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ON STANFORD'S QUESTIONS CONCERNING SINGULAR KNOTS

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Abstract. For each chord diagram D we define the collection $\mathcal{L}_D$ of knot diagrams, each respecting D, which represents only a finite set of knots. We construct an algorithm, defined by a sequence of crossing changes and local isotopies(the combinations of R1–R5 moves), for reducing any knot diagram K with the chord diagram D to the one of the collection $\mathcal{L}_D$ without increasing the number of crossings in all intermediate knot diagrams. The last is inspired by Stanford's Questions 4.3 and 4.4 of [**7**] which appeared in his analysis of a new combinatorial algorithm for computing Vassiliev knot invariants. We also give an upper bound for the number $|\mathcal{L}_D|$ of different knots in the collection $\mathcal{L}_D$, depending on the order n of the chord diagram D, and discuss the possibilities to extend some of our results for knots to the case of spatial graphs.

Introduction. The well-known algorithm for computing the Vassiliev knot invariants of knots is based on the so-called actuality tables [**9**]. An actuality table T for computing a Vassiliev invariant v of finite order n contains information about the values of v on some distinguished singular knots of orders $\leq n$, chosen in such a way, that for each chord diagram D of order $k, 0 \leq k \leq n$, there exists exactly one singular knot in T respecting D. The values of v on the knots in an actuality table are to be chosen certainly not arbitrarily, but in some consistent way satisfying the 4T and 1T relations (4T relations in the case of framed knots)[**7**]. It turns out that this information is sufficient to compute the value of v on an arbitrary knot (singular knot). In [**3**] Birman and Lin improved the method of actuality tables, originally suggested by Vassiliev [**9**], and outlined how one can determine inductively the space

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of the Vassiliev invariants of order n by solving the corresponding systems of linear equations at each inductive step. This method for computation of the Vassiliev invariants of order $\leq n$ is rather too complicated for practical goals. Stanford [**7**] suggested another method to determine the vector space (actually the abelian group) of Vassiliev invariants of order $\leq n$. Actually he has refined the method of actuality tables and showed that in order to determine correctly any Vassiliev invariant v of order n, it suffices to define its values on the distinguished knots from the actuality table so that v satisfies one 4T and one 1T relation for each chord diagram D of order $\leq n$. His algorithm for the computation of the values of any Vassiliev invariant v on any knot diagram K, respecting a chord diagram D, is based on the procedure of reducing K to a canonical knot which respects the same chord diagram D. The reduction is performed in two steps. At the first step one chooses an ordered spanning tree T of the graph of K and pushes it towards the initial vertex of T, using R1–R5 moves. At the second step the only crossing changes are applied to the resulting knot diagram. At the first step of reduction of the diagram the number of crossings in the intermediate knot diagrams can increase. This leads to some complications in the computing process. Therefore, Stanford [**7**] has raised the following two questions.

Question 4.3. Given a chord diagram D, does there exist a finite set of knots $K_1, \ldots, K_p$, each respecting chord diagram D such that any knot diagram respecting D can be made isotopic to one of the K_i by crossing changes only?

Question 4.4. Given a chord diagram D, does there exist a finite set of knot diagrams $K_1, \ldots, K_p$, each respecting D, such that any knot diagram K with the same chord diagram D can be made isotopic to one of the K_i by crossing changes and local R moves such that no intermediate knot diagram has more crossings than K?

An affirmative answer to any of the above two questions would allow us, in principle, to improve the effectiveness of the algorithm for computing the values of Vassiliev invariants on arbitrary knots. Another problem is how one can determine correctly the values of a Vassiliev invariant v of order n on distinguished knot diagrams in practice. One can start from any weight system w of order n and try to construct a Vassiliev invariant v of order n so that the weight system associated with v will be w. This can be done by using the so-called procedure of "integration" of the given weight system or, equivalently, by using the construction of the universal Vassiliev invariant Z (see [**4**] for analitical approach and [**2**] for more combinatorial approach to constructing the universal Vassiliev invariant of knots and links in $\mathbf{R}^3$). Even in the low orders the evaluation of Z is a problem of high computational complexity. One can also use the procedure of the so-called half-integration – going from an invariant, whose values on the singular knots with k singularities are already determined, to defining its values on $(k-1)$-singular knots. Until now this method turns out to be effective only in particular cases (for even weight systems) and does not work in general case ([**10**]). In this paper we will not touch these problems. Our goal is only to show how one can simplify the process of computing the values of any Vassiliev invariant v on knots, via the method of actuality tables, provided the values of v are already defined on the distinguished knot diagrams in a correct way.

In [**6**] we state a result, without giving any proof, which can be considered as an affirmative answer to some version of Question 4.4. This leads to a modified version of the algorithm for computing the Vassiliev invariants, based on actuality tables. Here we present the complete proof of this statement (Theorem 2.1). Our method of transforming any knot diagram K with the chord diagram D consists in reducing K, by application to K a sequence of isotopies and crossing changes, to one of the special class $\mathcal{L}_D$ of knot diagrams, which represents only a finite number of knot types. All intermediate knot diagrams which appeared in such process have no more crossings than the input knot diagram K. The proposed algorithm for reducing the knot diagrams does not already work, if one requires, in addition, that each local isotopy arising in the reduction process is to be one of R1–R5 moves (see below). Moreover, we show that there are infinitely many examples of knot diagrams, each respecting the same chord diagram D, for which a positive answer to Question 4.4 would yield actually a positive answer to Question 4.3 inside the given class of knot diagrams.

