ON AUTOMORPHISMS OF A JACOBI MANIFOLD

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Abstract. Jacobi manifolds generalize both the symplectic and contact manifolds. Groups of automorphisms of a Jacobi manifold are studied. It is shown that some of them determine the smooth and geometric structures.

1. Introduction. The notion of Jacobi manifolds has been introduced by A.Lichnerowicz in [**6**]. Our aim is to show that some groups of automorphisms associated with a Jacobi manifold determine the smooth and geometric structure of the manifold itself. Let us begin with basic definitions.

A *Jacobi structure* on a manifold M is a pair (Λ, E) where Λ is a 2-vector field on M, E is a vector field on M, and the equalities

$$[\Lambda,\Lambda] = 2E\wedge\Lambda, \quad \mathcal{L}_E\Lambda = [E,\Lambda] = 0 \tag{1.1}$$

are satisfied. Here $[.,.]$ is the Schouten-Nijenhuis bracket ([**4**]), and $\mathcal{L}$ is the Lie derivative. The manifold M equipped with a Jacobi structure is called a *Jacobi manifold.* The *Jacobi bracket* is then defined by

$$\{u,v\} = \Lambda(\mathrm{d}\,u,\mathrm{d}\,v) + uE(v) - vE(u), \quad \text{for } u,v\in C^\infty(M,\mathbb{R}). \tag{1.2}$$

It is visible that the bracket $\{.,.\}$ is skew-symmetric. It satisfies the Jacobi indentity iff (1.1) is fulfilled. Furthermore, we have the locality condition

$$\mathrm{supp}\{u,v\}\subset(\mathrm{supp}(u)\cap\mathrm{supp}(v)).$$

Therefore the space $C^\infty(M,\mathbb{R})$ endowed with the Jacobi bracket becomes a local Lie algebra in the sense of Kirillov (cf. [**4**]). Conversely, any structure of a local Lie algebra on $C^\infty(M,\mathbb{R})$ uniquely determines a Jacobi structure on M.

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If $E = 0$ then the bracket $\{.,.\}$ becomes a derivation on each argument and defines a *Poisson structure* on M. Equivalently, a Poisson structure is defined by a 2-vector field Λ fulfilling $[\Lambda, \Lambda] = 0$, and the pair (M, Λ) is a *Poisson manifold.*

We have the bundle homomorphism associated with Λ

$$\Lambda^\sharp : T^*M \to TM, \quad < \alpha^\sharp, \beta >= \Lambda(\alpha, \beta),$$

where $\alpha^\sharp = \Lambda^\sharp(\alpha)$, for any $\alpha, \beta \in T^*M$. The distribution generated by $\Lambda^\sharp(T_x^*M)$ and E_x, $x \in M$, integrates to a generalized foliation (cf. [**4**], [**9**]). This foliation is called *characteristic* and denoted by $\mathcal{F} = \mathcal{F}(\Lambda, E)$. It is well-known that the Jacobi structure induces a locally conformal symplectic (resp. contact) structure on each leaf of $\mathcal{F}(\Lambda, E)$ of even (resp. odd) dimension. Thus Jacobi manifolds generalize both the symplectic and contact manifolds.

The motivation for this note comes from the Erlangen Program in its modern version: the automorphism group of a geometric structure essentially determines the geometric structure itself. The group of all diffeomorphisms of a smooth manifold has been studied by R.Filipkiewicz in [**3**] but, contrary to the unimodular and symplectic case (cf.[**1**]), the techniques used there cannot be extended to Jacobi manifolds. The first reason is that the perfectness theorem is not known in this case. The second one is that the method from [**1, 3**] does not work in the case of the nontransitive groups of diffeomorphisms. Our argument follows a scheme invented in [**7, 8**] and used in [**10**]. To apply this scheme we make some observations on automorphisms of a Jacobi manifold. In particular, we use the canonical coordinates of a Jacobi structure which have been introduced about 1990, cf.[**2**].

Notice that in the context of applications of Jacobi manifolds to Mechanics Theorem 1 may be viewed as stating that a phase-space is uniquely defined by the group of its symmetries.

