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ON COHOMOLOGY OF FIBER BUNDLES WITH GRASSMANNIAN FIBERS

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Abstract. We present some new results on $\mathbb{Z}_2$-cohomology of smooth fiber bundles with Grassmannian fibers. Outlines of proofs are given; detailed proofs will appear elsewhere.

1. Introduction. Our main aim is to present some new results on cohomology of smooth fiber bundles with fiber the Grassmann manifold $G_{n,k}$ of all k-dimensional vector subspaces in $\mathbb{R}^n$ or with fiber the complex Grassmann manifold $\mathbb{C}G_{n,k}$ of all complex k-dimensional vector subspaces in $\mathbb{C}^n$. As particular cases of such fibers one has $G_{n,1} = \mathbb{R}P^{n-1}$, real projective space of dimension $n-1$ or $\mathbb{C}G_{n,1} = \mathbb{C}P^{n-1}$, complex projective space of complex dimension $n-1$. We shall suppose that $2k \leq n$ in the sequel; this is not a restriction, because there are obvious diffeomorphisms $G_{n,k} \cong G_{n,n-k}$ and $\mathbb{C}G_{n,k} \cong \mathbb{C}G_{n,n-k}$. Detailed proofs of the results presented here will appear elsewhere ([**12**], [**14**]).

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By a fiber bundle we shall understand a locally trivial fibration, hence a quadruple $(E, p, B; F)$, where $p : E \to B$ is a continuous map from the total space E to the base space B and p locally looks like the first projection

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$B \times F \to B$. Briefly, we speak about the fiber bundle $p : E \to B$. We shall assume that all the spaces involved are path connected. The space F is called "the" fiber of the fiber bundle $p : E \to B$. The fiber $p^{-1}(b)$ over any point $b \in B$ is homeomorphic to F; in this sense, to call F *the* fiber is just a slight abuse of language. Replacing topological spaces by smooth manifolds and continuous maps by smooth maps, one obtains the concept of a smooth fiber bundle.

More generally, one can consider Hurewicz fibrations, hence continuous maps $p : E \to B$ having the homotopy covering property (Spanier [**21**]). Any two fibers, $p^{-1}(b)$, $p^{-1}(b')$ $(b, b' \in B)$, are homotopy equivalent provided that the base space B is path connected. We then let F be any space homotopy equivalent to $p^{-1}(b)$, for some b in the path connected base space B. By an abuse of language, we also speak about F as about *the* fiber. Also in case of Hurewicz fibrations, we always shall assume that all the spaces involved are path connected.

There are numerous situations in topology or geometry, where one needs to derive cohomological information on the total space E of a fibration if the cohomology of the fiber is known. This is the case, for instance, when one needs to decide whether a given space can serve as the total space of a fibration with prescribed fiber.

In this context, if $i : F \to E$ is the fiber inclusion, one often asks whether or not the induced cohomology homomorphism $i^* : H^*(E; G) \to H^*(F; G)$ is surjective. If the answer is positive, then one says that the fiber F is totally non-homologous to zero in E with respect to G, where G is a field or the group $\mathbb{Z}$ of integers. This question has been studied by many authors, including J. Leray [**15**], G. Hirsch [**11**], J.-P. Serre [**19**], A. Borel [**4**], [**5**], [**6**], ever since the late 1940s. Note that if G is a field and E and F have finitely generated homology in all dimensions, then the induced cohomology homomorphism i^* is surjective if and only if the induced *homology* homomorphism $i_* : H_*(F; G) \to H_*(E; G)$ is injective. The latter is the original meaning of the statement that the fiber F is totally non-*homologous* to zero in the total space E (see e.g. Serre [**19**], Mimura, Toda [**16**]).

2. On fiber bundles with a simply connected homogeneous space as fiber. Let us first consider smooth fiber bundles having a prescribed 1-connected homogeneous space as fiber. Interesting results in this direction can be found for instance in a paper by Shiga and Tezuka [**20**]. For smooth fiber bundles, their main theorem can be stated in the following way.

THEOREM 1. ([**20**]) *Let G be a connected compact Lie group and U be a closed connected subgroup of G such that* $\mathrm{rank}(U) = \mathrm{rank}(G)$. *Let $p : E \to B$ (with E path connected) be a smooth fiber bundle with fiber G/U, such that the* *fundamental group* $\pi_1(B)$ *acts trivially on the cohomology groups* $H^*(G/U;\mathbb{Q})$ *with rational coefficients (in other words, let the fiber bundle be* $\mathbb{Q}$*-orientable). Then the fiber* G/U *is totally non-homologous to zero in* E *with respect to* $\mathbb{Q}$.

Modifying their argument used in the rational case, Shiga and Tezuka also proved the following.

THEOREM 2. ([**20**]) *Let* G *and* U *be as in Theorem 1. Let* q *be a prime such that* q *does not divide the order of the Weyl group* $W(G)$. *If* $p : E \to B$ *(with path-connected* B*) is a smooth fiber bundle such that* $\pi_1(B)$ *acts trivially on* $H^*(G/U;\mathbb{Z}_q)$ *(in other words, if the fiber bundle is* $\mathbb{Z}_q$*-orientable), then the fiber* G/U *is totally non-homologous to zero in* E *with respect to* $\mathbb{Z}_q$.