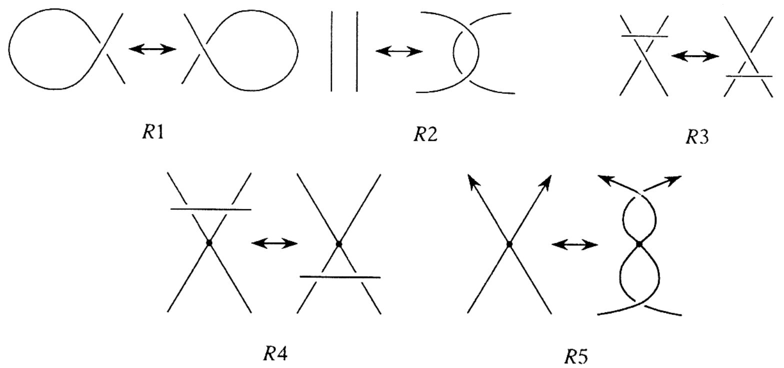
On the other hand, we show that, in general, there is no adequate positive answer to Question 4.3, having given the corresponding examples of knot diagrams. We also give some estimates for the number of different knot types represented by the knot diagrams inside each class $\mathcal{L}_D$. Finally, we notice that the suggested method of reducing any knot diagram to the one of the special class $\mathcal{L}_D$ can be extended to the case of spatial graphs in $\mathbf{R}^3$ [**5**]. We also raise some open questions.

The structure of the paper is the following. In the first section we give briefly the information about Vassiliev invariants, singular knots, the algorithm for computation Vassiliev invariants, based on actuality tables and so on. In the second section we state and prove the main results of the paper. In the third section we give some estimates for the number of distinct knot types in $\mathcal{L}_D$ and discuss the possibility of extending some of our results to the case of spatial graphs.

1. Preliminaries. It is known [**1**] that any knot invariant of order n can be computed in polynomial time and is polynomially bounded, the polynomials being of degree n and being considered as functions of the number of crossings in an input knot diagram. Let v be a Vassiliev knot invariant of order n. The well-known algorithm for computing the values v on any (singular) knot K, suggested by Vassiliev and improved by Birman and Lin [**3**], uses essentially the so-called actuality table for v. Let us describe briefly this algorithm. First recall some notions.

By a singular knot we mean an immersion of the oriented circle in $\mathbf{R}^3$ which is allowed to have a finite number of transverse self-intersections. The singular knots in $\mathbf{R}^3$ are considered up to the equivalence relation determined by the "rigid-vertex" isotopy [**3, 7**]. We shall work rather with knot diagrams than with knots (singular knots) in $\mathbf{R}^3$. A knot diagram K is an immersion $i\colon S^1 \to \mathbf{R}^2$ which has only transverse double points as its singularities. Moreover, some of double points are marked as vertices and a choice of over/under is made at each of the other double points, called the crossings. Any knot diagram, as generic curve in $\mathbf{R}^2$, is oriented. Knot diagrams, usually, are considered up to equivalence relation, the isotopy, determined by local moves R1–R5 [**7**] (see Figure 1).

FIGURE 1



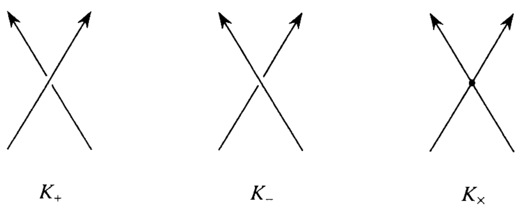
This equivalence relation for knot diagrams corresponds to the "rigid-vertex" isotopy of knots in $\mathbf{R}^3$. We define the order of a knot diagram K as the number of singular points in K. To each knot diagram K of order n there corresponds a chord diagram D_K with n chords and with oriented circle C (we shall say also that K respects D_K). The chord diagram D_K describes the cyclic order in which the vertices of K are encountered, when traveling along the oriented knot diagram K [**3**]. The number of chords in any chord diagram D is called the order of D. It follows from the definition of the chord diagram corresponding to a knot diagram that the equivalent knot diagrams have the same chord diagrams. Notice that any knot diagram K (or, correspondently, a singular knot in $\mathbf{R}^3$) can be considered as an oriented 4-valent topological graph (1-CW-complex) G_K, the vertices of G_K being the vertices of K and the oriented edges of G_K being the oriented paths of K joining its vertices. G_K is called the underlying graph of the knot diagram K or of the chord diagram of K. The local subsegments of a knot diagram K are called the strands of K. Since the knot diagrams we regard are oriented, each strand of any knot diagram is oriented in the natural way. Each vertex u of a knot diagram K has two strands, say x^+ and y^+, coming out u, and two strands x^- and y^-, going to u, so that, when traveling along K, we meet them in the order $\ldots, x^-, x^+, \ldots, y^-, y^+, \ldots$. Fix the couterclockwise orientation on the plane $\mathbf{R}^2$. This gives one of the two following orientations (the cyclic orders of the strands) at u: $x^-y^+x^+y^-$ or $x^-y^-x^+y^+$. Note that the local move R5, applied to a knot diagram K in the neighbouhood of u, changes the orientation at the vertex u.

Fix a field F. Denote by $\mathcal{K}$ the set of singular knots. A function $v\colon \mathcal{K} \to F$ is said to be a Vassiliev invariant, if it is isotopy invariant of knots (knot diagrams) and satisfies the following two axioms:

A1. $v(K_+) - v(K_-) = v(K_\times)$; here K_+, K_- and $K_\times$ are three knot diagrams, the same outside some open disc, inside of which they look in standard way, as over-crossing, under-crossing and singularity of the knot diagram (see Figure 2);

A2. There exists an integer n such that $v(K) = 0$ for any knot diagram K with more than n singularities; The smallest such non-negative integer n is called the order of v.

FIGURE 2



By a crossing change in a knot diagram K we mean a move which changes any over-crossing of K for under-crossing or conversely.

Let v be a Vassiliev knot invariant of order $\leq n$. Suppose that for each chord diagram D of order $\leq n$ we fix a knot diagram K_D respecting D, and suppose that we know all the values $v(K_D)$. This information is called an actuality table for v. Vassiliev's algorithm for computing v on an arbitrary knot diagram K looks as follows:

Find the knot diagram K_D in the actuality table with the same chord diagram D as K. We can express, using the axiom A.1, the value $v(K)-v(K_D)$ as the signed sum of the values of v on the knot diagrams encounterd in the sequence of crossing changes and local moves to pass from K to K_D. Since these all have one more vertices than K, the process finishes, thanks to the axiom A.2.

