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POSITIVENESS OF MIXED CURVATURE AND DIMENSION OF FOLIATION

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Abstract. We prove that if a totally geodesic foliation on a Riemannian manifold has positive sectional curvature in mixed directions and satisfies some additional condition containing "turbulence" and curvature pinching, then the dimension of a foliation is "small". The corollary is a decomposition theorem for compact foliation with nonnegative mixed curvature. On the other hand, we construct examples of foliations on even-dimensional manifolds by closed geodesics, which show that in above results for every curvature pinching $\delta < 1$ we need some additional conditions.

1. Introduction.

1.1. Since 1970 foliations have played an important role not only in topology [**1**], analysis [**2**], but also in Riemannian geometry, [**3**], [**4**] and [**5**]. The relationship between curvature and topology of manifolds plays the key role in global Riemannian geometry, see [**6**]. Over the last few years the interest of geometers in the problems of existence of adapted metrics and foliations (distributions) on a manifold with additional curvature restrictions has increased [**7**]:

FC1: *Does there exist a complete Riemannian metric g for a manifold M with a fixed foliation $\{L\}$ which induces the given geometrical and curvature properties of a foliation?*

FC2: *Does there exist a foliation $\{L\}$ with given metric and curvature properties for a fixed complete Riemannian manifold (M, g)? If yes, what is their classification?*

Let M^m be a *Riemannian manifold* and $T_1 \subset TM$ be a ν-dimensional *distribution.* Denote by $T_2 \subset TM$ the distribution on M which is completely orthogonal to T_1, i.e., $T_1 \oplus T_2 = TM$. In the particular case $\nu = 1$ we deal with *a line field* T_1, whose integral curves form *a 1-dimensional foliation* on M. For

$\nu > 1$ the distribution T_1 is tangent to the *foliation* $\{L\}$ only in the case when the integrability tensor for T_1 is zero. For simplicity, below we assume this condition, moreover, the leaves $\{L\}$ are supposed *totally geodesic submanifolds* (*geodesics* in the particular case $\nu = 1$).

Note that *totally geodesic* foliations and *Riemannian foliations* have the simplest external geometry of the leaves (respectively, the tangent or orthogonal distribution has the zero second fundamental form), are dual in a some sense, and are investigated in a number of works.

1.2. We shall concentrate in the work on the mixed sectional curvature K_{mix} of a foliation. For two nonzero vectors $x \in T_1$ and $y \in T_2$ the sectional curvature $K(x, y)$ is called *mixed*, it controls the relative behaviour of nearby leaves. Actually, the geometrical sense of K_{mix} follows from the fact that for a totally geodesic foliation the components of curvature tensor along any leaf geodesic $\gamma(t) \subset L$ are contained in *the Jacobi equation* $Y'' + R(t)Y = 0$ for the induced by a foliation *Jacobi tensor* $Y(t)$, and *the Riccati equation* $B' + B^2 + R(t) = 0$ for the *structural tensor* $B : T_1 \times T_2 \to T_2$ defined by the rule

$$B(x, y) = (\nabla_y \tilde{x})^{\perp}, \quad (x \in T_1,\ y \in T_2), \tag{1.1}$$

where $\tilde{x} \subset T_1$ is any local extension of x. The relationship between these tensors Y and B is the following: $Y' = B(\gamma', Y)$. Note that the equality $B = 0$ holds iff the distribution $T_2 = TL^{\perp}$ is also tangent to a totally geodesic foliation, and by the de Rham decomposition theorem M is locally a Riemannian product $L \times L^{\perp}$. In the case of constant curvature K_{mix} the solutions of the above Jacobi and Riccati ODE are known (and thus the relative behaviour of geodesics on nearby leaves), see Figure 2.

For $K_{mix} = k = const > 0$ D. Ferus proved a surprising theorem in 1970 using the Riccati equation with $R(t) = kE$.

THEOREM 1.1. *[8] If a totally geodesic foliation* $\{L\}$ *has* $K_{mix} = k > 0$ *along a complete leaf* L_0, *then*

$$\dim L < \rho(codim\, L), \tag{1.2}$$

where $\rho(n) - 1$ *is a number of vector fields on a sphere* S^{n-1}.

In [**9**] we showed that the estimate (1.2) is
– **exact**: *for all* $\nu < \rho(n)$: *take some neighbourhood of a* ν*-dimensional great sphere in a* $(n+\nu)$*-dimensional round sphere is foliated by* ν*-dimensional great spheres,*
– **wrong** *for nonconstant* $K_{mix} > 0$: *in any of projective spaces* KP^n, $(n \geq 2;\ K = \mathbb{C},\ \mathbb{H},\ \mathbb{C}a)$ *there exists a neighbourhood of a closed geodesic which can be foliated by closed geodesics,*

– **true** *for ruled submanifolds* $M^{n+\nu}$ *with* ν*-dimensional complete rulings* $\{L^\nu\}$ *and* $K_{mix} > 0$ *in a round sphere* S^N.

Thus the following questions are natural.

Which manifolds admit totally geodesic foliations with $K_{mix} > 0$ *?*

What is the structure of such foliations with $K_{mix} \geq 0$ *and large dimension, in particular, when they split?*

These questions are interesting even locally for a foliation by closed geodesics, i.e. when $\dim L = 1$ and $codim\, L = odd$.

PROBLEM 1.2. *Let* $M = S^1 \times B^{2m+1}$ *be the product of a circle and an odd-dimensional ball with the product foliation* $\{S^1 \times b\}$. *For what maximal* $\delta \in (\frac{1}{4}, 1)$ *there exists a geodesible metric on* M *with positive and* δ*-pinched* K_{mix}*?*

Note that Hopf fibrations (of round sphere by closed geodesics) have $codim\, L = even$, $\dim L = 1$ and $K_{mix} = const > 0$. V. Toponogov conjectured that for a totally geodesic foliation on a compact Riemannian manifold with $K_{mix} > 0$ the inequality (1.2) holds.

1.3. In the work we prove **Theorem 2.3** that if a totally geodesic foliation on a Riemannian manifold has positive sectional curvature in mixed directions and satisfies some additional condition containing turbulence and curvature pinching, then the dimension of a foliation is "small", i.e., inequality (1.2) holds. The corollary is a decomposition **Theorem 2.5** for compact foliation with nonnegative mixed curvature. On the other hand, in **Theorem 2.6** we construct examples of foliations on even-dimensional manifolds by closed geodesics, which show that in the above results for every curvature pinching $\delta < 1$ we need some additional conditions.

2. Main results.

2.1. *Main notations and the statement of results.* Along any *leaf geodesic* $\gamma \subset M$ the structural tensor $B(t) := B(\dot\gamma, *)$ of a foliation, defined in (1.1), satisfies the Riccati ODE

$$\dot B + B^2 + R_\gamma(t) = 0, \qquad (t \in I), \tag{2.1}$$

where $R_\gamma := R(\dot\gamma, \cdot)\dot\gamma$ is the *Jacobi operator.*

DEFINITION 2.1. A vector field $y : \gamma \to T_2(\gamma)$ along the unit speed leaf geodesic $\gamma \subset M$ is called *L-parallel* if the following first order ODE holds, see **[10]**,

$$\dot y = B(\dot\gamma, y). \tag{2.2}$$

A smooth $(1,1)$-tensor field $Y : T_2(\gamma) \to T_2(\gamma)$ is called *L-parallel Jacobi tensor* if the following first order ODE holds

$$\dot{Y} = B(\dot{\gamma}, Y). \tag{2.3}$$

Note that L-parallel vector field $y(t)$ along the leaf geodesic $\gamma \subset M$ never vanishes and, and in view of (2.1)–(2.2) satisfies the Jacobi ODE

$$\ddot{y} + R_\gamma y = 0. \tag{2.4}$$

Hence, L-parallel fields form linear subspace among all Jacobi fields along γ. The condition (2.3) is satisfied precisely when the action of Y on linearly independent parallel sections of $T_2(\gamma)$ gives rise to linearly independent L-parallel (Jacobi) vector fields.