2. The main result. Let us begin with definitions concerning automorphisms. To any function $u \in C^\infty(M, \mathbb{R})$ one assigns the vector field $\mathcal{H}(u)$ given by

$$\mathcal{H}(u) = \Lambda^\sharp(\mathrm{d}\, u) + uE = [\Lambda, u] + uE,$$

which is called the *hamiltonian* vector field associated with u. In particular, $\mathcal{H}(1) = E$. It can be shown, cf. [**2**], that the mapping $u \mapsto \mathcal{H}(u)$ is a Lie algebra homomorphism with respect to the Jacobi bracket (1.2) and the usual bracket of vector fields. Let $L^*(M) = L^*(M, \Lambda, E) = \operatorname{im} \mathcal{H}$ denote the Lie algebra of hamiltonian vector fields.

Now let $a \in C^\infty(M, \mathbb{R})$ be nowhere zero. It is visible that the expression

$$\{u, v\}_a = \frac{1}{a}\{au, av\}, \quad u, v \in C^\infty(M, \mathbb{R}),$$

defines the bracket of a new Jacobi structure which is conformally equivalent to the initial one. Namely, the 2-vector field Λ^a and the vector field E^a of the new structure are given by $\Lambda^a = a\Lambda$ and $E^a = \mathcal{H}(a) = [\Lambda, a] + aE$. Furthermore, the hamiltonian vector field $\mathcal{H}^a(u)$ satisfies $\mathcal{H}^a(u) = \mathcal{H}(au)$ for all $u \in C^\infty(M, \mathbb{R})$. Consequently the characteristic foliation is an invariant of the conformal class of a Jacobi structure. We will write $(M, \Lambda, E) \sim (M, \Lambda', E')$ if there is a such that $\Lambda' = \Lambda_a$ and $E' = E_a$.

Let $L(M) = L(M, \Lambda, E)$ be the Lie algebra of all *conformal* Jacobi infinitesimal automorphisms. That is $X \in L(M)$ iff there is $u \in C^\infty(M, \mathbb{R})$ such that

$$\mathcal{L}_X \Lambda = u\Lambda, \quad \mathcal{L}_X E = [\Lambda, u] + uE. \tag{2.1}$$

We write $L^t(M)$ for the subalgebra of $L(M)$ consisting of elements tangent to $\mathcal{F}$.

PROPOSITION 1. *$L^*(M)$ is a Lie subalgebra of $L^t(M)$.*

In fact, if $X = \mathcal{H}(v)$ then X satisfies (2.1) with $u = -\mathrm{d}\, v(E)$.

We say that $X \in L(M)$ is *strict* whenever $u = 0$ in (2.1). Notice that a strict infinitesimal automorphism need not be tangent to $\mathcal{F}$. It follows from the definition that a hamiltonian vector fields is not strict unless $Eu = 0$. Thus, contrary to the Poisson case, strict i.a. play a minor role.

A smooth mapping $f : (M_1, \Lambda_1, E_1) \to (M_2, \Lambda_2, E_2)$ is called a Jacobi morphism if

$$\{u \circ f, v \circ f\}_1 = \{u, v\}_2 \circ f \quad \text{for any } u, v \in C^\infty(M_2).$$

The following is easy to check.

PROPOSITION 2. *The following statements are equivalent:*
(1) ϕ is a Jacobi morphism;
(2) $\phi_* \mathcal{H}(u \circ \phi) = \mathcal{H}(u)$, $\forall u \in C^\infty(M_2)$;
(3) $\phi_* \Lambda_1 = \Lambda_2$, $\phi_* E_1 = E_2$ (in particular, Λ_1 and E_1 are related by ϕ to Λ_2 and E_2).

But in the theory of Jacobi manifolds conformal Jacobi morphisms are more important. A smooth map $\phi : (M_1, \Lambda_1, E_1) \to (M_2, \Lambda_2, E_2)$ is conformal Jacobi morphism if

$$(M_1, \Lambda_1, E_1) \sim (M_1, \phi^* \Lambda_2, \phi^* E_2).$$

Let $\tilde{G}(M) = \tilde{G}(M, \Lambda, E)$ stand for the group of all conformal Jacobi automorphisms of (M, Λ, E), and let $\tilde{G}_c(M)$ be its subgroup of compactly supported elements. (As usual the subscript "c" indicates the subspace of compactly supported elements.) Next by $\hat{G}(M)$ (resp. $G^*(M)$) we denote the subgroup of $\tilde{G}(M)$ generated by all $\exp(X)$ where $X \in L^t(M)$ (resp. $L^*(M)$) and X is complete.