Stanford [**7**] has given a more explicit version of Vassiliev's algorithm. This algorithm, when applied to any knot diagram K respecting a given chord diagram D, always yields, via local R moves and crossing changes, the same up to isotopy output diagram. This gives the canonical way for the choice of the distinguished knot diagram K_D for every chord diagram D. Moreover, the last circumstance allows to compute inductively, in the way defined above, the value $v(K)$ for any knot diagram K, if an actuality table is given. However, the intermidiate knot diagrams arising in the process, when going from K to K_D, can increase the number of crossings. This makes the process of computing the values $v(K)$ somewhat complicated. To avoid this Stanford in [**7**] has raised Question 4.4 formulated above.

Here we give an affirmative answer to Question 4.4, formulated in a slightly weaker form. Its modified version allows at each intermediate step of reduction, besides crossing changes, the application of the so-called local isotopies each of which is a combination of R1-R5 moves inside some region of the plane. Regarding the complexity of computing the Vassiliev invariants, the latter looks as being not too considerable weakening, because Vassiliev invariants are the isotopy invariants of knots.

To establish our result we proceed as follows. First we define, for every chord diagram D of order n, a special class $\mathcal{L}_D$ of knot diagrams L, each respecting D and satisfying the following properties:

a) no edge e of L, $L \in \mathcal{L}_D$, has in $\mathbf{R}^2$ self-intersections (for exception, maybe, its initial and terminal vertices);
b) for any pair of distinct edges e and f in L, the edge e intersects the internal part of f , i.e. f with its endpoints removed, at most at one point, i.e. e and f give at most one crossing of L;
c) L respects the chord diagram D;
d) the edge set of the graph G_L admits a colouring c by the numbers $1, \ldots, 2n$, so that if $c(e) > c(f)$, then e is over f at each crossing of L formed by these two edges.

Here under a colouring of a knot diagram K of order n we mean a function $c, c\colon E(K) \to \{1, \ldots, 2n\}$, defined on the set $E(K)$ of oriented edges of K, which satisfies the following condition: if e follows just after f, where $e, f \in$

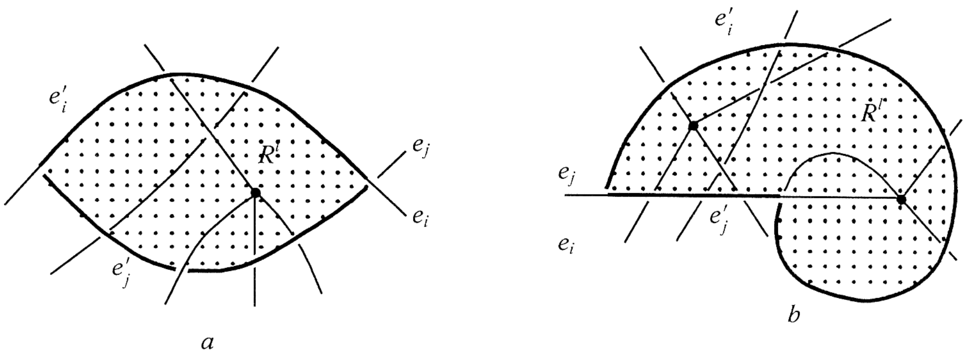
$E(K)$, then $c(e) = c(f) + 1$ $(mod\, 2n)$. Notice that all knot diagrams $L \in \mathcal{L}_D$ have the same underlying oriented graphs G_L which can differ from each other only by the orientations at the vertices. A colouring c of the edges of K is called perfect if it satisfies the above condition d). Next, we shall show that each knot diagram K with the chord diagram D can be transformed to one of the family $\mathcal{L}_D$ by applying to K a sequence of crossing changes and the so-called local isotopies, each being a composition of R1–R4 moves, so that no intermediate knot diagram has more crossings than K. At the second step of our proof, with each coloured knot diagram $L \in \mathcal{L}_D$ we associate a coloured rotation graph H_L, adding to L new vertices (the crossings of L) and subdividing the edges of L by the crossings of L. The rotation system on H_L is determined by the given natural embedding of its underlying graph into the plane. All the faces of this embedding are 2-cells, besides one, which is outer. Notice that the isotopy of $\mathbf{R}^2$ produces on the set of knot diagrams an equivalence relation more strong than one produced by local R moves (we shall call it the strong equivalence relation on (colored)knot diagrams). It turns out that any two strongly equivalent coloured knot diagrams $K, L \in \mathcal{L}_D$ have the equivalent (in the usual sense) coloured rotation graphs H_K, H_L. Finally, we note that for a given chord diagram D there exists only a finite number of equivalence classes of coloured rotation graphs associated with the elements of $\mathcal{L}_D$.

2. Proof of the main results. Let e be an edge of a knot diagram K. The edge e can be considered as the image of an arc $J = [a, b] \subset S^1$ under an immersion φ, $e = \varphi(J)$, where $\varphi(S^1) = K$. We shall say that $l = \varphi[c, d] \subset J$, is an inner loop of e, if $[c, d] \subset (a, b)$ and $\varphi(c) = \varphi(d)$. An inner loop l of e is called maximal, if there is no inner loop l' of e, such that $l \subset l'$ and $l' \neq l$. The operation of removing an inner loop l from the edge e consists in replacing the edge $e' = \varphi(J)$ by the edge $e' = \varphi(J - (c, d))$, where $l = \varphi[c, d]$. Clearly, given an edge e of a knot diagram K, one can choose a collection $L_e = \{l_1, \dots, l_k\}$ of maximal loops of e, where $l_i = \varphi[a_i, b_i]$ and $[a_i, b_i] \cap [a_j, b_j] = \emptyset$ if $i \neq j$, such that after removing all them from e, the resulting edge e' of the new knot diagram will be either a simple path, or a simple closed curve in $\mathbf{R}^2$. We shall call such L_e the complete collection of loops for e. Notice that for some edges f of K the collection L_f can be empty.

