\alpha} : W \cong B/i_B(B_1) \to C/i_C(C_1).$$

This map is injective. Indeed, assume that $\alpha(w) \in i_C(C_1)$. Then, $\alpha(w) = i_C(c_1)$. By exactness, $\beta\alpha(w) = 0 \implies \beta i_C(c_1) = 0 \implies \beta_1(c_1) = 0$, since η is an isomorphism. Again, by exactness, $c_1 = \alpha_1(b_1)$. Hence,

$$\alpha(w) = i_C(\alpha_1(b_1)) = \alpha(i_B(b_1)) \implies \alpha(w - i_B(b_1)) = 0.$$

Therefore

$$w - i_B(b_1) \in \mathrm{Ker}\,\alpha \subset i_B(B_1)$$

which is a contradiction. Hence, $\bar{\alpha}$ is injective.

To prove the surjectivity of $\bar{\alpha}$, one must consider the 'next' square in the given diagram:

$$\begin{array}{ccccc}
C & \xrightarrow{\beta} & D & \xrightarrow{\mu^+} & B^+ \\
i_C\uparrow & & \cong\uparrow\theta & & i_{B^+}\uparrow \\
C_1 & \xrightarrow{\beta_1} & D & \xrightarrow{\mu_1^+} & B_1^+
\end{array}$$

Let $c \in C$. Then $\beta(c) = d \in D$ and $d = \eta(d_1)$. Consider $i_{B^+}(\mu_1^+(d_1))$. Assume that $\mu_1^+(d_1) \neq 0$. Then $i_{B^+}(\mu_1^+(d_1)) \neq 0$. But $i_{B^+}\mu_1^+(d_1) = \mu^+\eta(d_1) = \mu^+(d) = \mu^+\beta(c) = 0$, because of the exactness. Hence, $\mu_1^+(d_1) = 0 \implies d_1 \in \operatorname{Ker} \mu_1^+ \implies d_1 = \beta_1(c_1)$. Therefore, $d = \eta(d_1) = \eta\beta(c_1) = \beta i_C(c_1)$. Finally,

$$\beta(c - i_C(c_1)) = 0 \implies c - i_C(c_1) \in \operatorname{Im} \alpha$$

which implies the surjectivity of $\bar{\alpha}$. Lemma 5.7 is proved. □

Combining all the results above together, we get an *additive* decomposition of $H^*(\tilde{X})$.

Now the proof of Theorem 5.4 goes exactly as the proof of the analogous theorem for complex blow-ups in [**9**].

Now we begin the proof of Theorem 5.5. Property (i) in the formulation of Theorem 5.5 is obvious. To prove (ii), note that the isomorphism $\operatorname{Ker} \phi! \cong H^*(\tilde{M})/g^*H^*(M)$ was induced by the map $\tilde{j}^*$ (and, hence, by $\tilde{j}_1^*$, since the form of the commutative diagram and the fact that in the proof W was arbitrary). The latter can be identified with $\tilde{k}^*$ (since $\tilde{N}$ is homotopy equivalent to $\tilde{M}$). It means that we have a sequence

$$H^*(T(\tilde{N})) \xrightarrow{\tilde{t}^*} H^*(\tilde{X}) \xrightarrow{\tilde{k}^*} H^*(\tilde{M}) \xrightarrow{\pi} H^*(\tilde{M})/g^*H^*(M)$$

(with π being the natural projection) which identifies $\operatorname{Ker} \phi!$ with $H^*(\tilde{M})/g^*H^*(M)$. Note that $\tilde{t}^*$ and $\tilde{k}^*$ are *multiplicative, but π is not.*

Now we are using Thom spaces. Since $\tilde{E} \cong \tilde{N}$ is a line bundle with the Euler class ζ, we can use the following fact proved in [**26**]:

$$\tilde{k}^*\tilde{t}^*(U) = \zeta$$

where U is the Thom class of the line bundle $\tilde{N}$. From the usual properties of the cup product

$$\tilde{k}^*\tilde{t}^*(U \cup \zeta^p y) = \tilde{k}^*\tilde{t}^*(U)\zeta^p y = \zeta^{p+1} y.$$

Therefore, for $\bar{k}^*\tilde{t}^* : H^*(T(\tilde{N}) \to H^*(\tilde{M})/g^*H^*(M)$ determined by $\tilde{k}^*\tilde{t}^*$ and the canonical projection

$$\bar{k}^*\tilde{t}^*(U \cup \zeta^p y) = [\zeta^{p+1} y]$$

and this proves (ii).

