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HOMOGENEOUS GEODESICS OF LEFT-INVARIANT METRICS

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Abstract. It is shown that there are infinitely many homogeneous geodesics issuing from the identity element of a compact connected semi-simple Lie group in case of any left-invariant Riemannian metric provided that the group is of rank greater than 1.

A geodesic $\gamma : \mathbf{R} \to M$ of a Riemannian manifold $(M, <,>)$ is said to be *homogeneous* if there is a 1-parameter group of isometries $\Phi_\tau : M \to M$, $\tau \in \mathbf{R}$ such that γ is an orbit of Φ; more precisely, if $\gamma(\tau) = \Phi_\tau(\gamma(0))$, $\tau \in \mathbf{R}$ holds. The concept of homogeneous geodesic plays a basic role in the theory of homogeneous Riemannian manifolds; namely, the assumption that all the geodesics of a homogeneous Riemannian manifold are homogeneous proved to be useful in their classification theory [**1**], [**3**]. On the existence of homogeneous geodesics there is a result obtained by V. V. Kajzer [**2**] stating that in the case of a connected Lie group and a left-invariant Riemannian metric $<,>$ on it, the Riemannian manifold $(G, <,>)$ has at least one homogeneous geodesic issuing from the identity element. It is shown below that if the Lie group G is also compact semi-simple and of $rank \geq 2$ then there are infinitely many homogeneous geodesics issuing from the identity element in the case of any left-invariant Riemannian metric.

DEFINITION. Let G be a connected Lie group, $\mathbf{g} = T_eG$ its Lie algebra identified with the tangent space at the identity element, $B : \mathbf{g} \times \mathbf{g} \to \mathbf{R}$ a euclidean inner product, i.e., a positive definite symmetric bilinear form and $<,>$ the left-invariant Riemannian metric induced by B on G. A tangent vector $X \in T_eG - \{0\}$ is said to be a *geodesic vector* if the 1-parameter subgroup $\tau \mapsto Exp(\tau X)$, $\tau \in \mathbf{R}$, is a geodesic of $<,>$. The geodesic defined by a geodesic vector is obviously a homogeneous one. Conversely, let γ be a geodesic with

$\gamma(0) = g$ which is homogeneous with respect to a 1-parameter group of left-translations, namely

$$\gamma(\tau) = Exp(\tau Y)g, \ \tau \in \mathbf{R},$$

then a homogeneous geodesic $\tilde{\gamma}$ is given by

$$\tilde{\gamma}(\tau) = \mathcal{L}_g^{-1} \circ \gamma(\tau) = \mathcal{L}_g^{-1} \circ \mathcal{R}_g \circ Exp(\tau Y)$$
$$= Exp(Ad(g^{-1})\tau Y) \cdot e = Exp(Ad(g^{-1})\tau Y)\tilde{\gamma}(0),$$

which means that $X = Ad(g^{-1})Y$ is a geodesic vector.

The following *lemma* has been applied in several papers, its subsequent simple proof is presented for convenience here.

LEMMA. *Let G be a connected Lie group, B a euclidean inner product on its Lie algebra $\mathbf{g}$ and $<,>$ the left-invariant Riemannian metric induced on G by B. Then $X \in \mathbf{g} - \{0\}$ is a geodesic vector if and only if $B(X,[Z,X]) = 0$ holds for $Z \in \mathbf{g}$.*