Before proving the main result we need to introduce some new notion. Suppose the edges of K are enumerated, say $e_1, e_2, \dots, e_{2n}$. Let e_i and e_j be any two edges of the knot diagram K without inner loops and let u and u' be the vertices of K, incident to e_i, and let v and v' be the vertices of K, incident to e_j (the possibility that some of the vertices u, u', v, v' coincide is not excluded). Let $w_1, \dots, w_k$ be all the crossings of K which are the common points of curves e_i and e_j. Suppose $k > 1$. The set $W_{e_i,e_j} = \{w_1, \dots, w_k\}$

decomposes the edge e_i into the arcs $f_1, \dots, f_{k+1}$ and decomposes the edge e_j into the arcs $f'_1, \dots, f'_{k+1}$. Choose among $f_1, \dots, f_{k+1}$ the internal arcs of e_i, i.e. the arcs which do not contain the points v and v'. Without loss of generality, we may assume that $f_1, \dots, f_{k-1}$ are the internal arcs of e_i and $f'_1, \dots, f'_{k-1}$ are the internal arcs of e_j. The internal arcs of e_i and e_j bound in $\mathbf{R}^2$ a finite number of compact regions, say $R^1, \dots, R^p$, so that the oriented circuit of the boundary of each region R^l is an alterning sequence of some internal arcs of e_i and e_j, $l = 1, \dots, p$. The interior of each R^l in $\mathbf{R}^2$ is an open 2-cell which does not contain any poins of internal arcs of e_i and e_j. The set $F_i = \bigcup_{l=1}^{p} R^l$ is called the covering region of the isotopy of the edge e_i with respect to the edge e_j (or the covering region for the pair $\{e_i, e_j\}$) and the regions R^l are called the components of the covering region. Denote by $\mathcal{F}_{i,j}$ the set of all components of the covering region $F_{i,j}$ for the pair $\{e_i, e_j\}$ of K. Note that for each i and j, $i \neq j$, we have $\mathcal{F}_{i,j} = \mathcal{F}_{j,i}$ and $F_{i,j} = F_{j,i}$. There are among $R^l \in \mathcal{F}_{i,j}$ the components which are bounded by only two arcs, one being an internal arc of e_i and another being an internal arc of e_j. Set the index of each such component be equal to 1. Any component $R^l \in \mathcal{F}_{i,j}$ of index 1 bounded by the two arcs e'_i and e'_j, $e'_i \subset e_i, e'_j \subset e_j$, has one of two types, a and b, indicated in Figure 3a and 3b respectively.

FIGURE 3



A component $R^k \in \mathcal{F}_{i,j}$ of index 1 is called minimal, if there is no other component R' of index 1 in any of the collections $\mathcal{F}_{i',j'}$ so that $R' \subset R^k$. Let $R^k \in \mathcal{F}_{i,j}$ be any component of index 1. Let $\partial R^k = l_i \cup l_j$ be the decomposition of the boundary of R^k into two arcs l_i and l_j, where $l_i \subset e_i$ and $l_j \subset e_j$. Denote by s_i and s_j the numbers of double points (crossings) of K which lie on l_i and l_j respectively. The number $s_{ij}(R^k) = s_i - s_j$ will be called the defect of the component $R^k \in \mathcal{F}_{i,j}$. Notice that the defect of the same component R^k in the collection $\mathcal{F}_{j,i}$ is $s_{ji}(R^k) = s_j - s_i = -s_{ij}(R^k)$. If $W_{e_i,e_j} = \emptyset$ or $|W_{e_i,e_j}| = 1$, we set $F_{i,j} = \emptyset$. In this case the collection $\mathcal{F}_{i,j}$ is the empty set.

A component R of the covering region for a pair $\{e_i, e_j\}$ of the edges of K is called essential, if its interior in $\mathbf{R}^2$ contains the vertices of K. Otherwise R is called non-essential.

THEOREM 2.1. *Each knot diagram K respecting the chord diagram D of order n can be reduced by crossing changes and combinations of local moves R1–R4 to one of the collection $\mathcal{L}_D$, so that no intermediate knot diagram has more crossings than K.*

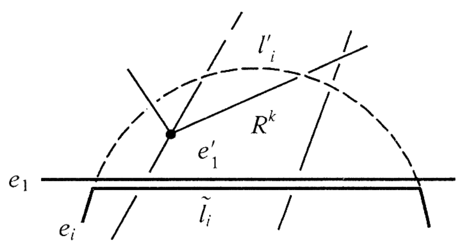
PROOF. Let K be any knot diagram which respects D. We can assume, without loss of generality, that each edge of K has no inner loop. Indeed, if some edges of K have inner loops then, using the crossing changes, we can "lift" all the maximal loops of the complete collections corresponding to such the edges over the knot diagram, make them unknotted and unlinked and remove them from the new knot diagram K_1. The last procedure can be performed consequently, using local isotopies (some combinations of R1–R4 moves) applied to the intermediate knot diagrams. Every such local isotopy deletes one inner loop of the complete collection for some edge of K, so that the number of crossings in the intermediate knot diagrams can only decrease (see below for the detailed description of such type moves applied, however, in somewhat different situation). Clearly, the resulting knot diagram respects the chord diagram D.