To prove (iii), we note that we must find the result of the multiplication $\phi^*x \cdot z$, where $z \in H^*(\tilde{X})$ corresponds to some $[y\zeta^p] \in H^*(\tilde{M})/g^*H^*(M)$ by the map $\pi\tilde{k}^*$. By Lemma 6.5, since $z \in \operatorname{Ker} \phi!$, $\phi^*x \cdot z \in \operatorname{Ker} \phi!$. Therefore, the result of multiplication of ϕ^*x and z again can be described as the image $\pi\tilde{k}^*$.

Therefore

$$\pi\tilde{k}^*(\phi^*x\cdot z) = \pi(\tilde{k}^*(\phi^*x\cdot z)) = \pi(g^*k^*x\cdot\tilde{k}^*(z)) = \pi(g^*k^*x\cdot y\zeta^p) = [k^*xy\zeta^p].$$

This yields the necessary identification.

Proof of Theorem 5.1. Recall that we need to prove the following. *Any compact symplectic blow-up $\tilde{X}$ of X along M is non-formal, if X has non-vanishing triple Massey products.*

PROOF. By Theorem 5.5,

$$H^*(\tilde{X}) = H^*(X)\oplus\tilde{H}^*(M)[\zeta]/(z^k - c_1\zeta k - 1 + \ldots + (-1)^{k-1}c_{k-1}\zeta + (-1)^k c_k)$$

with the multiplication rules determined by $\tilde{t}^*$ and k^*. The property of this multiplication, which we need, can be reformulated as follows:

$$H^*(\tilde{X}) = \phi^*H^*(X)\oplus_{v.s.} W$$

where

$$\phi^*(x)\cdot w = w' \in W, \quad \text{for any} \quad x\in H^*(X), w\in W.$$

Now denote $(A, d_A) = A_{PL}(X)$ and $(B, d_B) = A_{PL}(\tilde{X})$ and consider the following algebraic situation. Let (A, d_A) and (B, d_B) be differential graded algebras such that there exists a DGA-morphism $\gamma : (A, d_A)\to(B, d_B)$ inducing *monomorphism* in cohomology

$$\gamma^* : H^*(A) \xrightarrow{\text{mono}} H^*(B)$$

Assume that

$$H^*(B) = \gamma^*H^*(A)\oplus_{v.s.} W$$

and

$$\gamma^*[a]\cdot w = w' \in W, \quad \text{for any } [a]\in H^*(A), w\in W.$$

Then the minimal model $(\mathcal{M}_B, \delta_B)$ of B inherits non-vanishing triple Massey products of $(\mathcal{M}_A, \delta_A)$. We are going to show that any non-trivial Massey product in $H^*(A)$ determines a non-trivial Massey product in $H^*(B)$. Note that in general this is not the case, since the *indeterminancy* in the definition of the cohomology classes representing Massey products. But the restrictions on $H^*(B)$ force this indeterminancy to be 'the right one'. So let $[a], [b], [c]$ be cohomology classes such that $[a][b] = [b][c] = 0$ and choose a representative cocycle in $\mathcal{M}_A$

$$u = ay + xc, \quad \delta_A x = ab, \quad \delta_A y = bc.$$

We don't write the right signs here, since it does not influence the argument. Since we have assumed that there exists a non-trivial Massey product, this means that one can choose u in a way to get

$$[u]\notin([a],[c])\subset H^*(A).$$

Now consider the map $\hat{\gamma}$ induced by $\gamma : A \to B$ with $\gamma^* = \hat{\gamma}^*$, i.e. $\hat{\gamma} : \mathcal{M}_A \to \mathcal{M}_B$. Then the cohomology class

$$\hat{\gamma}^*[u] = [\hat{\gamma}(a)\hat{\gamma}(y) + \hat{\gamma}(c)\hat{\gamma}(x)] \in \gamma^* H^*(A)$$

represents some Massey product of the pair $(\mathcal{M}_B, H^*(B))$. In general, as we have already noticed, this cohomology class may represent a vanishing Massey product in $H^*(B)$. However, in our case we can perform the following argument. Assume that $\hat{\gamma}^*[u] = \langle \hat{\gamma}^*[a], \hat{\gamma}^*[b], \hat{\gamma}^*[c] \rangle = 0$. Then, by definition,

$$\hat{\gamma}^*[u] \in (\hat{\gamma}^*[a], \hat{\gamma}^*[c]) \subset H^*(B).$$

Hence

$$\hat{\gamma}^*[u] = \hat{\gamma}^*[a]\theta + \hat{\gamma}^*[c]\eta, \quad \theta, \eta \in H^*(B).$$