DEFINITION 2.2. The *turbulence of the foliation along the leaf geodesic* γ (a rotation component of the tensor B) is defined by the formula

$$a(\gamma) = \sup\{(B(\dot{\gamma}, y), z) : y,\, z \in T_2(\gamma),\ y \perp z,\ |y| = |z| = 1\}, \tag{2.5}$$

(see **[11]**, **[12]** for Riemannian submersions and Riemannian foliations). The *turbulence* $a(L)$ *of the foliation along a leaf* L is defined by an analogous formula

$$a(L) = \sup\{(B(x, y), z) :\ y \perp z,\ |x| = |y| = |z| = 1\}.$$

Note that the equality $a(L) = 0$ for all L means that $B(x, y) = \lambda(x)y$, in this case the orthogonal distribution T_2 is tangent to an umbilic foliation.

Let $\rho(n) - 1$ be the *number of continuous pointwise linearly independent vector fields on the* $(n-1)$*-dimensional sphere*; see Table 1, names known from topology. We write n as a product of odd number and a power of 2

$$n = (\mathrm{odd})\, 2^{4b+c}, \qquad (b \geq 0,\ 0 \leq c \leq 3).$$

Then $\rho(n) = 8b + 2^c$, and the following inequality holds:

$$\rho(n) \leq 2\, \log_2 n + 2 \leq n\ .$$

Since $\rho(odd) = 1$, then $\dim T_2$ in Theorem 1.1 by D. Ferus is even.

$n-1$	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	$\cdots$
$\rho(n)-1$	1	3	1	7	1	3	1	8	1	3	1	7	1	3	1	$\cdots$

Table 1. The values of function $\rho(n) - 1$

In the following Theorem that generalizes the Theorem 1.1 by D. Ferus, the condition for the curvature is given along a sheaf of leaf geodesics.

THEOREM 2.3. *Suppose $\{L\}$ be a totally geodesic foliation and there exists such point $m \in M$ that along any leaf geodesic $\gamma : [0, \frac{\pi}{\sqrt{k}}] \to M$, $(\gamma(0) = m)$ we have inequalities*

$$k_2 \geq K(\gamma', y) \geq k_1 > 0, \qquad (y \in T_2(\gamma)), \tag{2.6}$$

$$(1 - \frac{k_1}{k_2}) \max\{\frac{a(\gamma)^2}{k}, 1\} \leq 0.337\ , \tag{2.7}$$

where $k = \frac{k_1+k_2}{2}$. Then the inequality $\dim L < \rho(codim\, L)$ *holds.*

REMARK 2.4. From (2.7) follows $\frac{k_1}{k_2} \geq 0.663$, but the coefficient in (2.7) is obtained by the method of the proof and perhaps may be improved.

IDEA OF PROOF OF THEOREM 2.3.
Step 1. D. Ferus [**8**] showed that the inequality (1.2) follows from the property
for every $x \neq 0$ the operator $B(x, *)$ does not have eigenvectors.
Step 2. Suppose the opposite, i.e., $B(x_0, y_0) = \lambda y_0$ holds for some unit vectors $x_0 \in T_1(m)$, $y_0 \in T_2(m)$ and the real number $\lambda \leq 0$. The *"extremal" leaf geodesic* $\gamma : [0, \frac{\pi}{\sqrt{k}}] \to M$, $(\gamma'(0) = x_0)$ and the L-parallel Jacobi field $y(t)$ along γ with the initial value $y(0) = y_0$ play the key role in the proof. We decompose the vector field onto standard and "small" terms $y(t) = \left(\cos(\sqrt{k}t) + \frac{\lambda}{\sqrt{k}}\sin(\sqrt{k}t)\right) y_0 + u(t)$, where $u(0) = u'(0) = 0$. Note that for $k_2 = k_1$ we have $u(t) \equiv 0$ and hence the L-parallel Jacobi vector field $y(t)$ vanishes at the point $t_0 = \text{arcctg}(-\frac{\lambda}{\sqrt{k}})/\sqrt{k}$. The proof of Theorem 1.1 by D. Ferus is based on this contradiction. In the case $k_2 > k_1$ we prove that under assumption (2.7) (actually, for $\delta \geq 0.5821$) the function

$|y(t)|$ – *the length of L – parallel vector field $y(t)$*

has a local minimum at some point $t_m \in (0, \frac{\pi}{\sqrt{k}}]$, see Figure 3.
Step 3. Further we observe that the function

$V(t)$ – *the area of a parallelogram on the vectors $y(t)$, $y'(t)$*

varies "slowly" along the geodesic γ. (This function is constant in the case $k_2 = k_1$). From these we shall obtain a contradiction, because the function $V(t)$ can not increase from zero value $V(0)$ to "large" value $V(t_m)$ on the interval with length t_m bounded above by $\frac{\pi}{\sqrt{k}}$. □

From Theorem 2.3 we obtain (as in [13]) a number of corollaries with *metric decomposition* of compact foliations and ruled submanifolds with $K_{mix} \geq 0$. We give one such result without proof.

THEOREM 2.5. *(decomposition). Suppose $\{L\}$ be a compact (i.e., with compact leaves) totally geodesic foliation on a Riemannian manifold M with* *the conditions*

$$k_2 \geq K(x,y) \geq k_1 \geq 0, \qquad (x \in TL,\ y \in TL^\perp), \tag{2.6'}$$

$$(k_2 - k_1)\max\{a(L_0)^2, k\} \geq 0.337\, kk_2, \tag{2.7'}$$

where $k = \frac{k_1+k_2}{2}$, L_0 *is some leaf. If* $\dim L \geq \rho(codim\, L)$, *then* $k_1 = k_2 = 0$ *and* M *is locally isometric to the product* $L \times L^\perp$.

The Hopf fibrations serve as examples of foliations on *odd-dimensional* Riemannian manifolds by closed geodesics $\{L = S^1\}$ with constant positive K_{mix}. The Problem 1.2 of the existence of *even-dimensional* Riemannian manifold foliated on closed geodesics with ($\delta \approx 1$)-pinched positive K_{mix} is important in view of Theorems 2.3 and 2.5 where the foliations with nonnegative K_{mix} are studied under some additional assumptions. The Theorem 2.6 below gives the positive answer on the above problem for any $\delta \in (0,1)$.

THEOREM 2.6. *a) For any* $\delta \in (0,1)$ *there exists a Riemannian manifold* M^{2n+2}, *where* $n \geq \frac{\sqrt{\delta}}{1-\sqrt{\delta}}$, *with the fibration on closed geodesics* $\{L = S^1\}$ *and with positive* δ*-pinched* K_{mix}.

REMARK 2.7. In the proof of the Theorem 2.6 we construct a metric on the product $M = S^1 \times B^{2m+1}$ with the following properties a) $(\frac{n}{n+1})^2 \leq K_{mix} \leq 1$, b) the length of circles (closed geodesics) $l(L) = 2\pi(n+1) \to \infty$. Our hypothesis is that *Theorem 2.6 holds when* $\delta \to 1$ *and* n *is fixed.* b) Theorem 2.6 is *local* in the sense of directions transversal to the leaf $L = S^1$, and the problem of the existence of analogue geodesic foliation on *compact* Riemannian manifold M^{2n+2} is open.

2.2. *Proof of Theorem 2.3.* The following comparison Lemma is used for estimating the length of an L-parallel field in terms of mixed sectional curvature pinching.