Let K be any knot diagram which has no inner loops and respects the chord diagram D and let $u_1, \ldots, u_n$ be the vertices of K. Choose an oriented edge of G_K coming from the vertex u_1 and denote it by e_1. Let us enumerate all the edges of G_K as they appear, when traveling along K, starting from $e_1 : e_1, \ldots, e_{2n}$. This is actually a colouring c of the edges of K with $c(e_i) = i, i = 1, \ldots, 2n$. Apply to K crossing changes to make of K a new knot diagram K' which respects D and satisfies the following property. For any distinct numbers i, j the following holds. If the edge e_i intersects the internal part of the edge e_j at the double points (crossings) $p_1, \ldots, p_m$ of K' then e_i is over e_j at each of such crossings $p_s, s = 1, \ldots, m$. Notice that K' inherits the colouring of its edges from the colouring of the corresponding edges of K. With this colouring, K' will be always a perfectly coloured diagram. If K' satisfies additionally the condition b) of the definition of the collection $\mathcal{L}_D$, then $K' \in \mathcal{L}_D$ and we have nothing to do with K'. Suppose K' does not satisfy the property b). This means that at least one of the collections $\mathcal{F}_{s,t}, s, t \leq 2n$, is non-empty set. Now we want to apply to K' a sequence T_1 of local moves R1–R4 (a combination of local moves R1–R4) and crossing changes to transform K' into a new knot diagram $\tilde{K}$ which respects D and has fewer crossings than K'. Let us consider in the collections $\mathcal{F}_{i,j}$ with $\mathcal{F}_{i,j} \neq \emptyset$ all the minimal components of index 1 (here we should point out that at least one such component exists). Choose among them a component with maximal value of $|s_{ij}|$, the module of defect, and denote it by R^k. We may assume, for instance, that $R^k \in \mathcal{F}_{1i}$ and $s = s_{1i} < 0$. First consider the case, when R^k is a component of type a. Let $\partial R^k = l_1 \cup l_i$ be the corresponding decomposition of the boundary of R^k into two arcs l_1 and l_i, where l_1 is an internal arc of e_1 and l_i is an internal arc of e_i, and $l_1 \cap l_i = \{w_1, w_2\}$, where w_1 and w_2 are two crossings of K'. Consider a bigger arc l'_i in e_i which is the closed ε-neighbourhood of l_i in e_i with small ε, so that l'_i contains the same double points of K' as l_i. Let c_1 and c_2 be the endpoints of the arc l'_i. The first step of reducing the knot diagram K' consists in replacing the arc l'_i of e_i by an arc $\tilde{l}_i$ which is "parallel" to e_1 and satisfies the following properties:

a): $\tilde{l}_i$ is contained in a δ-neighbourhood of l_1 for some small δ;
b): $\tilde{l}_i$ has the same endpoints as l'_i;
c): $\tilde{l}_i \cap e_1 = \emptyset$.

Therefore, in the new knot diagram all the edges $\tilde{e}_j, j \neq i$, are the same as in K', while $\tilde{e}_i = (e_i - l'_i) \cup \tilde{l}_i$. This procedure can be performed consecutively. First we apply crossing changes to the double points of K' lying on l'_i in order to lift the arc l'_i over the remaining part of K'. After that we apply to the new knot diagram K'_1 in appropriate neighbourhood $\mathcal{N}$ of R^k in $\mathbf{R}^2$ a sequence of R1–R4 moves (a combination of R1–R4 moves) in order to push the arc l'_i along R^k out the arc l_1 (see Figure 4).

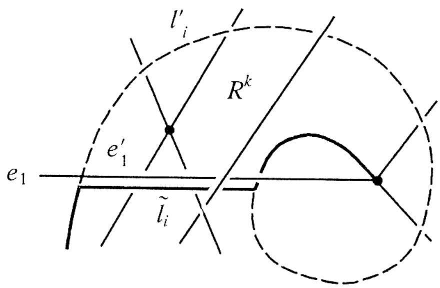
FIGURE 4



In the resulting knot diagram K'_2 the deformed arc $\tilde{l}_i$ has the same endpoins as l'_i. Finally, we apply to the double points of K'_2 lying on $\tilde{l}_i$ the crossings changes, if needed, in order to preserve the levels of the edges of the resulting knot diagram $\tilde{K}$ according to their colours, where the colouring of the edges of $\tilde{K}$ is inherited from the colouring of K'. Notice that if K' has d crossings, then the intermediate knot diagrams K_1', K_2' and $\tilde{K}$ have, respectively, d, $d-s_{i,1}-2$ and $d-s_{i,1}-2$ crossings. It is easy to see that if R^k is non-essential, we can accomplish all the above elementary local moves R1–R4, when going from K_1' to K_2', without increasing, with respect to K', the numbers of crossings in all intermediate knot diagrams. If R^k is essential, the last remark, in general, is not true (see Figure 7 and the discussion below).

The case when R^k is of type b can be regarded in the similar way as before. First, we apply to K' in the neibourhood of R^k, if needed, the crossing changes. Then we apply to the resulting knot diagram K_1' in some neighbourhood $\mathcal{N}$ of R^k an appropriate combination of R1–R4 moves (local isotopy), so that the new knot diagram K_2' has no more crossings than K_1'. In the neighbourhood $\mathcal{N}$ the knot diagram K_2' looks like in Figure 5.

FIGURE 5



Finally, we apply to K_2' the crossing changes, if needed, as before. The colouring of the edges of the obtained knot diagram $\tilde{K}$ is inherited from the colouring of K'. Notice that the number of crossings in the resulting knot diagram $\tilde{K}$ is less than in K' by $2+s_{i,1}$, when R^k is of type a, and by $1+s_{i,1}$, when R^k is of type b, where $s_{i,1}\geq 0$. It is clear that $\tilde{K}$ respects the chord diagram D. Notice also that for the new knot diagram $\tilde{K}$, all the covering regions $\tilde{F}_{j,1}, j\neq i$, and the corresponding collections $\tilde{\mathcal{F}}_{j,1}, j\neq i$, remain the same as for K', while $|\tilde{\mathcal{F}}_{i,1}| = |\mathcal{F}_{i,1}|-2$ or $|\tilde{\mathcal{F}}_{i,1}| = |\mathcal{F}_{i,j}|-1$. We now proceed by induction on the number of crossings in the input knot diagram K' (or the total number of components of the covering regions for K'). Proceeding at each inductive step in such a way as before, we may achieve that for the resulting knot diagram L each collection $\mathcal{F}_{i,j}$ of components of the covering region $F_{i,j}$ will be the empty set, $i,j=1,\ldots,2n$. Otherwise, let us consider in L a non-empty collection $\mathcal{F}_{i',j'}$ of components. We may choose in $\mathcal{F}_{i',j'}$ a minimal component R' of index 1 with the maximal defect and apply to it the reducing procedure described above, with respect to R', after which we shall obtain a knot diagram with fewer crossings than in L. Thus, the final knot diagram L respects D and satisfies the property b) of the definition of the collection $\mathcal{L}_D$. The colouring of the edges of L is inherited from the colouring of the previous intermediate knot diagrams which appeared in the process of reduction. Our choice of the crossing changes at each inductive step assures that the colourings of the intermediate knot diagrams are perfect. Finally, we note that the suggested procedure of reducing the knot diagrams does not lead to the appearence of any inner loops in the intermediate knot diagrams. Thus $L \in \mathcal{L}_D$. This completes the proof. □