Because of the assumptions on the multiplication in $H^*(B)$, one can write

$$\hat{\gamma}^*[u] = \hat{\gamma}^*[a](\hat{\gamma}^*(x_1) + w_1) + \hat{\gamma}^*[c](\hat{\gamma}^*(x_2) + w_2)$$

where

$$\theta = \hat{\gamma}^*(x_1) + w_1, \quad \eta = \hat{\gamma}^*(x_2) + w_2, \quad x_i \in H^*(XA), \quad w_i \in W.$$

Therefore

$$\hat{\gamma}^*[u] = \hat{\gamma}^*[a]\hat{\gamma}^*(x_1) + \hat{\gamma}^*[c]\hat{\gamma}^*(x_2) + (\hat{\gamma}^*[a]w_1 + \hat{\gamma}^*[c]w_2)$$

with the last bracket *belonging to* W, because of our assumptions. However, since $H^*(B)$ is a direct sum of $\gamma^* H^*(A)$ and W, the last bracket must be zero and we have obtained an equality

$$\hat{\gamma}^*[u] = \hat{\gamma}^*[a]\hat{\gamma}^*(x_1) + \hat{\gamma}^*[c]\hat{\gamma}^*(x_2)$$

which can be transferred to $H^*(A)$, but in $H^*(A)$ such equality is impossible.

To complete the proof of the first part of Theorem 1.5, it is enough to specialize A and B as was indicated above, as a de Rham polynomial algebras of X and $\tilde{X}$.

Thus, we have proved that non-vanishing triple Massey products survive under blow-ups along *any* submanifold. □

Proof of Theorem 5.2. Recall that we need to prove the following statement.
Let $\tilde{X}$ be the symplectic blow-up of X along a symplectic submanifold M of codimension $2k$, $k > 3$. If $(\mathcal{M}_M, H^*(M))$ has non-vanishing triple Massey products, the same is valid for $(\mathcal{M}_{\tilde{X}}, H^*(\tilde{X}))$.

PROOF. The proof of this theorem follows from Lemmas 5.8 and 5.9 below. □

LEMMA 5.8. *Let $\tilde{M}$ denote the projectivization of the complex vector bundle $E \to M$ of rank k. If $k > 3$, the non-zero Massey triple products in $H^*(M)$ determine non-zero Massey triple products in $H^*(\tilde{M})$.*

PROOF. It is not difficult to show using the theory of KS-extensions [**15**], that

$$\mathcal{M}_{\tilde{M}} = (\mathcal{M}_M \otimes \Lambda(\zeta, y), D)$$

$$D\zeta = 0, \quad Dy = \zeta^k - c_1\zeta^{k-1} + ... + (-1)^{k-1}c_{k-1}\zeta + (-1)^k c_k, \quad |\zeta| = 2, |y| = 2k-1$$

where $c_j \in \mathcal{M}_M$ represent the rational Chern classes $c_j(E)$ via isomorphism $H^*(\mathcal{M}_M) \cong H^*(M)$. (alternatively, one look at the proof of the Lupton-Oprea theorem in [**21**], or at the proof given in [**1**]). Let $[a], [b], [c] \in H^*(M)$ denote cohomology classes determining some Massey triple product, i.e.

$$[a][b] = [b][c] = 0.$$

Assume that $\langle [a], [b], [c] \rangle \neq 0$, i.e. there exists at least one cocycle

$$\bar{a}y + \bar{x}c, \quad dx = ab, \quad dy = bc$$

such that

$$[\bar{a}y + \bar{x}c] \notin ([a], [c]) \subset H^*(M).$$

Consider the cohomology classes

$$[a]\zeta, [b]\zeta, [c]\zeta \in H^*(\tilde{M}).$$

Note that $H^*(\tilde{M}) = H^*(M) \otimes \mathbb{R}[\zeta]/I$, where $I = (\zeta^k - c_1\zeta^{k-1} + ... + (-1)^k c_k)$, which shows that the multiplication rule in $H^*(\tilde{M})$ is the 'natural' one and that these cohomology classes are non-zero. Also, obviously, $[a]\zeta \cdot [b]\zeta = [b]\zeta \cdot [c]\zeta = 0$ and the triple Massey product

$$\langle [a]\zeta, [b]\zeta], [c]\zeta \rangle$$

is well defined. Assume that it is zero. Then, any cohomology class representing it must belong to the ideal $([a]\zeta, [c]\zeta) \subset H^*(\tilde{M})$. However, consider the cocycle

$$\bar{a\zeta}(y\zeta^2) + (\bar{x\zeta^2})c\zeta$$

representing the above Massey product. We get

$$[\bar{a\zeta}(y\zeta^2) + (\bar{x\zeta^2})c\zeta] = [\bar{a}y + \bar{x}c]\zeta^3 =$$
$$[a](\alpha_1 + \alpha_2\zeta + ... + \alpha_k\zeta^{k-1}) +$$
$$+[c](\beta_1 + \beta_2\zeta + ... + \beta_k\zeta^{k-1})$$

which implies that $[ay + cx]\zeta^3 = [a]\alpha_3\zeta^3 + [c]\beta_3\zeta^3$ and the rest of the brackets is zero. Since $\zeta, ..., \zeta^{k-1}$ yield no relations, the coefficients must be equal and $[ay + cx] \in ([a], [c]) \subset H^*(M)$, a contradiction. □

The next lemma is completely algebraic.