These theorems can be applied for instance to ($\mathbb{Q}$, resp. $\mathbb{Z}_q$) orientable smooth fiber bundles with fiber $\tilde{G}_{n,k}$, the Grassmann manifold of *oriented* k-dimensional vector subspaces in $\mathbb{R}^n$, provided $k(n-k) = \dim(G_{n,k})$ is even. Indeed, one can identify $\tilde{G}_{n,k} \cong SO(n)/(SO(k) \times SO(n-k))$, and one has $\operatorname{rank}(SO(n)) = \operatorname{rank}(SO(k) \times SO(n-k))$ if and only if k or $n-k$ is even. On the other hand, to check if a given fibration is orientable is, in general, a hard problem (if it is manageable at all). In addition to this, Theorem 2 does not say anything about $i^* : H^*(E;\mathbb{Z}_2) \to H^*(\tilde{G}_{n,k};\mathbb{Z}_2)$, because the Weyl group $W(SO(n))$ $(n \geq 3)$ is of even order.

3. On fiber bundles with fiber $G_{n,k}$. Now we concentrate on situations where the fiber is the Grassmann manifold of all k-dimensional vector subspaces in $\mathbb{R}^n$. Note that $G_{n,k}$ is not simply connected (its fundamental group is $\mathbb{Z}_2$).

In this area, there is for instance a result due to J. C. Becker and D. H. Gottlieb ([**3**]) stating that for any Hurewicz fibration $p : E \to B$ with fiber $G_{2n+1,1} = \mathbb{R}P^{2n}$ (with E path connected, locally path connected, and semi-locally 1-connected), the fiber is totally non-homologous to zero with respect to $\mathbb{Z}_2$.

For smooth fiber bundles, we have the following more general result.

THEOREM 3. ([**12**]) *Let* $p : E \to B$ *be a smooth fiber bundle with* E *a closed connected manifold and with fiber the Grassmann manifold* $G_{n,k}$ $(2 \leq 2k \leq n)$. *If* n *is odd, then the fiber is totally non-homologous to zero in* E *with respect to* $\mathbb{Z}_2$.

PROOF. First, it is known that the cohomology algebra $H^*(G_{n,k};\mathbb{Z}_2)$ is generated by the Stiefel-Whitney characteristic classes of the canonical k-plane bundle ξ_k over $G_{n,k}$. But, using a result of Bartík and Korbaš [**1**], one sees that if n is odd, then $H^*(G_{n,k};\mathbb{Z}_2)$ is also generated by the first k Stiefel-Whitney classes of the tangent bundle, $w_1(G_{n,k}), \ldots, w_k(G_{n,k})$. Now the tangent bundle

TE is related to the tangent bundle TB by

$$TE \cong p^*(TB) \oplus \eta,$$

where η is the vector bundle of vectors tangent to the fibers. From this, denoting by $i : G_{n,k} \to E$ the fibre inclusion, one derives that $i^*(w_j(E)) = w_j(G_{n,k})$ for all j. This means that i^* is an epimorphism or, in other words, that the fiber is totally non-homologous to zero if n is odd. □

Theorem 3 has several consequences. For instance, it implies (Serre [**19**], Mimura, Toda [**16**]) that any smooth fiber bundle $p : E \to B$ such as in the theorem, with fiber $G_{n,k}$ for n odd, is $\mathbb{Z}_2$-orientable, the corresponding Serre spectral sequence collapses, and the Leray-Hirsch theorem is applicable.

As an application of Theorem 3, we also obtain the following result on cup-lengths (see [**12**]), which can be useful when one studies the Lyusternik-Shnirel'man category.

THEOREM 4. ([**12**]) *Let $p : E \to B$ be a smooth fiber bundle with fiber $G_{n,k}$ $(2 \leq 2k \leq n)$. If n is odd, then one has*

$$\operatorname{cup}(E) \geq \operatorname{cup}(G_{n,k}) + \operatorname{cup}(B),$$

where $\operatorname{cup}(X)$ *denotes the $\mathbb{Z}_2$ cup-length of X.*

For some values of k, Theorem 4 can be made still more explicit. Namely, for $k \leq 4$ the values of $\operatorname{cup}(G_{n,k})$ $(2k \leq n)$ are known, due to H. Hiller [**10**] and mainly due to R. Stong [**22**]. Here is a sample corollary of Theorem 4 (for this and others cf. [**12**]).

COROLLARY. *Let $p : E \to B$ be a smooth fiber bundle with E a closed connected manifold and with fiber the Grassmann manifold $G_{n,2}$ $(n \geq 4)$. Let s be such that $2^s < n \leq 2^{s+1}$. If n is odd, then* $\operatorname{cup}(E) \geq n + 2^s - 3 + \operatorname{cup}(B)$.