PROOF. Consider the Levi-Cività covariant derivation ∇ defined by $<,>$ which is obviously left-invariant. If $X \in \mathbf{g} - \{0\}$ then let $\bar{X} : G \to TG$ be the left-invariant vector field defined by $\bar{X}(e) = X$. Then X is a geodesic vector if and only if $(\nabla_{\bar{X}}\bar{X})(e) = 0$; in other words, $< \nabla_{\bar{X}}\bar{X}, \bar{U} > |_e = 0$ is valid in case of any left-invariant field $\bar{U}$. But by *Koszul's formula* the following holds:

$$2 < \nabla_{\bar{X}}\bar{X}, \bar{U} > = \bar{X} < \bar{X}, \bar{U} > + \bar{X} < \bar{U}, \bar{X} > - \bar{U} < \bar{X}, \bar{X} >$$
$$- < \bar{X}, [\bar{X}, \bar{U}] > + < \bar{X}, [\bar{U}, \bar{X}] > + < \bar{U}, [\bar{X}, \bar{X}] >$$
$$= 2 < \bar{X}, [\bar{U}, \bar{X}] >,$$

since the functions $< \bar{X}, \bar{U} >$, $< \bar{X}, \bar{X} >$ are constant. Therefore the following is obtained

$$< \nabla_{\bar{X}}\bar{X}, \bar{U} > |_e = < \bar{X}, [\bar{U}, \bar{X}] > |_e = B(X, [U, X]).$$

Consequently the assertion of the *lemma* follows. □

The following *corollary* is a simple consequence of the preceding *lemma* by polarization and an application of *Schur's lemma*.

COROLLARY. *Let G be a connected compact simple Lie group and $K : \mathbf{g} \times \mathbf{g} \to \mathbf{R}$ the negative of the Cartan-Killing form of $\mathbf{g}$. Let B be a euclidean inner product on $\mathbf{g}$ and $<,>$ the left-invariant Riemannian metric induced by it. Then all the geodesics of $<,>$ are homogeneous if and only if $B = \lambda\ K$ for some $\lambda \in \mathbf{R}^+$.*

DEFINITION. Let G be a connected Lie group and B a euclidean inner product on its Lie algebra $\mathbf{g}$ then a quadratic form $Q : \mathbf{g} \to \mathbf{R}$ is defined by $Q(X) = B(X, X)$, $X \in \mathbf{g}$. Consider now the adjoint action $Ad : G \times \mathbf{g} \to \mathbf{g}$ and the orbit $G(U) \subset \mathbf{g}$ of an element $U \in \mathbf{g}$. The corresponding isotropy subgroup $G_U < G$ being closed, the canonical smooth manifold structure of the coset space G/G_U exists, the canonical left-action $\Lambda : G \times (G/G_U) \to G/G_U$ is smooth and also the canonical equivariant bijection $G/G_U \to G(U)$ is smooth and yields an equivariant injective immersion $\rho : G/G_U \to G(U) \subset \mathbf{g}$. The smooth function $q = Q \circ \rho : G/G_U \to \mathbf{R}$ will be called the *pull-back of quadratic form* Q.

PROPOSITION. *Let G be a connected Lie group and B a euclidean inner product on its Lie algebra $\mathbf{g}$. For $X \in \mathbf{g} - \{0\}$ let $U \in \mathbf{g}$ be such that $X \in G(U)$ for the corresponding adjoint orbit and let $gG_U \in G/G_U$ be the unique coset with $\rho(gG_U) = X$. Then X is a geodesic vector if and only if gG_U is a critical point of $q = Q \circ \rho$ the pull-back of the quadratic form Q.*

PROOF. The coset gG_U is a critical point of q if and only if $vq = 0$ for $v \in T_{gG_U}(G/G_U)$. But as G/G_U is homogeneous, for each v there is a $Z \in \mathbf{g}$ such that $v = \tilde{Z}(gG_U)$ where $\tilde{Z} : G/G_U \to T(G/GU)$ is the infinitesimal generator of the action Λ corresponding to Z. Consider also the infinitesimal generator $\hat{Z} : \mathbf{g} \to T\mathbf{g}$ of the adjoint action corresponding to Z. Since the injective immersion ρ is equivariant with respect to the actions Λ and Ad the following holds: $\hat{Z} \circ \rho = T\rho \circ \tilde{Z}$. But then the following is valid:

$$vq = \tilde{Z}q|_{gG_U} = \tilde{Z}(Q \circ \rho)|_{gG_U} = (T\rho\tilde{Z})|_{gG_U} Q$$

$$= \frac{d}{d\tau}(B(Ad(Exp\ \tau Z)X, Ad(Exp\ \tau Z)X))|_{\tau=0} = 2B([Z, X], X).$$