LEMMA 2.8. *Let* $y(t) = Y(t) + u(t) \subset \mathbb{R}^n$ *be the solution of the Jacobi ODE*

$$y'' + R(t)y = 0, \qquad (0 \leq t \leq \frac{\pi}{\sqrt{k}}), \tag{2.8}$$

where $Y(t) = y(0)\cos(\sqrt{k}t) + \frac{y'(0)}{\sqrt{k}}\sin(\sqrt{k}t)$. *Then*

$$u(t) = -\int_0^t \frac{\sin(\sqrt{k}(t-s))}{\sqrt{k}}(Du(s) + f(s))ds, \tag{2.9}$$

where $D(t) = R(t) - kE$ *and* $f(t) = (R(t) - kE)Y(t)$. *In particular, if the norm* $||D(t)|| \le \varepsilon < \frac{k}{2}$, *then*

$$|u(t)| \le \frac{\varepsilon}{k - \left(1 - \cos(\sqrt{k}t)\right)\varepsilon} \int_0^t \sqrt{k}|Y(s)| \sin(\sqrt{k}(t-s))\, ds\ . \tag{2.10}$$

PROOF. **Step 1**. Denote $z(t) = y'(t)$ and $\sqrt{k}v = u'$. Then $z' = -R(t)y$ and $\sqrt{k}v' = -R(t)u - f(t)$. We rewrite the given Jacobi ODE (2.8) using the matrices

$$w = \begin{pmatrix} u \\ v \end{pmatrix},\ A = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},\ F(t) = \begin{pmatrix} 0 \\ -D(t)u - f(t)/\sqrt{k} \end{pmatrix}, \tag{2.11}$$

as the first order ODE with a constant matrix $\sqrt{k}A$

$$w' = \sqrt{k}Aw + F, \qquad (0 \le t \le \frac{\pi}{\sqrt{k}}). \tag{2.12}$$

Note that $w(0) = 0$ in view of $u(0) = u'(0) = 0$. Using the formula for a solution of a linear ODE (2.12) we obtain the integral equality

$$w(t) = \int_0^t \exp(\sqrt{k}(t-s)A)F(s)\, ds. \tag{2.13}$$

It is easy to calculate that

$$\exp(\sqrt{k}(t-s)A) = \begin{pmatrix} \cos(\sqrt{k}(t-s))E & \sin(\sqrt{k}(t-s))E \\ -\sin(\sqrt{k}(t-s))E & \cos(\sqrt{k}(t-s))E \end{pmatrix}$$

and then

$$\exp(\sqrt{k}(t-s)A)F(s) = -\frac{1}{\sqrt{k}} \begin{pmatrix} \sin(\sqrt{k}(t-s))(D(t)u + f(t)) \\ \cos(\sqrt{k}(t-s))(D(t)u + f(t)) \end{pmatrix}.$$

So, from (2.13) we deduce the integral expression (2.9) for the vector-function $u(t)$.

Step 2. We now estimate the norm $|u|(t) := \sup\{|u(s)| : s \le t\}$

$$\begin{aligned} |u(t)| \le & \frac{\varepsilon}{k} \int_0^t \sqrt{k}(|Y(s)| + |u(s)|) \sin(\sqrt{k}(t-s))\, ds \le \\ & \frac{\varepsilon}{k} \int_0^t \sqrt{k}|Y(s)| \sin(\sqrt{k}(t-s))\, ds + \left(1 - \cos(\sqrt{k}t)\right) \frac{\varepsilon}{k}|u|(t), \end{aligned}$$

i.e., $|u|(t)\left(1 - \left(1 - \cos(\sqrt{k}t)\right)\frac{\varepsilon}{k}\right) \le \frac{\varepsilon}{k}\int_0^t \sqrt{k}|Y(s)| \sin(\sqrt{k}(t-s))\, ds$, which is equivalent to the inequality

$$|u|(t) \le \frac{\varepsilon}{k - \left(1 - \cos(\sqrt{k}t)\right)\varepsilon} \int_0^t \sqrt{k}|Y(s)| \sin(\sqrt{k}(t-s))\, ds \;.$$

□

PROOF OF THEOREM 2.3.

Step 1. We now prove that for every $x \neq 0$ the operator $B(x,*)$ does not have eigenvectors, because from this property follows the inequality (1.2). Let $p \in M$ be an arbitrary point and $\{e_i\} \subset T_1(p)$ be any base. Then the following $\dim L$ continuous vector fields $\{w_i\}$

$$w_i(y) = B(e_i, y) - (B(e_i, y), y)y,$$

are tangent to the unit sphere $S^{codim\, L-1} \subset T_2(p)$. If these vector fields are linearly dependent at some point $y \in S^{codim\, L-1}$, i.e. $\sum_i \lambda_i w_i(y) = 0$, for some real numbers $\{\lambda_i\}$ then the equality $B(\sum_i \lambda_i e_i, y) = [\sum_i \lambda_i (B(e_i, y), y)]y$ holds which is impossible by our assumption. But if $\{w_i\}$ are $\dim L$ continuous pointwise linearly independent vector fields on $(codim\, L - 1)$-dimensional sphere, then $\dim L < \rho(codim\, L)$.

Step 2. Suppose the opposite, i.e. there exist unit vectors $x_0 \in T_1(p)$, $y_0 \in T_2(p)$ and a real number $\lambda \le 0$ with the property $B(x_0, y_0) = \lambda y_0$. Let $\gamma(t) : [0, \frac{\pi}{\sqrt{k}}] \to M$, $(\gamma'(0) = x_0)$ be a leaf geodesic, $y(t) : \gamma \to T_2(\gamma)$ be an L-parallel vector field along the geodesic γ containing the vector y_0. Note that the Jacobi ODE $y'' + R(t)y = 0$ holds with $||R(t) - kE|| \le \frac{k_2 - k_1}{2}$, where $y' = \nabla_{\gamma'} y$, $y'' = \nabla_{\gamma'}(\nabla_{\gamma'} y)$.

Step 3. We now prove that the area of a parallelogram $V(t)$ in $T_2(\gamma(t))$, whose sides are vectors $y(t)$, $y'(t)$, satisfies to inequality

$$|V(t)'| \le \left(\frac{k_2 - k_1}{2}\right) |y(t)|^2. \tag{2.14}$$

In view of (2.8) the derivative of the function $V^2 = y^2(y')^2 - (y, y')^2$ is

$$(V^2)' = -2 \begin{vmatrix} (R(t)y, y') & (R(t)y, y) \\ (y, y') & (y, y) \end{vmatrix}.$$

By using linear combinations of columns in this 2×2-determinant, we obtain

$$(V^2)' = -2(R(t)y, \widetilde{y'})y^2 = -2(\widetilde{R(t)y}, \widetilde{y'})y^2, \tag{2.15}$$

where $\widetilde{..}$ denotes the orthogonal to vector $y(t)$ component. Since

$$(V(t)^2)' = 2V'(t)V(t), \quad V(t) = |y(t)| \cdot |\widetilde{y'}(t)|, \tag{2.16}$$

then from (2.15) follows that $|V(t)'| \leq |\widetilde{R(t)y(t)}| \cdot |y(t)|$. Thus we obtain (2.14) with the help of

$$|\widetilde{Ry}(t)| \leq |(R(t) - kE)y(t)| \leq \left(\frac{k_2 - k_1}{2}\right) |y(t)|.$$

Step 4. Suppose that the first local minimum t_m of a function $|y(t)|$ for $t \geq 0$ belongs to the interval $(0, \frac{\pi}{\sqrt{k}}]$ and let $q = \gamma(t_m)$. We consider (in this step) the opposite parameterization of the geodesic γ with parameter $s = t_m - t$, i.e. $\gamma(0) = q$, $\gamma(t_m) = p$. Since $\gamma(0)$ is also a local minimum of the function $|y(s)|^2$, then

$$y'(0) \perp y(0), \quad (y, y)'' \geq 0$$

and from (2.8) and the equality $\frac{1}{2}(y, y)'' = (y, y'') + (y', y')$ we get

$$V(0) = |y'(0)| \cdot |y(0)| \geq \sqrt{k_1}|y(0)|^2. \tag{2.17}$$

Moreover $(y'(q), y'(q)) = (B(\gamma', y(q)), y'(q))$, i.e.

$$\frac{|y'(q)|}{|y(q)|} \leq |(B\left(\gamma', \frac{y(q)}{|y(q)|}\right), \frac{y'(q)}{|y'(q)|})| \leq a(\gamma).$$