Possibilities of extending Theorem 3 seem to be quite limited. Indeed, an example pointed out to me by R. Stong (see [**12**]) shows that the theorem cannot be extended to cover all smooth fiber bundles with fiber $G_{n,k}$ with n even and k odd ($(n,k) \neq (2,1)$). In addition to this, another example due to Stong shows that Theorem 3 cannot be generalized to cover all smooth fiber bundles with fiber $G_{n,k}$ when n is even and k is even; cf. [**14**]. Perhaps the following can be the most general version of Theorem 3:

Let $p : E \to B$ be a smooth fiber bundle with E a closed connected manifold and with fiber the Grassmann manifold $G_{n,k}$ $(2 \leq 2k \leq n)$. If each power of 2 dividing n also divides k (that is, when $G_{n,k}$ is not bordant to zero; see [**17**], [**2**]), then the fiber is totally non-homologous to zero with respect to $\mathbb{Z}_2$.

Some comments on this can be found in [**14**]. Here we outline one of possible methods of searching for examples which could disprove the above conjecture. Let $T : G_{n,k} \to G_{n,k}$ be a smooth involution (that is, we have $T \circ T = \mathrm{id}$; for instance, sending any k-dimensional vector subspace in $\mathbb{R}^{2k}$ to its orthogonal complement, one obtains an involution on the Grassmannian $G_{2k,k}$; cf. [**2**] for other examples of involutions on $G_{n,k}$ with n even). If $a : S^m \to S^m$ ($m \geq 1$) is the antipodal involution on the m-sphere S^m, then the diagonal involution $T \times a$ on the manifold $G_{n,k} \times S^m$ is free. The map

$$p : \frac{G_{n,k} \times S^m}{T \times a} \to \mathbb{R}P^m,$$

$$p([D, s]) = [s],$$

is a smooth fiber bundle with fiber $G_{n,k}$ (see Husemoller [**13**]).

Now for any $t \in S^m$ the map

$$i_t : G_{n,k} \to \frac{G_{n,k} \times S^m}{T \times a},$$

$$i_t(D) = [D, t],$$

is the inclusion of the fiber over the point $[t] \in \mathbb{R}P^m$ into the total space. Since S^m is path connected, we have $i_t{}^* = i^*_{a(t)} := i^*$ for the induced homomorphisms in cohomology. Since $i_t \circ T = i_{a(t)}$, then we have $T^* \circ i^* = i^*$, hence the image of

$$i^* : H^*(\frac{G_{n,k} \times S^m}{T \times a}; \mathbb{Z}_2) \to H^*(G_{n,k}; \mathbb{Z}_2)$$

is contained in the set of elements which are invariant with respect to T^*. Keeping this in mind, one could try, for instance, to look (using naturality) for Stiefel-Whitney characteristic classes of suitable vector bundles which are not invariant under T^*. Of course, if a non-invariant class exists, then the fiber is not totally non-homologous to zero.

4. On fiber bundles with fiber $\mathbb{C}G_{n,k}$. For q a prime, D. Gottlieb proved in [**9**] that if $p : E \to B$ is a $\mathbb{Z}_q$-orientable Hurewicz fibration with fiber $\mathbb{C}G_{n+1,1} = \mathbb{C}P^n$ such that $n + 1 \not\equiv 0 \pmod{q}$, then the fiber is totally non-homologous to zero with respect to $\mathbb{Z}_q$. In [**14**], we generalize this for $q = 2$ and smooth fiber bundles to the following.

THEOREM 5. ([**14**]) *Let $p : E \to B$ be a smooth fiber bundle with E a closed connected manifold and with fiber the Grassmann manifold $\mathbb{C}G_{n,k}$ ($2 \leq 2k \leq n$). If n is odd, then the fiber is totally non-homologous to zero in E with respect to $\mathbb{Z}_2$.*

The proof, in substance similar to the proof of Theorem 3, is based on the following result ([**14**]).

LEMMA. *If n is odd and $2k \leq n$, then the cohomology algebra $H^*(\mathbb{C}G_{n,k};\mathbb{Z}_2)$ is generated by the Stiefel-Whitney classes $w_{2i}(\mathbb{C}G_{n,k})$ $(i = 1, \dots, k)$ of the realification of the complex tangent bundle of the manifold $\mathbb{C}G_{n,k}$.*

Theorem 5 has similar applications as Theorem 3. For instance, one can try to use Theorem 5 when one wishes to know if a given manifold can be fibered with fiber $\mathbb{C}G_{n,k}$ with n odd. By the way, J. Ferdinands and R. Schultz ([**7**], [**8**], [**18**]) have studied a different but related question of when a Grassmannian can be the total space of a fibration.

As observed by R. Stong, in contrast to Theorem 5, there are many fiber bundles with fiber $\mathbb{C}G_{n,k}$ (n arbitrary) not totally non-homologous to zero with respect to $\mathbb{Z}$ or $\mathbb{Z}_q$ with $q \neq 2$; see [**14**] for an example.

In the above, we mainly have been speaking about smooth fiber bundles with Grassmannian fibers. Several natural questions appear in this context. One of them is, whether Theorem 3 or Theorem 5 remain valid for continuous fiber bundles or even for Hurewicz fibrations.

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