LEMMA 5.9. *Let* $f\colon A \to B$ *be a DGA-morphism. Assume that* $[b_1], [b_2], [b_3] \in H^*(B)$ *are cohomology classes determining a triple Massey product* $\langle [b_1], [b_2], [b_3] \rangle$. *Assume that there exist cohomology classes* $[a_1], [a_2], [a_3] \in H^*(A)$ *such that*

(1) $[a_1][a_2] = [a_2][a_3] = 0$
(2) $f^*[a_i] = [b_i]$ *for all* i.

If $\langle [b_1], [b_2], [b_3] \rangle \neq 0$, *then* $\langle [a_1], [a_2], [a_3] \rangle \neq 0$.

PROOF. It is necessary to prove that there exists at least one cocycle, say, $v \in Z(A)$ representing $\langle [a_1], [a_2], [a_3] \rangle$ such that $[v] \notin ([a_1], [a_3])$. Assume that $[v] \in ([a_1], [a_3])$ for any v. Note that $f^*[v]$ represents (obviously), the triple Massey product

$$\langle f^*[a_1], f^*[a_2], f^*[a_3] \rangle = \langle [b_1], [b_2], [b_3] \rangle.$$

Note also that, by the assumption,

$$f^*[v] \in (f^*[a_1], f^*[a_3]) = ([b_1], [b_3]) \subset H^*(B). \tag{$*$}$$

However, any two cohomology classes representing the given triple Massey product, differ by an element in $([b_1], [b_3])$ (by definition). Since $\langle [b_1], [b_2], [b_3] \rangle \neq 0$, there exists $[u] \in H^*(B)$ such that $[u]$ represents the above Massey product and does not belong to the ideal generated by $[b_1]$ and $[b_3]$. By the previous remark

$$f^*[v] = [u] + \alpha[b_1] + \beta[b_3]$$

which together with $(*)$ yields a contradiction. Lemma is proved. □

Let's consider $\tilde{X}$, X and M as in the assumptions of the Theorem. Let $[b_1], [b_2], [b_3] \in H^*(M)$ be cohomology classes representing non-zero triple Massey product. By Lemma 5.8,

$$\langle [b_1]\zeta, [b_2]\zeta, [b_3]\zeta \rangle \neq 0$$

is a non-zero triple Massey product in $H^*(\tilde{M})$. Consider cohomology classes

$$[a_1] = \tilde{t}^*(U \cup [b_1]), \quad [a_2] = \tilde{t}^*(U \cup [b_2]), \quad [a_3] = \tilde{t}^*(U \cup [b_3]) \in H^*(\tilde{X}).$$

From the standard properties of cup-products

$$[a_1][a_2] = \tilde{t}^*(U \cup [b_1]) \cdot t^*(U \cup [b_2]) = \tilde{t}^*(U^2 \cup ([b_1][b_2])) = 0$$

$$[a_2][a_3] = \tilde{t}^*(U \cup [b_2]) \cdot \tilde{t}^*(U \cup [b_3]) = \tilde{t}^*(U^2 \cup ([b_2][b_3])) = 0$$

which yields the triple Massey product $\langle [a_1], [a_2], [a_3] \rangle \in H^*(\tilde{X})$. Consider $\tilde{k}^* : H^*(\tilde{X}) \to H^*(\tilde{M})$ and evaluate it on $[a_i]$. It will give

$$\tilde{k}^*[a_i] = \tilde{k}^*\tilde{t}^*(U \cup [b_i]) = [b_i]\zeta.$$

Applying Lemma 5.9 to $\tilde{k}^*$ and $\langle [a_1], [a_2], [a_3] \rangle$, one completes the whole proof.

6. Homotopy properties of symplectic G-manifolds

This section is devoted to homotopy properties of symplectic manifolds (M, ω) endowed with an action of a Lie group G preserving the given symplectic form. It appeares that the existence of a compact Lie group of symplectomorphisms forces some restrictions on the topology of such manifold. The 'tendency' is that larger symmetry groups of symplectic form force the topology of M to be 'closer' to the topology of Kähler manifolds (see [**33**]). If the action is *hamiltonian*, the restrictions are stronger.