Write the vector field $y(s)$ in the form $y(s) = Y_1(s) + u_1(s)$, where $Y_1(s) = y(0)\cos(\sqrt{k}s) + \frac{y'(0)}{\sqrt{k}}\sin(\sqrt{k}s)$ is the standard component. In view of Lemma 2.8 with $\varepsilon = \frac{k_2 - k_1}{2}$ we have

$$\begin{aligned} 1 =& |y(p)| \leq |Y_1(t_m)| + |u_1(t_m)| \leq \max\{\frac{a(\gamma)}{\sqrt{k}}, 1\}|y(q)| \cdot \\ & \left(1 + (1 - \cos(\sqrt{k}t_m))\frac{k_2 - k_1}{\cos(\sqrt{k}t_m)(k_2 - k_1) + 2k_1}\right), \end{aligned}$$

i.e. the inequality

$$|y(q)| \geq \frac{\cos(\sqrt{k}t_m)(k_2 - k_1) + 2k_1}{k_2 + k_1} / \max\{\frac{a(\gamma)}{\sqrt{k}}, 1\}. \tag{2.18}$$

From the eigenvector condition $B(x_0, y_0) = \lambda y_0$ it follows that $V(t_m) = 0$. Thus in view of (2.14) and the estimate $|y(s)| \leq 1$ for $0 \leq s \leq t_m$ we obtain the inequality

$$V(0) \leq \int_0^{t_m} |V'(s)|ds \leq \frac{k_2 - k_1}{2} \int_0^{t_m} |y(s)|^2 ds \leq \left(\frac{k_2 - k_1}{2}\right) t_m.$$

From these and (2.17), (2.18), in view of $\cos(\tau+\tau_0) \geq \cos(\tau+\frac{\pi}{2})$ we obtain the inequality

(2.19)
$$\left(\frac{\pi}{2}+\tau\right)(1-\delta)\left(\frac{1+\delta}{\cos(\tau+\frac{\pi}{2})(1-\delta)+2\delta}\right)^2 \max\{\frac{a(\gamma)^2}{k},1\} \geq \sqrt{2\delta(1+\delta)},$$

where $\tau = \sqrt{k}(t_m - t_0)$, $\delta = \frac{k_1}{k_2}$, $\tau_0 = \sqrt{k}t_0 = \operatorname{arcctg}(-\frac{\lambda}{\sqrt{k}}) \in (0, \frac{\pi}{2}]$.

Step 5. We now go back to the initial parameterization of γ and consider the problem, when the function $|y(t)|$ has a local minimum in interval $(0, \frac{\pi}{\sqrt{k}}]$.

The vector field $y(t)$ can be written in form $y(t) = Y(t) + u(t)$, where $t \in [t_0, 2t_0]$ and

$$Y(t) = \left(\cos(\sqrt{k}t) + \frac{\lambda}{\sqrt{k}}\sin(\sqrt{k}t)\right) y_0 = \frac{\sin(\sqrt{k}(t_0-t))}{\sin(\sqrt{k}t_0)} y_0.$$

In view of Lemma 2.8 with $\varepsilon = \frac{k_2-k_1}{2}$, we have

(2.20) $|u(t)| \leq$
$$\frac{k_2-k_1}{(\cos(\sqrt{k}t)(k_2-k_1)+2k_1)\sin(\sqrt{k}t_0)} \int_0^t \sqrt{k}\sin(\sqrt{k}(t-s))\sin(\sqrt{k}|t_0-s|)\,ds.$$

With the help of a trigonometry identity

$$f = \sin(\sqrt{k}(t-s))\sin(\sqrt{k}(t_0-s)) = \frac{\cos(\sqrt{k}(t-t_0)) - \cos(\sqrt{k}(t+t_0-2s))}{2},$$

and abbreviations

$$\tau = \sqrt{k}(t-t_0),\ \tau_0 = \sqrt{k}t_0,\ S = \sin,\ C = \cos,$$

we transform the integral in the RHS of (2.20):

$$I(t) = \sqrt{k}\left(\int_0^{t_0} f\,ds - \int_{t_0}^t f\,ds\right) = \frac{1}{2}(\tau_0-\tau)C(\tau) + \frac{3}{4}S(\tau) - \frac{1}{4}S(\tau+2\tau_0).$$

Since

$$\tau_0,\ \tau \in [0, \frac{\pi}{2}],\ \tau_0 - \tau \geq 0,\ C(\tau) \geq 0,\ S(2\tau_0+\tau) \geq S(\pi+\tau) = -S(\tau),$$

the function $I(t)$ has the largest upper value for $\tau_0 = \frac{\pi}{2}$:

$$I(t) \leq \left(\frac{\pi}{2}-\tau\right)\frac{C(\tau)}{2} + S(\tau), \qquad I(t_0) \leq \frac{\pi}{4} \text{ (since } \tau = 0).$$

Consequently, (assuming $\delta = \frac{k_1}{k_2}$)

$$|u(t)| \le \frac{\left((\frac{\pi}{2} - \tau)\frac{C(\tau)}{2} + S(\tau)\right)(1-\delta)}{(C(\tau+\tau_0)(1-\delta) + 2\delta)S(\tau_0)},$$

$$|u(t_0)| \le \frac{\frac{\pi}{4}(1-\delta)}{(C(\tau+\tau_0)(1-\delta) + 2\delta)S(\tau_0)}.$$

Note that $|Y(t)| = \frac{S(\tau)}{S(\tau_0)}$ and $Y(t_0) = 0$. Since $|y|'(0) = \lambda \le 0$, and in the case of equality $\lambda = 0$ from (2.27) it follows that $|y|''(0) < 0$, then the function $|y(t)|$ decreases for small values $t \ge 0$. Thus for the property that the function $|y(t)|$ has a local minimum at some $t_m \in (0, t]$ it is sufficient to require that $|y(t_0)| \le |y(t)|$, i.e.

$$|u(t_0)| + |u(t)| \le |Y(t)|.$$

The last inequality (in view of the above estimates) is reduced to

$$\left(\frac{\pi}{4} + (\frac{\pi}{2} - \tau)\frac{C(\tau)}{2} + S(\tau)\right)\frac{1-\delta}{C(\tau+\tau_0)(1-\delta) + 2\delta} \le S(\tau),$$

that is equivalent to

$$\delta \ge 1 - \frac{2S(\tau)}{\frac{\pi}{4} + (\frac{\pi}{2} - \tau)\frac{C(\tau)}{2} + (3 - C(\tau+\tau_0))S(\tau)}.$$

For $\tau_0 \in [0, \frac{\pi}{2}]$ and $\tau \in [0, \frac{\pi}{2}]$ we have $C(\tau_0 + \tau) \ge -S(\tau)$ and hence the last inequality follows from

$$\delta \ge 1 - \frac{2S(\tau)}{\frac{\pi}{4} + (\frac{\pi}{2} - \tau)\frac{C(\tau)}{2} + (3 + S(\tau))S(\tau)}. \tag{2.21}$$

Remember that from (2.7) it follows that $\delta \ge 0.663$. With the help of a computer we deduce for all $\tau \in [0.533, \frac{\pi}{2}]$ the inequality, see Figure 4,

$$f(\tau) := \frac{2S(\tau)}{\frac{\pi}{4} + (\frac{\pi}{2} - \tau)\frac{C(\tau)}{2} + (3 + S(\tau))S(\tau)} \ge 0.337$$

i.e. for $\delta \ge 0.663$ the function $|y(t)|$ has a local minimum at $t_m \in [0, (\frac{\pi}{2} + 0.533)/\sqrt{k}]$.

But for $\delta \in [0.663, 1]$ and $\tau = 0.533$ the inequality

$$0.337\left(\frac{\pi}{2} + \tau\right)\left(\frac{1+\delta}{C(\frac{\pi}{2} + \tau)(1-\delta) + 2\delta}\right)^2 < \sqrt{2\delta(1+\delta)} \tag{2.22}$$

follows from the inequality (where we substituted $\cos(\frac{\pi}{2} + 0.533) \approx -0.508$)

$$g(\delta) := \left(\frac{2.508\,\delta - 0.508}{1+\delta}\right)^2 \sqrt{2\delta(1+\delta)} > 0.337\,(\frac{\pi}{2} + 0.533) \approx 0.709\ . \tag{2.23}$$

Since the function $\frac{2.508\,\delta-0.508}{1+\delta}$ is increasing, then also function $g(\delta)$ is increasing for $\delta > \frac{1}{3}$. From the calculation $g(0.663) > 0.71$ we see that (2.23) and also (2.22) are satisfied for $\delta \geq 0.663$. From (2.22) and (2.19) the inequality follows, which contradicts (2.7). □

2.3. *Proof of Theorem 2.6.* The Theorem 2.6 follows from Lemmas 2.9 and 2.10 below.