THEOREM 1. *Let* (M_i, Λ_i, E_i) $(i = 1, 2)$ *be a Jacobi manifold with no leaves of dimension 0. Let* $G(M_i)$ *be one of the four groups* $\hat{G}(M_i)$, $\hat{G}_c(M_i)$, $G^*(M_i)$, $G^*_c(M_i)$. *If there exists a group isomorphism* Φ *of* $G(M_1)$ *onto* $G(M_2)$ *then there exists a unique diffeomorphism* ϕ *of* M_1 *onto* M_2 *such that* $\Phi(f) = \phi \circ f \circ \phi^{-1}$ *for any* $f \in G(M_1)$. *Moreover,* ϕ *is a conformal Jacobi morphism.*

REMARK. We do not know whether $\hat{G}(M)$ is equal to the identity component of the subgroup of $\tilde{G}(M)$ of all diffeomorphisms preserving the leaves of $\mathcal{F}$. Another question is whether any diffeomorphism of M which can be joined to id by a hamiltonian isotopy belongs to $G^*(M)$. In the symplectic case the answer is affirmative but it depends on difficult simplicity theorems and the existence of the flux homomorphism (cf.[**1**]). We believe that Theorem 1 is true for $\tilde{G}(M)$, but our techniques are not sufficient to show this.

3. Pseudo-n-transitivity of Automorphism Groups. Let $G(M) \subset \mathrm{Diff}^\infty(M)$ be any diffeomorphism group. By an isotopy in $G(M)$ we mean any family $\{f_t\}_{t\in\mathbb{R}}$ with $f_t \in G(M)$ such that the map $(t, x) \mapsto f_t(x)$ is smooth. Next, $G(M)_0$ denotes the subgroup of all $f \in G(M)$ such that there is a smooth diffeotopy $\{f_t\}_{t\in\mathbb{R}}$ with $f_t = \mathrm{id}$ for $t \leq 0$ and $f_t = f$ for $t \geq 1$. Notice that $G(M)_0$ is the connected component of id if $G(M)$ is locally contractible.

Observe that $\hat{G}(M)_0 = \hat{G}(M), G^*(M)_0 = G^*(M)$, and the same for compactly supported subgroups.

Let A be an arbitrary set of isotopies, and let A^* be the totality of diffeomorphisms ψ such that $\psi = \phi(t, .)$ for some $\phi \in A$, $t \in \mathbb{R}$. Next $\bar{A}$ denotes the set consisting of all local diffeomorphisms being finite compositions of elements from A^* or $(A^*)^{-1} = \{\psi^{-1} : \psi \in A^*\}$, and of the identity. For $p \in M$ we let $\bar{A}(p)$ be the vector subspace of T_pM generated by

$$\{d_q\psi(v) : \psi \in \bar{A}, \psi(q) = p, v \in A(q)\},$$

where $A(q)$ is spanned by all $\frac{\mathrm{d}}{\mathrm{d}t}\phi(t, q)|_{t=0}$, $\phi \in A$. The orbits of $\bar{A}$ determine uniquely an equivalence relation on M, and its equivalence classes are called accessible sets. More precisely, we have

THEOREM 2. *(P.Stefan [**12**].) Every accessible set of* A *admits a (unique) smooth structure of a connected weakly imbedded submanifold of* M, *and the accessible sets form a generalized foliation* $\mathcal{F} = \mathcal{F}(A)$. *Furthermore,* $\bar{A}(p)$ *is the tangent distribution at* p *of the foliation* $\mathcal{F}(A)$.

Now let $G(M)$ be a locally contractible group of diffeomorphisms. One considers the set of all isotopies lying in $G(M)$, denoted by A_G. It is then easily observed that the accessible sets of A_G coincide with the orbits of the group $G(M)_0$. In case of $\hat{G}(M)$ and $G^*(M)$ the orbits are the leaves of the characteristic foliation.

DEFINITION 1. $G(M)$ satisfies *(L)-condition* (locality) if for any open relatively compact $U, V \subset M$ with $\bar{U} \subset V$, and a smooth diffeotopy $\{f_t\}$ in $G(M)$ with $f_0 = id$, there exist $\epsilon > 0$ and a smooth diffeotopy $\{g_t\}$ such that $g_t = f_t$ on U for $|t| < \epsilon$, and $\text{supp}(g_t) \subset V$.