Since the map $\mathbf{g} \ni Z \mapsto \tilde{Z}(gG_U) \in T_{gG_U}(G/G_U)$ is an epimorphism, the assertion of the *proposition* follows. □

THEOREM. *Let G be a compact connected semi-simple Lie group and B a euclidean inner product on its Lie algebra $\mathbf{g}$. Then each orbit of the adjoint action $Ad : G \times \mathbf{g} \to \mathbf{g}$ contains at least two geodesic vectors.*

PROOF. Consider an orbit $G(U)$ of the adjoint action, the corresponding coset manifold G/G_U and the injective immersion $\rho : G/G_U \to \mathbf{g}$. Then the pull-back q of the quadratic form Q has at least 2 critical points since G/G_U is compact. □

In the next *corollary* two geodesics are considered different if their images are different.

COROLLARY. *Let G be a compact connected semi-simple Lie group of rank ≥ 2 and B a euclidean inner product on its Lie algebra. Then the left-invariant Riemannian metric $<,>$ induced by B on G has infinitely many homogeneous geodesics issuing from the identity element.*

PROOF. As two geodesics are considered different if they have different images, two geodesic vetors $X, Y \in \mathbf{g} - \{0\}$ yield different homogeneous geodesics issuing from the identity element if and only if there is no $\lambda \in \mathbf{R}$ with $Y = \lambda X$. Consequently it is sufficient to show that a K-sphere of $\mathbf{g}$ contains infinitely many geodesic vectors, where K is the negative of the Cartan-Killing form. Therefore by the preceding *theorem* it is enough to see that a K-sphere includes infinitely many orbits of the adjoint action. But since

$$\begin{aligned} codim\ G(U) &= dim\ \mathbf{g} - dim\ G(U) = dim\ \mathbf{g} - (dim\ \mathbf{g} - dim\ G_U) \\ &= dim\ G_U \geq rank\ G \geq 2, \end{aligned}$$

the number of adjoint orbits included in a K-sphere cannot be finite. □

REMARK. If a compact connected semi-simple Lie group G has rank 1 then there are left-invariant Riemanniann metrics on G which have only a finite number of homogeneous geodesics issuing from the identity element.

PROOF. Consider an arbitrary euclidean inner product B on $\mathbf{g}$. Then there is a K-symmetric vector space automorphism $\kappa : \mathbf{g} \to \mathbf{g}$ such that $B(U, V) = K(\kappa U, V),\ U, V \in \mathbf{g}$ holds. But then

$$B(X, [Z, X]) = K(\kappa X, [Z, X]) = K([X, \kappa X], Z),\ \ X, Z \in \mathbf{g}.$$

Consequently, X is a geodesic vector if and only if $[X, \kappa X] = 0$. In other words, X is a geodesic vector if and only if $\kappa X \in \mathbf{g}_X$ where $\mathbf{g}_X < \mathbf{g}$ is the Lie subalgebra corresponding to the isotropy subgroup $G_X < G$. Since $\mathbf{g}_X$ is the union of those Cartan subalgebras which contain X, if $rank\ G = 1$ then $\mathbf{g}_X$ is the 1-dimensional subalgebra spanned by X. But then X is a geodesic vector if and only if $\kappa X = \lambda X$ for some $\lambda \in \mathbf{R}$, in other words if and only if X is an eigenvector of κ. If B is chosen so that all the eigenvalues of κ are different then a K-sphere can contain only a finite number of eigenvectors. □

The idea to consider the K-symmetric automorphism in case of a semi-simple Lie group is due to Kajzer and his argument yields the existence of at least m homogeneous geodesics issuing from the identity element in case of an m-dimensional semi-simple Lie group; although, this is not explicitely stated in his paper.

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