LEMMA 2.9. *Let the symmetric matrix $R(t)$ and the nondegenerate matrix $Y(t)$ have the order $n \times n$ and are T-periodic, and the Jacobi equation is satisfied*

$$\ddot{Y}(t) + R(t) \cdot Y(t) = 0, \qquad (0 \leq t \leq T).$$

Then there exists Riemannian metric on the product $M^{n+1} = S^1 \times B^n(r)$ of the circle S^1 and n-dimensional ball $B^n(r)$ of radius r with the following properties:
a) the closed curves $\{\gamma_z(t) = (t, Y(t)z)\}_{z \in B^n(r)}$ are the geodesics,
b) the components of mixed sectional curvature $R(*, \dot{\gamma}_o)\dot{\gamma}_o$ along $\gamma_o(t)$ are expressed by the formula $R(z, \dot{\gamma}_o)\dot{\gamma}_o = R(t)z, \quad (z \perp \dot{\gamma}_o)$,
c) the Jacobi tensor of the foliation $\{\gamma_z\}$ has the form $Y(t)$ for some parallel orthonormal base along γ_o.

PROOF. Together with the coordinate system $t = z_0, z = (z_1, \ldots, z_n)$ of the *direct product* we consider on M^{n+1} the *adapted coordinates* $t = y_0, y = (y_1, \ldots, y_n)$, with the following relation $z = Y(t)\,y$. For the adapted coordinates the curves $\{\gamma_z\}$ are the coordinate lines $\{y = const\}$. The property a) means that the following Christoffel symbols vanish

$$\Gamma^i_{oo}(t, y) \equiv 0, \qquad (0 \leq i \leq n). \tag{2.24}$$

In view of the formulas, which express the Christoffel symbols through the coefficients of the metric $\{g_{ij}(t, y)\}$ and their derivatives [6]

$$2\sum_s g_{sk}\Gamma^s_{ij} = g_{jk,i} + g_{ki,j} - g_{ij,k}\ , \tag{2.25}$$

we obtain from (2.24) the equalities

$$g_{oo,i}(t, y) \equiv 2g_{oi,o}(t, y) \qquad (i \geq 0). \tag{2.26}$$

In view of (2.26) for $i = 0$, the coefficient $g_{oo}(t, y)$ does not depend on t. Also we need the following "initial" conditions along γ_o:

$$g_{oi}(t, 0) \equiv \delta_{oi}, \qquad (i \geq 0), \tag{2.27}$$

$$g_{ij}(t, 0) \equiv \left(Y(t)^\top \cdot Y(t)\right)_{ij} \qquad (i, j \geq 1). \tag{2.28}$$

The conditions (2.27)–(28) mean that the coordinate vector fields along γ_o in the product coordinates (t,z) are orthonormal. Since the matrix $B(t) := \dot{Y}(t)\cdot Y^{-1}(t)$ must correspond to the structural tensor of foliation $\{\gamma_z\}$ along γ_o in the product coordinate system (t,z), we need the following properties for the matrix $\Gamma_o(t,0) := \{\Gamma^k_{oi}(t,0)\}$ in the adapted coordinates (t,y):

$$\Gamma_o(t,0) = Y^{-1}(t)\cdot B(t)\cdot Y(t) = Y^{-1}(t)\cdot \dot{Y}(t). \tag{2.29}$$

We write the formulas (2.25), which relate the metric with the above Christoffel symbols, in the form comfortable for our purposes

$$2\sum_s g_{sj}(t,0)\cdot\Gamma^s_{oi}(t,0) - g_{ij,o}(t,0) = g_{jo,i}(t,0) - g_{io,j}(t,0). \tag{2.30}$$

We will show that the RHS of (2.30) does not depend on t. To prove this we first calculate that the matrix $C := \dot{Y}^\top(t)\cdot Y(t) - Y^\top(t)\cdot\dot{Y}(t)$ is constant, since its derivative with respect to t vanishes:

$$\begin{aligned}\dot{C} = \ldots Y^\top(t)\cdot Y(t) + \dot{Y}^\top(t)\cdot\dot{Y}(t) - \dot{Y}^\top(t)\cdot\dot{Y}(t) - Y^\top(t)\cdot\ldots Y(t) = \\ (-R(t)\cdot Y(t))^\top\cdot Y(t) - Y(t)^\top\cdot(-R(t)\cdot Y(t)) = \\ -Y(t))^\top\cdot R(t)\cdot Y(t) + Y(t)^\top\cdot R(t)\cdot Y(t) = 0.\end{aligned}$$

Then we put $C := \dot{Y}^\top(0)\cdot Y(0) - Y^\top(0)\cdot\dot{Y}(0)$. In view of (2.28) the derivatives of the coefficients of the metric $g_{ij,o}(t,0)$ form the following matrix

$$(Y^\top(t)\cdot Y(t))' = \dot{Y}^\top(t)\cdot Y(t) + Y^\top(t)\cdot\dot{Y}(t) = 2\dot{Y}^\top(t)\cdot\dot{Y}(t) + C,$$

and, in view of (2.28) and (2.29), the functions $2\sum_s g_{sj}(t,0)\cdot\Gamma^s_{oi}(t,0)$ are the elements of the matrix

$$2(Y^\top(t)\cdot Y(t))\cdot(Y^{-1}(t)\cdot\dot{Y}(t)) = 2\,Y^\top(t)\cdot\dot{Y}(t).$$

Hence the RHS of (2.30) represents the coefficients of the constant matrix $-C$, and does not depend on t. Moreover, the matrix C is skew-symmetric. Since (2.30) was reduced to the equalities

$$g_{jo,i}(t,0) - g_{io,j}(t,0) = -C_{ij}, \tag{2.31}$$

we can assume $g_{io,j}(t,y) := \begin{cases}0, & j\le i\\ C_{ij}, & j>i\end{cases}.$ We integrate these expressions and, in view of (2.27), have

$$g_{io}(t,y) = \sum_{j>i} C_{ij}y_j, \qquad (i\ge 1). \tag{2.32}$$

Let substitute (2.32) into (2.26), then in view of (2.27) we obtain that the functions $g_{oo}(t,y)$ are constant and equal to 1. The condition (2.27), obviously, holds. Finally, in view of (2.28), we can define the other coefficients by the formula

$$g_{ij}(t,y) := \left(Y(t)^\top \cdot Y(t)\right)_{ij}, \qquad (i,j \geq 1). \tag{2.33}$$

For sufficiently small $r > 0$ the metric $\{g_{ij}(t,y)\}$ on M^{n+1} is positive. Note, that the property b) follows from the formulas (2.29) and the property c) follows from the formulas (2.28). □

LEMMA 2.10. *For any $n \in \mathbb{N}$ there exists the matrix $Z_{n,n+1}(t,s)$ of the order $(2n+1) \times (2n+1)$ with the elements*

$$z_{jk} = \begin{cases} a_{jk}\cos(\frac{t}{n}) + b_{jk}\sin(\frac{t}{n}), & j \leq n, \\ a_{jk}\cos(\frac{s}{n+1}) + b_{jk}\sin(\frac{s}{n+1}), & n+1 \leq j \leq 2n+1, \end{cases}$$

with the property $\det Z_{n,n+1}(t,s) = \cos(t-s)$. *In particular, the determinant of the matrix $Y_{2n+1}(t) := Z_{n,n+1}(t,t)$ is identically* 1.