DEFINITION 2. A diffeomorphism group $G(M)$ is *pseudo-n-transitive* if for any two n-tuples of pairwise distinct points $(p_1, \dots, p_n)$ and $(q_1, \dots, q_n)$ of M such that p_i, q_i belong to the same orbit of $G(M)_0$ and each orbit of dimension ≤ 1 contains at most one p_i there exists $f \in G(M)$ satisfying $f(p_i) = q_i$, $i = 1, \dots, n$.

Notice that this definition coincides with the n-transitivity (i.e. $T(n)$-property in [**3, 7**]) if $G(M)_0$ acts transitively. That is, the pseudo-n-transitivity is the n-transitivity along leaves. The following theorem, which generalizes a theorem of Boothby, connects the two concepts.

THEOREM 3. *Let $G(M) \subset Diff^\infty(M)$ satisfy the (L)-condition. Then $G(M)$ is pseudo-n-transitive for each $n \geq 1$. In particular, $G(M)$ is n-transitive provided $G(M)_0$ is transitive.*

For the proof see [**9**]. Notice that Theorem 3 still holds in the C^r-smooth and real analytic categories. However, the formulation of the (L)-condition must be changed in the real-analytic case.

COROLLARY 1. *Let (M, Λ, E) be a Jacobi manifold. Then $G^*_c(M)$ (and a fortiori $G^*(M)$, $\hat{G}_c(M)$, $\hat{G}(M)$) is pseudo-n-transitive for each $n \geq 1$.*

This is so since the group in question satisfies the (L)-condition (see the proof of Proposition 4 below).

4. Splitting theorems. Let us recall the concept of *homogeneous* Poisson manifold. This is a Poisson manifold (M, Λ) equipped with a vector field Z, called a vector field of homotheties, such that

$$[Z, \Lambda] = -\Lambda.$$

This concept plays a clue role in the theory of Jacobi manifolds. Namely, any submanifold of codimension 1 in (M, Λ) which is transversal to Z possesses an induced Jacobi structure, and each Jacobi structure can be obtained in this way. On the other hand, any submanifold of codimension 1 in a Jacobi manifold (M, Λ, E) transversal to E admits an induced structure of a homogeneous Poisson manifold, and, again, each homogeneous Poisson structure can be obtained in such a way.

Now we turn our attention to the local description of Jacobi manifolds, which will be useful for our purpose. In full generality this description has been given for the first time in [**2**]. According to the dimension of leaves one has two splitting theorems.

We begin with standard structures on $\mathbb{R}^n$. If $n = 2p$ we write

$$\Lambda_{2p} = \sum_{i=1}^{p} \frac{\partial}{\partial x^{i+p}} \wedge \frac{\partial}{\partial x^i}, \quad Z_{2p} = \sum_{i=1}^{p} x^{i+p} \frac{\partial}{\partial x^{i+p}}.$$

Likewise, for $n = 2p+1$ we set

$$E_{2p+1} = \frac{\partial}{\partial x^0}, \quad \Lambda_{2p+1} = E_{2p+1} \wedge Z_{2p} + \Lambda_{2p}.$$

Now let (M, Λ, E) be a Jacobi manifold of dimension n.

THEOREM 4. *Suppose $x_0 \in M$ lies on a leaf L of even dimension $2p$. Then there is a neighborhood W of x_0 which is identified, up to a conformal Jacobi diffeomorphism ϕ, with the product $U_{2p} \times N$ of an open neighborhood of 0 in $\mathbb{R}^{2p}$ and a Jacobi manifold (N, Λ_N, E_N) of dimension $n-2p$. The restrictions $\Lambda_{U_{2p}\times N}$ and $E_{U_{2p}\times N}$ of Λ and E to $U_{2p} \times N$ assumes the form*

$$\Lambda_{U_{2p}\times N} = \Lambda_{2p} + \Lambda_N - Z_{2p} \wedge E_N, \quad E_{U_{2p}\times N} = E_N.$$

The diffeomorphism ϕ sends x_0 to $(0, \tilde{x}_0)$ and $L \cap W$ to $U_{2p} \times \{\tilde{x}_0\}$, and the Jacobi structure on N has rank 0 at $\tilde{x}_0$.