REMARK 2.1. Notice that the equivalence (isotopy) class of the resulting knot diagram L depends, in general, on the choice of a minimal component of the covering region R^k at each inductive step of the reduction process and depends also on the colouring of edges of the input knot diagram K. We have also to point out that our manipulations with knot diagrams, performed in the course of the reduction, differ from the procedure of reducing the knot diagrams, provided by Stanford's algorithm.

REMARK 2.2. Notice that the graph of any knot diagram of order n has exactly $2n$ edges. Theorem 2.1 now implies that any knot diagram $L \in \mathcal{L}_D$ has no more than $C_{2n}^2 = n(2n-1)$ crossings.

For a given chord diagram D denote by $\mathcal{C}_D$ the subcollection of the collection $\mathcal{L}_D$ consisting of coloured knot diagrams with only positively oriented vertices.

THEOREM 2.2. *Let K be any coloured knot diagram having* $\operatorname{cr}(K)$ *crossings and respecting a chord diagram D of order $n > 0$. Then K can be reduced by sequence of crossing changes and combinations of moves R1–R5 (local isotopies) to one of the collection $\mathcal{C}_D$, so that the number of crossings of each intermediate diagram is no more than* $\max\{\operatorname{cr}(K), n(2n-1)+2\}$.

PROOF. Let K be any coloured knot diagram respecting a chord diagram D of order n and having $\operatorname{cr}(K)$ crossings. Let $u_{i_1}, \dots, u_{i_k}, k \leq n,$, be all the vertices of K having the negative orientations (with respect to the given colouring of K). Apply to K the reducing algorithm given by the proof of Theorem 2.1 to obtain a coloured knot diagram $K' \in \mathcal{L}_D$. Since the reduction process given by the algorithm does not change the colouring of the graph of K and does not use R5 move, the orientations at all vertices of K' are the same as the ones of K. Thus u_{i_1} is a negatively oriented vertex of K'. It follows from Remark 2.2 that the number of crossings of K' is $\leq n(2n-1)$. Apply to K an R5 move in a small neighbourhood of u_{i_1}. Then the resulting knot diagram K_1 has no more than $n(2n-1)+2$ crossings. Applying to K_1 the reducing algorithm again, we shall obtain a knot diagram K_1' having no more than $n(2n-1)$ crossings and exactly $k-1$ negatively oriented vertices $u_{i_2}, \dots, u_{i_k}$. Using the induction on the number k of negatively oriented vertices and proceeding at each inductive step i in the same way as before, we shall obtain a sequence q of perfectly coloured knot diagrams $K = K_0, \ldots, K_1, \ldots, K_2, \ldots, K_k$ so that each $K_i \in \mathcal{L}_D$, every intermediate knot diagram of the sequence q has no more than $\max\{n(2n-1)+2, \mathrm{cr}(K)\}$ crossings and all vertices of K_k are positively oriented, i.e. $K_k \in \mathcal{C}_D$. Notice also that all knot diagrams of the sequence q respect the chord diagram D and the colouring of them is inherited from the colouring of K. This completes the proof. □

It is of interest to know whether Theorems 2.1 and 2.2 can be strengthened in the form given by the original formulation of Question 4.4. In other words, does Theorem 2.1 (Theorem 2.2) still hold, if we require that each "combination of moves R1–R4 (R1–R5)", as it defined in the statement of the theorem, is to be one of the moves R1–R4 (R1–R5 respectively)? The following examples show that, in general, this is not true.

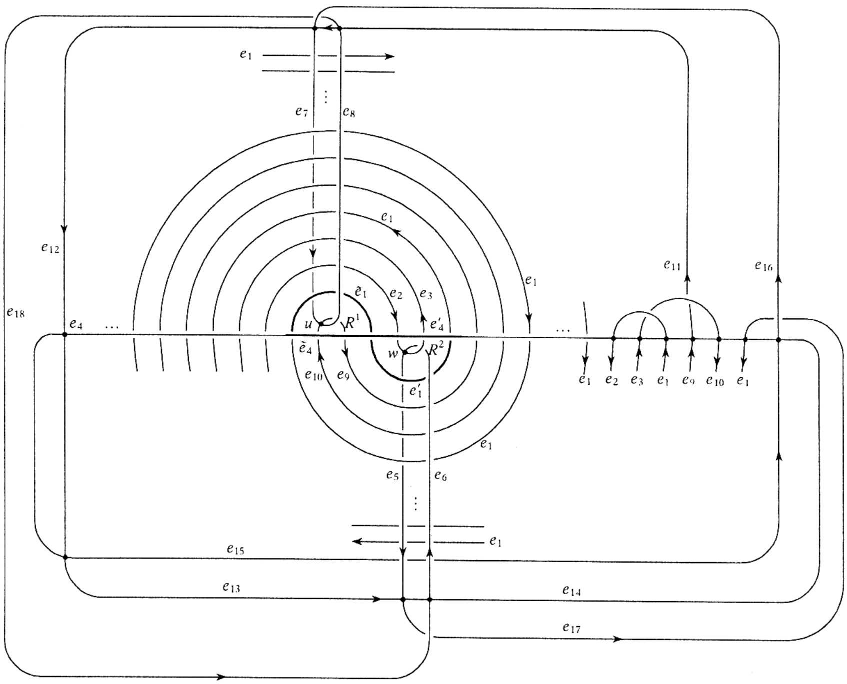
Example 2.1. We present in Figure 6 an example of a coloured knot diagram K_l so that any application of R1–R4 moves lead to increasing the number of crossings in the resulting knot diagram. Let us make some explanations to this example. Here R^1 and R^2 denote the two minimal components of index 1 of the covering region $F_{1,4}$. R^1 is bounded by the two arcs $\tilde{e}_1$ and $\tilde{e}_4$ of the edges e_1 and e_4, respectively. Similarly, R^2 is bounded by the arcs e_1' and e_4' of the edges e_1 and e_4. For the components R^1, R^2 we have $s_{14}(R^1) = s_{14}(R^2) = 0$. The edge e_1 of the knot diagram K turns around the vertices u and w l times, where the natural number l is chosen arbitrarily. Therefore the pairs of edges e_3, e_4 and e_2, e_4 yield both $4l$ crossings. Starting from the edge e_1 and traveling along the oriented knot K_l, we have the following sequence q of its edges (for simplicity, some of them are omitted here):