Topology of closed hamiltonian G-manifolds. Let us start with a *symplectic G-manifold* (M, ω). For any smooth function $H : M \to \mathbb{R}$ on a symplectic manifold (M, ω) the vector field $X_H : M \to TM$ determined by the identity

$$i(X_H)\omega = dH$$

is called a *hamiltonian* vector field. If we are given a G-action, i.e. a homomorphism $G \to Symp(M, \omega)$, there exists a homomorphism on the Lie algebra level

$$\mathfrak{g} \to \chi(M, \omega), \quad \xi \to X_\xi$$

defined by

$$(X_\xi)_p = \frac{d}{dt}|_{t=0} \exp(t\xi) \cdot p, \quad p \in M.$$

Definition. We say that the action of G is *weakly hamiltonian*, if each vector field X_ξ is hamiltonian. The action is hamiltonian, if it is weakly hamiltonian and the correspondence

$$\xi \to H_\xi, \text{ where } i(X_\xi)\omega = dH_\xi$$

determines a *Lie algebra homomorphism*

$$\mathfrak{g} \to C^\infty(M)$$

with respect to the Poisson structure on $C^\infty(M)$.

The role of hamiltonian actions in symplectic topology is explained in [**24**]. For us, however, the most important fact is that hamiltonian G-actions are related to the *moment map*

$$\mu : M \to \mathfrak{g}^*.$$

By definition, a moment map is any map as above satisfying the property that the relation

$$H_\xi(p) = \langle \mu(p), \xi \rangle$$

defines a Lie algebra homomorphism $\xi \to H_\xi$. Here $\langle \cdot, \cdot \rangle$ is the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. In fact, the existence of a moment map provides a link to the topology of M through the 'generalized' Morse theory. Namely, μ determines the function $f(p) = |\mu(p)|^2$ which, (although not Morse in the classical sense), yields some 'controllable' stratification of M and gives some information about the cohomology of M as well as the degeneration of the Leray-Serre spectral sequence and some 'formality-like' properties. These results are contained in the works of Ginzburg [**8**], Kirwan [**17**], Goresky-Kottwitz-MacPherson and others [**11**].

As an example we describe one particular problem in this area.

Cohomogeneity 1 symplectic manifolds. A G-manifold M is called *of cohomogeneity 1*, if the G-action has an orbit of cohomogeneity 1. Cohomogeneity 1 manifolds constitute an important class of manifolds for geometry, since they are not homogeneous but only 'one-step' removed from the homogeneous case and, therefore, allow testing various geometric conjectures via the Lie group techniques. In particular, it is natural to analyze the topology of closed *symplectic* (hamiltonian) cohomogeneity 1 manifolds. In [**19**] the author started this analysis. Since all closed symplectic homogeneous spaces are Kähler (see [**32**]), they are also formal. Hence the problem: *Are cohomogeneity 1 closed symplectic (hamiltonian) G-manifolds formal?* We present a result which shows that at least the first obstruction to formality vanishes in this case.

THEOREM 6.1. *Any closed symplectic G-manifold with a hamiltonian action of a compact Lie group G of cohomogeneity 1 has vanishing all triple Massey products.*

PROOF. It was proved in [**19**] that any such manifold is

(1) either a $\mathbb{C}P^n$-bundle over a co-adjoint orbit G/G_ξ
(2) or a symplectic blow-down of fiber bundle (1) along two singular symplectic orbits.

It is known that any rational $\mathbb{C}P^n$-bundle is formal if and only if the base is formal [**21, 32**]. Since the base G/G_ξ is symplectic and homogeneous, it is formal, which implies the formality of M in case (1).

Now, in the second case, the vanishing of the triple Massey products follows from Theorem 1.5 of the present article. □

Groups of symplectomorphisms and symplectic fibrations. We complete this Section with a recent result of McDuff [**23**], which suggests that there may exist new phenomena of topological nature related to hamiltonian group actions.

THEOREM 6.2. [**23**] *Let $M \to P \to S^2$ be any symplectic fibration. Then the Leray-Serre spectral sequence of this fibration degenerates.*

McDuff conjectured that this theorem holds for any symplectic base B under the additional assumption that the fibration is hamiltonian (i.e. the structure group of the fibration consists of hamiltonian symplectomorphisms). The proof of this theorem uses the hard machinery of pseudoholomorphic curves and quantum cohomology, but the result itself is completely homotopic, and this fact suggests further analysis.

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Received December 17, 1999

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