COROLLARY 2. *Under the above assumption there is a chart (x^λ, x^μ, x^a) at x_0 with $1 \le \lambda, \mu \le p$, $\bar{\mu} = \mu + p$, $2p+1 \le a, b \le n$, such that the only possible nonzero components are E^a, $\Lambda^{\lambda\bar{\lambda}} = -\Lambda^{\bar{\lambda}\lambda} = -1$, Λ^{ab}, and $\Lambda^{\bar{\lambda}a} = -\Lambda^{a\bar{\lambda}} = -x^{\bar{\lambda}}E^a$. Moreover, E^a and Λ^{ab} are independent of $x^\lambda, x^{\bar{\mu}}$ and vanish at x_0.*

Next we consider the case of odd dimension.

THEOREM 5. *Assume that x_0 lies on a leaf L of dimension $2p+1$. Then there is a neighborhood W of x_0 which is identified, up to a Jacobi diffeomorphism ϕ, with the product $U_{2p+1} \times N$ of an open neighborhood of 0 in $\mathbb{R}^{2p+1}$ and a homogeneous Poisson manifold (N, Λ_N, Z_N) of dimension $n-2p-1$. The restrictions $\Lambda_{U_{2p+1}\times N}$ and $E_{U_{2p+1}\times N}$ of Λ and E to $U_{2p+1} \times N$ have the form*

$$\Lambda_{U_{2p+1}\times N} = \Lambda_{2p+1} + \Lambda_N + E_{2p+1} \wedge Z_N, \quad E_{U_{2p+1}\times N} = E_{2p+1}.$$

The diffeomorphism ϕ sends x_0 to $(0, \tilde{x}_0)$ and $L \cap W$ to $U_{2p+1} \times \{\tilde{x}_0\}$, and the Poisson structure on N has rank 0 at $\tilde{x}_0$.

COROLLARY 3. *There exists a chart $(x^0, x^\lambda, x^{\bar{\mu}}, x^a)$ at x_0 with $1 \le \lambda, \mu \le p$, $\bar{\mu} = \mu + p$, $2p+1 \le a, b \le n-1$, such that the only possible nonzero* *components are* $E^0 = 1$, $\Lambda^{\lambda\bar{\lambda}} = -\Lambda^{\bar{\lambda}\lambda} = -1$, Λ^{ab}, $\Lambda^{\bar{\lambda}0} = -\Lambda^{0\bar{\lambda}} = -x^{\lambda}$, *and* $\Lambda^{0a} = -\Lambda^{a0}$. *Moreover,* Λ^{a0} *and* Λ^{ab} *are independent of* $x^0, x^{\lambda}, x^{\bar{\mu}}$ *and vanish at* x_0.

If $f \in G(M)$, $\mathrm{Fix}(f)$ denotes the set of fixed points, $\{x \in M : f(x) = x\}$. The following will be useful in the sequel.

PROPOSITION 3. *For any sufficiently small neighborhood* V *of* $x_0 \in M$ *there is* $f \in G_c^*(M)$ *such that* $\mathrm{Fix}(f) \cap L_{x_0} = (L_{x_0} - U) \cup \{x_0\}$ *for some open ball* U *with* $\bar{U} \subset V$. *Here* L_{x_0} *is the leaf passing through* x_0.

PROOF. We consider two cases with respect to the dimension of L_{x_0}. Let $(x^{\lambda}, x^{\bar{\mu}}, x^a)$ with $1 \leq \lambda, \mu \leq p$, $\bar{\mu} = \mu + p$, $2p + 1 \leq a \leq n$, be a chart at x_0 having all the properties from Corollary 2. In this chart let $r > 0$ be such that the $B(0, r)$ is in the chart domain. Choose a smooth $\alpha : \mathbb{R} \to [0, 1]$ such that $\alpha(0) = 1$, $\alpha(\xi) = 0$ for $|\xi| \geq \frac{r}{2}$, and $\alpha'(\xi) = 0$ iff $\xi = 0$ or $|\xi| \geq \frac{r}{2}$. We define

$$u(x^1, \dots, x^n) = \alpha((x^1)^2 + \dots + (x^{2p})^2).$$

Likewise for a chart $(x^0, x^{\lambda}, x^{\bar{\mu}}, x^a)$ at x_0 with $1 \leq \lambda, \mu \leq p$, $\bar{\mu} = \mu + p$, $2p + 1 \leq a \leq n - 1$, from Corollary 3 we put

$$u(x^0, \dots, x^{n-1}) = \alpha((x^0)^2 + \dots + (x^{2p})^2).$$

Then $f = \exp(X)$, where $X = \mathcal{H}(u) = (du)^{\sharp} + uE$, verifies the claim. We check this in the second case which is less immediate.