PROOF. **Step 1**. The structure of the matrix $Z_{n,n+1}(t,s)$ is given for n odd and n even on Table 2. Consider the case of odd n. Note, that $z_{1,1} = \cos(\frac{t}{n})$, $z_{1,n+1} = -\sin(\frac{t}{n})$ and in both cases of n the quadratic matrices A_1 and A_2 have even order. Let the matrix $\tilde{A}_1$ be obtained from A_1 (with the size $n-1$) by deleting the last column and completing with the column $\{z_{j,1}\}_{2\leq j \leq n}$ inside of its first column. Analogously, let the matrix $\tilde{A}_2$ be obtained from A_2 (with the size $n+1$) by replacing of its first column by the column $\{z_{j,n}\}_{j>n}$. For shortened notations let

$$c_n := \cos\frac{t}{n}, \quad s_n := \sin\frac{t}{n}, \qquad c_{n+1} := \cos\frac{s}{n+1}, \quad s_{n+1} := \sin\frac{s}{n+1}.$$

The elements of initial n rows of the matrix $Z_{n,n+1}(t,s)$ are linear combinations of c_n, s_n, and the elements in the other (lower) $n+1$ rows are linear combinations of c_{n+1}, s_{n+1}. Obviously, we have, see the Table 2,

$$\det Z_{n,n+1}(t,s) = c_n \det A_1 \cdot \det A_2 + s_n \det \tilde{A}_1 \cdot \det \tilde{A}_2.$$

To deduce the equality $\det Z_{n,n+1}(t,s) = \cos(t)\cdot\cos(s) + \sin(t)\cdot\sin(s)$ we will provide (in steps 2–4) the following equalities

$$\begin{aligned} \det A_1 &= \cos(t)/c_n, & \det A_2 &= \cos(s), \\ \det \tilde{A}_1 &= \sin(t)/s_n, & \det \tilde{A}_2 &= \sin(s). \end{aligned}$$

c_n	O	$-s_n$	
$z_{2,1}$ $\dots$ $z_{n,1}$	$\boldsymbol{A}_1$	O	O
O	$z_{n+1,n}$ $\dots$ $z_{2n+1,n}$	$\boldsymbol{A}_2$	

n odd

c_n	O	$-s_n$	
$z_{2,1}$ $\dots$ $z_{n+1,1}$	$\boldsymbol{A}_1$	O	O
O	$z_{n+2,n+1}$ $\dots$ $z_{2n+1,n+1}$	$\boldsymbol{A}_2$	

n even

Table 2. The matrix $Z_{n,n+1}(t,s)$

Step 2. We separate the real and imaginary parts in Moivre's formula

$$\cos(mx) + \boldsymbol{i}\sin(mx) = (\cos x + \boldsymbol{i}\sin x)^m = \sum_{k=0}^{m} \boldsymbol{i}^k C_m^k \cos^{m-k}(x)\cdot\sin^k(x)$$

and find the polynomials $p_m(a,b)$, $q_m(a,b)$ of degree m

$$p_m(a,b) = a^m - C_m^2 a^{m-2}b^2 + C_m^4 a^{m-4}b^4 - C_m^6 a^{m-6}b^6 + \dots,$$
$$q_m(a,b) = C_m^1 a^{m-1}b - C_m^3 a^{m-3}b^3 + C_m^5 a^{m-5}b^5 + \dots,$$

with the following properties

$$\cos(t) = p_n(c_n, s_n), \qquad \sin(t) = q_n(c_n, s_n),$$
$$\cos(s) = p_{n+1}(c_{n+1}, s_{n+1}), \quad \sin(s) = q_{n+1}(c_{n+1}, s_{n+1}).$$

It is easy to check that the roots of the polynomial $p_n(\lambda, 1)$ are the following $\lambda_i = \cot(\frac{\pi(2i+1)}{2n})$, where $i = 1..n$, and the roots of the polynomial $q_n(\lambda,1)$ are $\lambda_i = \cot(\frac{\pi i}{n})$, where $i = 1..n$.

Proposition 2.11. *For any polynomial $P_m(\lambda)$ of degree m there exists the constant matrix D_m of order $m\times m$, whose characteristic polynomial* $\det(D_m - \lambda\cdot I_m)$ *is $P_m(\lambda)$.*

Proof of Proposition 2.11. Consider the factoring of $P_m(\lambda)$ over the field $\mathbb{R}$ onto (linear and quadratic) multipliers, assuming for simplicity the absence of multiple roots, $P_m(\lambda) = \prod_\alpha(\mu_\alpha - \lambda)\cdot\prod_\beta((\lambda_\beta - \lambda)\cdot(\bar\lambda_\beta - t))$, where $\lambda_\beta = u_\beta + \boldsymbol{i}v_\beta$, and let $D_\beta = \begin{pmatrix} u_\beta & -v_\beta \\ v_\beta & u_\beta \end{pmatrix}$ be the matrices of the order

2×2. Then the block-diagonal matrix $D_n = [\ldots \mu_\alpha, \ldots D_\beta, \ldots]$ have the property which we wish. $\square$

In view of Proposition 2.11, we can take the matrices A_1, A_2 with the properties $c_n \det A_1 = \cos(t)$, $\det A_2 = \cos(s)$. Namely let $A_1 = s_n \cdot D_{n-1} - c_n \cdot I_{n-1}$, where the characteristic polynomial of the matrix D_{n-1} is $p_n(\lambda, 1)/\lambda$, and let $A_2 = s_{n+1} \cdot D_{n+1} - c_{n+1} \cdot I_{n+1}$, where the characteristic polynomial of the matrix D_{n+1} is $p_{n+1}(\lambda, 1)$. Note that A_1, A_2 are the block-diagonal matrices, all their blocks have the size 2×2.

Step 3. We now deduce the equalities $s_n \det \tilde{A}_1 = \sin(t)$, $\det \tilde{A}_2 = \sin(s)$. Note that the determinants of the matrices A_1, A_2 keep their values, if we replace some zero elements in the right side of blocks by arbitrary real numbers. Obviously, the determinant of the matrix $\tilde{A}_2$ is the linear function of variables $\{a_{j,n}, b_{j,n}\}_{j=n+1\ldots 2n+1}$. We collect the coefficients at the monomials $(c_{n+1})^k \cdot (s_{n+1})^{n+1-k}$, where $k = 0 \ldots n+1$, in the equation $\det \tilde{A}_2 = q_{n+1}(c_{n+1}, s_{n+1})$ and obtain the system of $n+1$ equations with $2n-2$ variables $\{a_{j,n}, b_{j,n}\}$. We do not show here that under the replacing of some elements of A_2 keeping its block-triangle form, this linear system would remain compatible. Its arbitrary solution $\{a_{j,n}, b_{j,n}\}$ (which was calculated using the package *MAPLE* in the examples with $n = 3, 4$ and 5 below) defines the desired matrix $\tilde{A}_2$.

Analogously, we obtain for $\tilde{A}_1$ the compatible system of $n+2$ equations with $2n+2$ variables $\{a_{j,1}, b_{j,1}\}_{j=2\ldots n}$. Its arbitrary solution (calculated using *MAPLE* in the examples with $n = 3, 4$ and 5 below) defines the desired matrix $\tilde{A}_1$.

So for odd n the matrix $Z_{n,n+1}(t,s)$ with the determinant $\cos(t-s)$ is constructed. The case of even n is analogous. $\square$

Note that the matrix $Y_{2n+1}(t) := Z_{n,n+1}(t,t)$ satisfies the Jacobi ODE

$$\ddot{Y}_{2n+1}(t) + R_{2n+1} \cdot Y_{2n+1}(t) = 0,$$

where the curvature matrix R_{2n+1} is diagonal with elements $\frac{1}{n^2}$ and $\frac{1}{(n+1)^2}$ on its diagonal. Hence $\delta = (\frac{n}{n+1})^2 \to \infty$ when $n \to \infty$.