$e_1, \ldots, e_2, e_3, \ldots, e_{10}, e_9, e_{11}, \ldots, e_{12}, \ldots, e_{13}, \ldots, e_{14}, \ldots, e_4, \ldots, e_{15}, e_{16}, e_7, e_8, e_{18}, e_6, e_5, e_{17}, e_1$.

Clearly, changing $l > 0$ arbitrarily, we shall obtain the sequence $K_1, \ldots, K_l, \ldots$ of knot diagrams each respecting the same chord diagram D. For $l \gg$ the order of D, to make of K_l, by using only the crossing changes and R1–R5 moves, a knot diagram with fewer crossings, we need to perform, after some changes of crossings, an isotopy with respect to the component R^1 or the component R^2. But we see from the figure that it is impossible to make the isotopy, which pushes e_1' or $\tilde{e}_1$ out of the edge e_4 by a sequence of R1–R5 moves so that no intermediate knot diagram has more crossings that K_l.

Example 2.2. We depict in Figure 7 the piece of a knot diagram L. Here R denotes the minimal component of index 1 of the covering region $F_{1,2}$ which is bounded by the arcs e_1' and e_2' of the edges e_1 and e_2. We have $s_{12}(R) = 0$. In order to decrease the number of crossings of L and "delete" the covering region

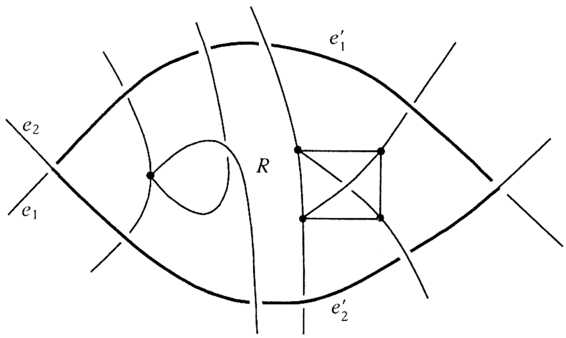
FIGURE 6



R, we have first to make some crossing changes and then to isotopy e_1' or e_2' along R, i.e to push e_1' out of e_2' in the same fashion as in the proof of Theorem 2.1. Let $q = q_1, q_2, \ldots, q_m$ be any sequence of local moves R1–R4 (the move R5 is here irrelevant) which gives, taken together, such an isotopy along R. It is easy to see from the picture in Figure 7 that some intermediate knot diagrams appearing in the process will have more crossing changes that the input knot diagram L. The reason is the same as in the case of Example 2.1 - starting from the given knot diagram, one may apply to L in the ε-neighbourhood of R only "one-side" local moves R1–R4 which increase the number of crossings.

REMARK 2.3. Theorem 2.1 remains true, if we replace in its statement "the combinations of the moves R1–R4" for "the moves R1–R4" and allow some little increase of the number of crossings in the intermediate knot diagrams. Similarly, Theorem 2.2 remains true, if we replace in the statement "the combinations of the moves R1–R5" for "the moves R1–R5" and allow to increase slightly the number of crossing changes in the intermediate knot diagrams. The increase depends each time on the configuration of the faces formed by the knot diagram in $\mathbf{R}^2$ and involved in the component R, along which an isotopy is established. For the minimal component R of index 1

FIGURE 7



shown in the Figure 7 the increase of the number of crossings does not exceed 2. Therefore each time, when isotopying a knot diagram K with d crossings along any minimal component R of index 1 by a sequence of R1–R5 moves, the number of crossings in intermediate knot diagrams can first increase slightly, after which it decreases to a number $d-1$ or $d-2$.