First observe that in the chart domain the leaf L_{x_0} is given by $x^a = 0$ for any $a > 2p$. By definition $\mathcal{H}(u)$ can be expressed as follows

$$\mathcal{H}(u) = \Big(u(x) + \alpha'(x)\sum_{\lambda=1}^{p}(x^{\bar{\lambda}})^2\Big)\frac{\partial}{\partial x^0} + \alpha'(x)\sum_{\lambda=1}^{p} x^{\bar{\lambda}}\frac{\partial}{\partial x^{\lambda}} + (x^0 x^{\bar{\lambda}} - x^{\lambda})\frac{\partial}{\partial x^{\bar{\lambda}}}.$$

It is clear that $\mathcal{H}(u)(x) = 0$ if $x = x_0$ or $|x| \geq \frac{r}{2}$. Now if $|x| < \frac{r}{2}$ and $x \neq x_0$, one has $x^0 \neq 0$ or there is λ such that $x^{\lambda} \neq 0$ or $x^{\bar{\lambda}} \neq 0$. $x^{\bar{\lambda}}$ is the coefficient of $\frac{\partial}{\partial x^{\lambda}}$. In the case $x^{\bar{\lambda}} = 0$ and $x^{\lambda} \neq 0$ the coefficient of $\frac{\partial}{\partial x^{\bar{\lambda}}}$ is nonzero. Finally, if all $x^{\lambda}, x^{\bar{\mu}}$ are zero then the coefficient of $\frac{\partial}{\partial x^0}$ is nonzero. Therefore $\mathcal{H}(u)(x) \neq 0$ for $x \in B(0, \frac{r}{2})$, $x \neq x_0$, as required. □

5. Further properties of automorphism groups.

DEFINITION 3. (Fragmentation property) For any finite family of open balls $\{U_i\}$ and any $h \in G(M)_0$ such that $\mathrm{supp}(h) \subset \bigcup U_i$ there exists a decomposition $h = h_s \circ \dots \circ h_1$ such that $\mathrm{supp}(h_j) \subset U_{i(j)}$ for $i = 1, \dots, s$.

PROPOSITION 4. *Let (M, Λ, E) be a Jacobi manifold. Then $G^*(M)$ fulfills the fragmentation property.*

PROOF. Choose a new family of open balls, $\{V_j\}_{j=1}^s$, satisfying $\mathrm{supp}(f) \subset V_1 \cup \ldots \cup V_s$ and which is starwise finer than $\{U_i\}_{i=1}^r$: $\forall j\, \exists i\, \mathrm{star}(V_j) \subset U_{i(j)}$. Any element of $G^*(M_1)$ is a finite composition of diffeomorphisms of the form $g = \exp(X)$ where $X \in L^*(M)$. Now it is easily seen from the definition that one may have $X = X_1 + \cdots + X_s$, where $X_j \in L^*(M)$ and $\mathrm{supp}\, X_j \subset V_j$. Set $f_0 = id$ and $f_j = \exp(X_1 + \cdots + X_j)$ for $j \geq 1$. We get

$$g = f_s = g_s \circ \cdots \circ g_1 \quad \text{where} \quad g_j = f_j \circ f_{j-1}^{-1}.$$

We have also

$$\mathrm{supp}(g_j) = \mathrm{supp}(f_j \circ f_{j-1}^{-1}) \subset \mathrm{star}(V_j) \subset U_{i(j)},$$

as required. □

PROPOSITION 5. *(i) If $x, y \in \Sigma \cap U, \Sigma \in \mathcal{F}, U$ being an open ball, then there is $g \in G^*(M)_0$ with $\mathrm{supp}(g) \subset U$ such that $g(x) = y$.*

(ii) For any $x \in M$ and for any $g \in \hat{G}(M)$ such that $g(x) = x$ there exists $h \in G_c^*(M)$ such that $h = g$ on a neighbourhood of x.

PROOF. (i) This is a consequence of the locality (Def.1).

(ii) Suppose $g = \exp(X)$ such that $X \in \hat{L}(M)$. We make use of the equality

$$X|_U = [\Lambda|_U, w_U] + w_U E$$

for some smooth w_U defined on U. By multiplying w_U by a smooth function α with $\alpha = 1$ near p and $\mathrm{supp}\, \alpha \subset U$, the property follows. □

6. The proof of Theorem 1. The proof consists in applying a main result of [**11**] (the proof is essentially in [**8**]). First we recall our "axiomatization" which describes what kind of automorphism groups can determine the underlying geometric structure. Plausibly our axioms are "only" local but it seems that such a local approach is appropriate in case of diffeomorphism groups.