EXAMPLE 2.12. For $n = 1$ the matrix $Z_{1,2}(t,s)$ and matrices $Y_3(t) = Z_{1,2}(t,t)$, $B_3(t) = Y_3(t)' \cdot (Y_3(t))^{-1}$ realize the example with $\delta = 0.25$ and correspond to the foliation by closed geodesics of some open domain in the complex projective plane CP^2

(2.34)

$$Z_{1,2} = \begin{vmatrix} \cos(t) & 2\sin(t) & 0 \\ 0 & \sin(\frac{s}{2}) & \cos(\frac{s}{2}) \\ \sin(\frac{s}{2}) & -\cos(\frac{s}{2}) & -\sin(\frac{s}{2}) \end{vmatrix}, \quad B_3 = \begin{vmatrix} 0 & -2\sin(\frac{t}{2}) & -2\cos(\frac{t}{2}) \\ -\frac{1}{2}\sin(\frac{t}{2}) & -\frac{1}{2}\sin(t) & -\frac{1}{2}\cos(t) \\ \frac{1}{2}\cos(\frac{t}{2}) & -\frac{1}{2}\cos(t) & \frac{1}{2}\sin(t) \end{vmatrix}$$

Consider some more examples, where for simplicity we denote by

$$C_n := \cos(\frac{t}{n}), \quad S_n := \sin(\frac{t}{n}).$$

EXAMPLE 2.13. The following matrix $Y_5(t) := Z_{2,3}(t,t)$ was derived using the package *DERIVE* independently from the proof of Lemma 2.10, it provides $\delta = 4/9 \approx 0.44$

$$Y_5(t) = \begin{pmatrix} 0 & S_3 & -C_3 & 0 & 0 \\ S_3 & C_3 & 0 & 3C_3 & 0 \\ C_3 & 3S_3 & 0 & S_3 & 0 \\ 0 & 0 & -2S_2 & C_2 & S_2 \\ 0 & 0 & 0 & S_2 & C_2 \end{pmatrix}.$$

The following matrices were constructed using *MAPLE* by the scheme in the proof of Lemma 2.10.

EXAMPLE 2.14. The following matrix $Y_7(t) := Z_{3,4}(t,t)$ provides $\delta = \frac{9}{16} \approx 0.56$

$$Y_7(t) = \begin{pmatrix} C_3 & 0 & 0 & -S_3 & 0 & 0 & 0 \\ S_3 & C_3 & 3S_3 & 0 & 0 & 0 & 0 \\ 3C_3 & S_3 & C_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_4 + S_4 & -2S_4 & 0 & 0 \\ 0 & 0 & 0 & C_4 & C_4 - S_4 & 0 & 2C_4 \\ 0 & 0 & 0 & 0 & 0 & C_4 + S_4 & 2S_4 \\ 0 & 0 & C_4 - S_4 & 0 & 0 & C_4 & C_4 - S_4 \end{pmatrix}.$$

EXAMPLE 2.15. The following matrix $Y_9(t) := Z_{4,5}(t,t)$ provides $\delta = \frac{16}{25} \approx 0.64$

$$Y_9(t) = \begin{pmatrix} C_5 & 0 & 0 & 0 & 0 & -S_5 & 0 & 0 & 0 \\ -\frac{1}{5}S_5 & C_5 & 0 & 0 & S_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_5 & S_5 & 0 & 0 & 0 & 0 & 0 \\ 15S_5 & C_5 & 5S_5 & C_5 & 5S_5 & 0 & 0 & 0 & 0 \\ -5C_5 & S_5 & 0 & S_5 & C_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_4+S_4 & -2S_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_4 & C_4-S_4 & 0 & 2C_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_4+S_4 & 2S_4 \\ 0 & 0 & 0 & 0 & S_4 - C_4 & 0 & 0 & C_4 & C_4-S_4 \end{pmatrix}$$

In view of the small size of the page we can not place here the matrix $Y_{11}(t) := Z_{5,6}(t,t)$ of the order 11×11, which realizes the case $\delta = \frac{25}{36} \approx 0.7$, etc. We also do not give here the matrices $B_{2n+1}(t) = Y_{2n+1}(t)' \cdot (Y_{2n+1}(t))^{-1}$, where $n = 2, 3, 4 \ldots$.

Author conjectures that the above matrix solutions of the Riccati and Jacobi equations can be deformed (keeping the dimension of matrix) in the direction of increasing of the curvature pinching. The deformation $B(t)+X(t)$ of the solution $B(t)$ of the Riccati equation $B' + B^2 + R(t) = 0$ satisfies the Riccati equation $(B+X)' + (B+X)^2 + (R+D) = 0$ with curvature matrix $R(t)+D(t)$. Thus we have the equality $X' + BX + XB + X^2 = -D$.
Since $D(t)$ is a symmetric matrix, then the "admissible" variation $X(t)$ satisfies some conditions, which we do not write here. For example, consider $n = 3$ and the solution (2.34) of the Riccati equation with diagonal matrix $R_3 = [1, \frac{1}{4}, \frac{1}{4}]$ and assume

$$X(t) = \begin{pmatrix} x_1(t) & x_2(t) & x_3(t) \\ x_2(t)+x_4(t) & x_5(t) & x_6(t) \\ x_3(t)+x_7(t) & x_6(t)+x_8(t) & x_9(t) \end{pmatrix}.$$

We obtained using *MAPLE* the following nonlinear system of ODE of "admissible" variation $\{x_i(t)\}$, $(1 \le i \le 9)$ (providing the symmetry conditions for $D(t)$):

(2.35)

$$\begin{cases} x_4' = -\frac{5}{2}\sin(\frac{t}{2})x_5 - \frac{5}{2}\cos(\frac{t}{2})x_6 - 2\cos(\frac{t}{2})x_8 - \frac{5}{2}\sin(\frac{t}{2})x_1 + \\ \qquad x_3x_8 + \frac{1}{2}\sin(t)x_4 + \frac{1}{2}\cos(t)x_7 - x_4x_1 - x_5x_4 - x_6x_7, \\ x_7' = -\frac{5}{2}\sin(\frac{t}{2})x_6 - \frac{5}{2}\cos(\frac{t}{2})x_9 - \frac{5}{2}\cos(\frac{t}{2})x_1 + \frac{1}{2}\cos(t)x_4 - \\ \qquad \frac{1}{2}\sin(t)x_7 - \frac{1}{2}\sin(\frac{t}{2})x_8 - x_7x_1 - x_6x_4 - x_8x_2 - x_8x_4 - x_9x_7, \\ x_8' = \frac{5}{2}\sin(\frac{t}{2})x_3 - \frac{5}{2}\cos(\frac{t}{2})x_2 - 2\cos(\frac{t}{2})x_4 + x_4x_3 + 2\sin(\frac{t}{2})x_7 - \\ \qquad x_7x_2 - x_8x_5 - x_9x_8. \end{cases}$$

Suppose that the three functions are constant $x_4(t) = x_4$, $x_7(t) = x_7$, $x_8(t) = x_8$. Then the system (2.35) is linear with respect to functions $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_5(t)$, $x_6(t)$, $x_9(t)$ and its solution (obtained with *MAPLE*) is the following:

$$x_2(t) = \frac{1}{2}((-5x_9(t) - 5x_1(t))\cos(\frac{t}{2}) + \cos(t)x_4 + (-5x_6(t) - x_8)\sin(\frac{t}{2}) - \\ 2x_1(t)x_7 - 2x_6(t)x_4 - 2x_8x_4 - 2x_9(t)x_7 - \sin(t)x_7)/x_8,$$

$$\begin{aligned}
x_3(t) =&((-20x_4^2x_6(t) - 4x_4^2x_8 - 20x_6(t)x_8^2 + 10x_4^2 - 16x_8^3 - 40x_4x_1(t)x_7 - \\
&40x_4x_9(t)x_7)\cos(\frac{t}{2}) + (-25x_4x_9(t) + 25x_7x_8 + 4x_4^2x_7 + 4x_7x_8^2 - \\
&25x_1(t)x_4 + 25x_6(t)x_7)\cos(t) + (-\frac{25}{2}x_8 - 10x_7^2 - 10x_4^2 - \\
&\frac{125}{2}x_6(t)) \cdot \sin(\frac{t}{2})\sin(t) + \frac{25}{2}x_4\cos(t)\sin(t) + (-10x_4x_7 - \\
&40x_4x_6(t)x_7 - 20x_7^2x_1(t) + 20x_9(t)x_8^2 - 20x_1(t)x_8^2 - 20x_7^2x_9(t) - \\
&\frac{125}{2}x_1(t) - \frac{125}{2}x_9(t) - 40x_7x_8x_4)\sin(\frac{t}{2}) + (-\frac{125}{2}x_1(t) - \\
&\frac{125}{2}x_9(t))\cos(t)\sin(\frac{t}{2}) + 8x_4x_9(t)x_8^2 - 8x_4x_7^2x_9(t) - 8x_4x_7^2x_1(t) - \\
&8x_1(t)x_4x_8^2 - 8x_6(t)x_7x_8^2 - 25x_1(t)x_4 - 8x_4^2x_7x_6(t) - \\
&25x_4x_9(t) - 8x_4^2x_7x_8 - 25x_7x_8 - 25x_6(t)x_7 + (-50x_1(t)x_7 - 10x_8x_4 + \\
&4x_4x_8^2 - 4x_4x_7^2 - 50x_9(t)x_7 - 50x_6(t)x_4)\sin(t) - \frac{25}{2}x_7\sin(t)^2)/ \\
&(x_8(-25\cos(t) + 40\sin(\frac{t}{2})x_4 + 8x_4^2 - 8x_8^2 + 25)).
\end{aligned}$$

$$\begin{aligned}
x_5(t) =&((-10x_8x_4 + 5x_4 - 20x_1(t)x_7 - 20x_6(t)x_4 - 20x_9(t)x_7)\cos(\frac{t}{2}) + \\
&(-\frac{25}{2}x_9(t) + 4x_4x_7)\cos(t) + (-5x_7 - 10x_7x_8 - 20x_1(t)x_4 - \\
&20x_6(t)x_7)\sin(\frac{t}{2}) - \frac{25}{2}x_9(t) - 25x_1(t) - 4x_1(t)x_4^2 - 4x_7^2x_1(t) - \\
&4x_7^2x_9(t) - 4x_7x_8x_4 + 4x_9(t)x_8^2 - 8x_4x_6(t)x_7 + (-\frac{25}{2}x_8 - 25x_6(t) + \\
&2x_4^2 - 2x_7^2)\sin(t))/(-\frac{25}{2}\cos(t) + 20\sin(\frac{t}{2})x_4 + 4x_4^2 - 4x_8^2 + \frac{25}{2}),
\end{aligned}$$

Author hopes that there exists the control by admissible functions $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_5(t)$, $x_6(t)$, $x_9(t)$ and constants x_4, x_7, x_8 for some $\delta > \frac{1}{4}$.

FIGURE 1. Foliation

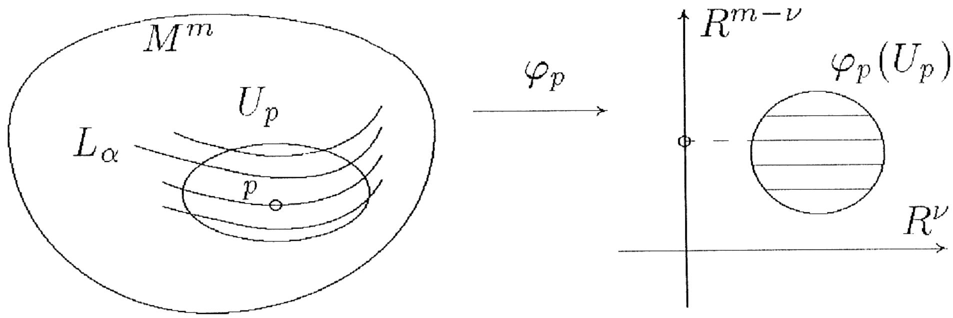


FIGURE 2. Behaviour of geodesics (Jacobi fields) for K_{mix} = const

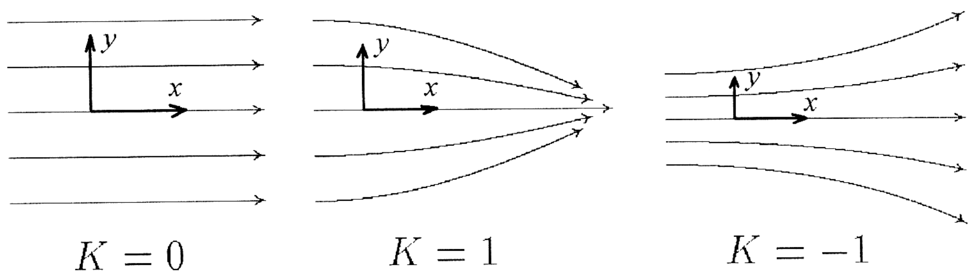


FIGURE 3. Behaviour of L-parallel field $y(t)$ along "extremal" geodesic

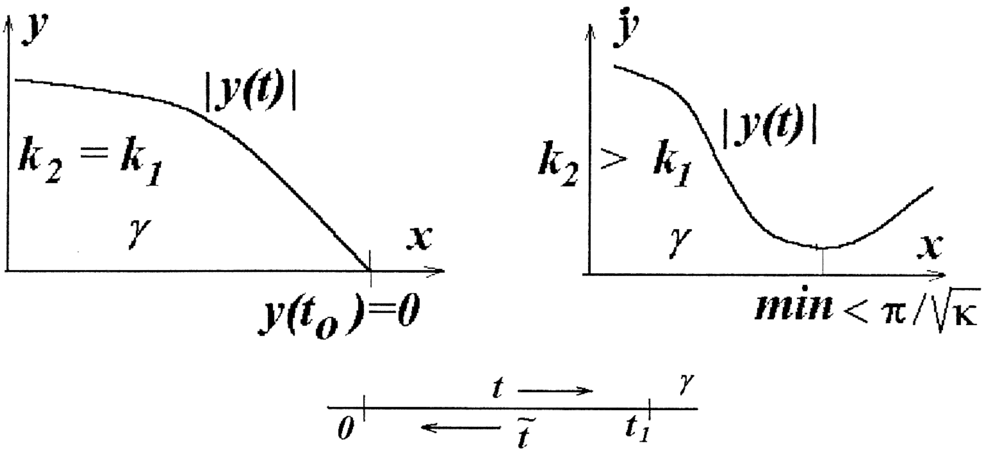
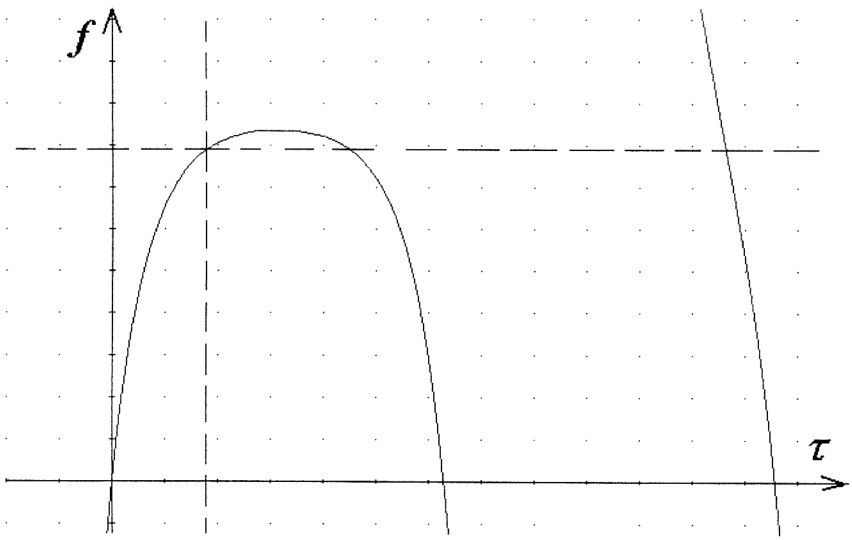


FIGURE 4. Graph of the function $f(\tau) = \frac{2\sin(\tau)}{\frac{\pi}{4}+(\frac{\pi}{2}-\tau)\frac{\cos(\tau)}{2}+(3+\sin(\tau))\sin(\tau)}$



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