AXIOM 1. *is just the fragmentation property (Def.3).*

It is fulfilled by $G^*(M)$. It can be checked that this is sufficient in the proof of Theorem 1.

AXIOM 2. *For any $x \in M$ and a sufficiently small open ball U with the center at x there exists $f \in G(M)_o$ with $\mathrm{Fix}(f) = (M - U) \cup \{x\}$. In addition, for any $x \in U$, U open, there is $f \in G(U)_o$ such that $f(x) \neq x$.*

Observe that the second assertion follows from the (L)-property (Def.1) for $G^*(M)$.

AXIOM 2'. *For any sufficiently small neighborhood V of $x \in M$ there is $f \in G_c^*(M)$ such that* $\mathrm{Fix}(f) \cap L_x = (L_x - U) \cup \{x\}$ *for some open ball U with $\bar{U} \subset V$. Here L_x is the leaf passing through x.*

This axiom is satisfied due to Proposition 1 and, a fortiori, Axiom 2 holds true for transitive Jacobi structures.

AXIOM 3. *$G(M)_0$ acts 3-transitively on M.*

AXIOM 3'. *$G(M)$ is pseudo-3-transitive.*

Again this is fulfilled by Corollary 1.

Theorems of Whittaker-Filipkiewicz type are "integral" counterparts of Pursell-Shanks type theorems. The theorem of Pursell-Shanks states that the Lie algebra of vector fields of a manifold M determines completely the smooth structure of M itself. Several generalizations concerning various geometric structures are also true. Our next AXIOM 4 requires the existence of a Pursell-Shanks type theorem. This is the case of Jacobi structures due to J. Grabowski [**5**].

The set of all isotopies of a diffeomorphism group $G(M)$ defines a generalized foliation $\mathcal{F}$ (Theorem 2). We impose further axioms to deal with the nontransitivity.

AXIOM 5. *$\mathcal{F}$ has no leaves of dimension 0, that is $G(M)_0$ fixes no points.*

AXIOM 6. *$G(M)$ preserves the leaves of $\mathcal{F}$.*

Observe that this is true for $G(M)_0$, and that $G(M)$ preserves $\mathcal{F}$, cf.[**12**].

AXIOM 7. *If $x, y \in L \cap U, L \in \mathcal{F}, U$ being an open ball, then there is $g \in G(M)_0$ with* $\mathrm{supp}(g) \subset U$ *such that $g(x) = y$.*

The following two last axioms are connected with the case of noncompact manifolds.

AXIOM 8. *For any $x \in M$ and for any $g \in G(M)$ such that $g(x) = x$ there exists $h \in G_c(M)$ such that $h = g$ on a neighbourhood of x.*

AXIOM 9. *If $\{U_i\}$ is a pairwise disjoint locally finite family of open balls and $g_i \in G(M)$ with* $\mathrm{supp}(g_i) \subset U_i$, *then $g = \prod g_i \in G(M)$, where $\prod g_i = g_i$ on U_i for any i, and $\prod g_i = id$ on $M - \bigcup U_i$.*

Note that in the infinitesimal case this condition is contained in the definition of a so-called quasi-foliation.

In the case of Jacobi structures and the groups of Theorem 1 Axioms 6 and 9 follows from the definition, Axiom 5 by assumption, and Axioms 7 and 8 by Proposition 5. Now we can recall the main result of [**11**] which implies Theorem 1.

THEOREM 6. *Let* (M_i, α_i), $i = 1, 2$, *be a geometric structure such that its group of automorphisms* $G(M_i, \alpha_i)$ *satisfies either Axioms 1,2 and 3, or Axioms 1,2',3',5,6 and 7, and* M_i *is compact, or Axioms 1,2',3',5,6,7,8 and 9. Then if there is a group isomorphism* $\Phi : G(M_1, \alpha_1) \to G(M_2, \alpha_2)$ *then there is a unique* C^∞*-diffeomorphism* $\phi : M_1 \to M_2$ *such that* $\Phi(f) = \phi f \phi^{-1}$ *for each* $f \in G(M_1, \alpha_1)$. *Moreover, it preserves* α_i *whenever Axiom 4 holds.*

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