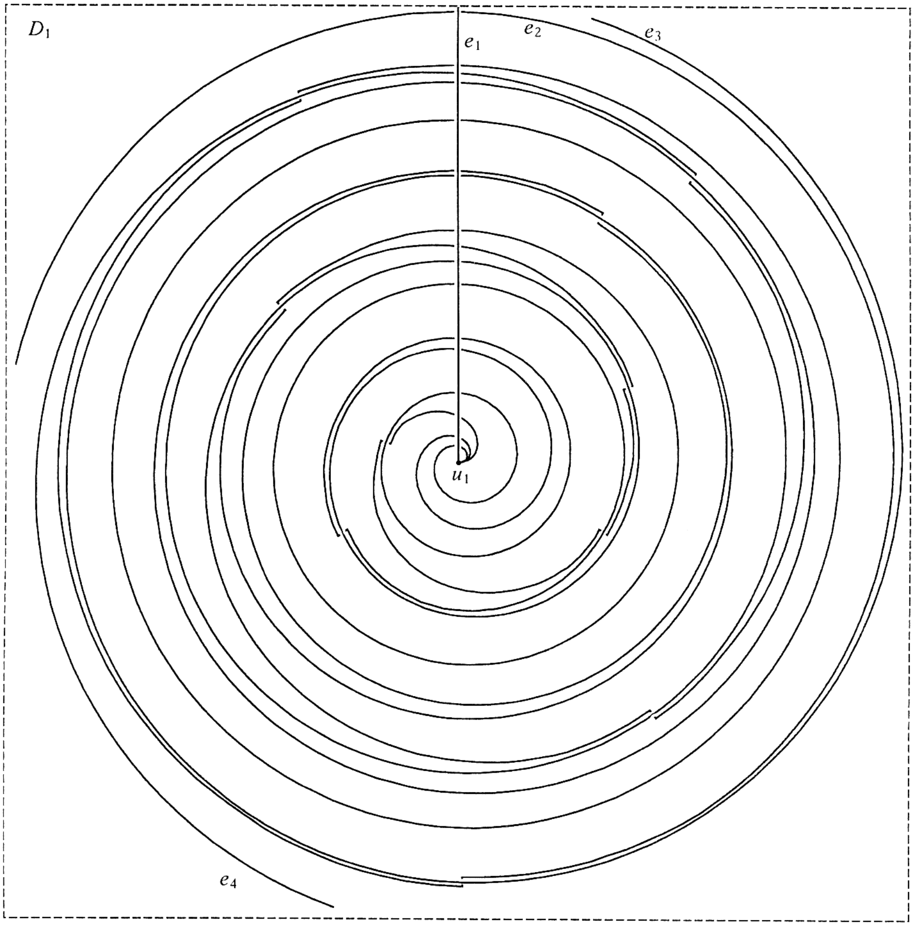
REMARK 2.4. Let us turn to the knot diagrams K_l from Example 2.1. Each K_l respects the chord diagram D. Suppose there is a finite set of knot diagrams $L_1, \ldots, L_s$ each with chord diagram D such that any knot diagram with the same chord diagram can be reduced to one of L_i by crossing changes and the local moves R1–R5 without increasing the number of crossings in all intermediate diagrams. If l is enough large, the only possibility to reduce K_l to one of L_i without increasing the crossings at all intermediate steps is to make crossing changes in K_l in such a way (if such exists), so that the resulting knot diagram will become isotopic to one of the set $L_1, \ldots, L_s$.

The following example shows that the answer to Stanford's Question 4.3 is, in general, negative.

Example 2.3. Let us consider the piece of a knot diagram $K_l, l = 1, \ldots, m, \ldots$, shown in Figure 8.

Here the four edges e_1, e_2, e_3, e_4 of the knot diagram are coloured by the colours 1, 2, 3 and 4 respectively. The colouring of the edges K_l is irrelevant. In the disc $D_1 \subset \mathbf{R}^2$ the pieces of the edges e_2, e_3 and e_4, as the curves in $\mathbf{R}^2$, are determined by the pieces of Archimedean spirals $r_2 = \frac{a_2}{l}\phi, r_3 = \frac{a_3}{l}\phi$ and $r_4 = \frac{a_4}{l}\phi$ coming from the point u_1, with $\phi \geq 0$, where a_2, a_3 and a_4 are three small positive numbers such that $a_2 < a_3 < a_4$. In the disc $D_2 \subset \mathbf{R}^2 - D_1$ the edges e_2, e_3 and e_4 look like inside the disc D_1, i.e. as the spirals coming from the point u_2. The edge e_1 is an interval of the line joining u_1 to u_2. Define $\mathcal{K}_l$ to be the collection of all knot diagrams obtained from K_l by all possible crossing

FIGURE 8



changes. Notice that for each number l all knot diagrams of the collection $\mathcal{L}_D$ respect the same chord diagram D of order 2, D being a unique non-trivial chord diagram of order 2 with the two non-parallel chords. Suppose there exists a finite set of knot diagrams $L_1, \ldots, L_n$, each respecting D, which satisfies the conditions specified by Question 4.3. Denote by $\mathrm{cr}(L_i)$ the number of crossings in the knot diagram $L_i, i = 1, \ldots, n$. Set $d = \max\{\mathrm{cr}(L_i), i = 1, \ldots, n\}$. Let us consider any knot diagram K_l with $l \gg d$. By the assumption, K_l can be made isotopic to one of the L_i, say L_j, by crossing changes only. Let K_l' be the knot diagram obtained from K_l by applying to it the appropriate crossing changes. Now the idea to get the contradiction is the following. Since K_l' has much more crossings than L_j, the only way to isotopy K_l' to L_j is first to deform slightly each of the edges e_2, e_3, e_4 inside the disc D_1 and then to remove, by using R1–R5 moves, the inner loops of them appeared after the first step. As for the edge e_2, the last can be made only if e_2 is either over or under the edges e_1, e_3, e_4 of K_l inside the disc D_1. Assume e_2 is over the other edges. Applying to the edge e_3 of K_l' and the resulting knot diagram the same arguments as just for e_2, we see that e_3 should be under the other edges of K_l' inside the disc D_1. Then the edge e_4 would be over the edge e_3 and under the edge e_2 of K_l'. The last condition however make it impossible to arrange an isotopy of e_4 inside D_1, which would lead to essential decreasing the number of crossings formed by e_4 with the other edges of the resulting knot diagram. Notice that our definition of the knot diagram K_l inside the disc D_2 provides that we cannot arrange the required isotopy in another way.

To extend the above example to the case of the other chord diagrams, we could insert in the disc D_2 an appropriate part of a knot diagram, more complicated than in the previous case, and leave the part of K_l inside D_1 without changing.

3. Estimates and generalization to spatial graphs. Below under an open face of the embedding $\varphi\colon G \to M$ of a graph G into a closed surface M we understand any connected component of the space $M - \varphi(G)$. With each knot diagram $K \in \mathcal{L}_D$ we associate a coloured oriented rotation graph H_K. As a rotation graph H_K is uniquely determined by a 2-cell embedding of its underlying graph $\hat{H}_K$ into the plane with fixed counterclockwise orientation on it. Let us recall the definition of a rotation graph (see [**8**] for details). Let H be a non-oriented finite graph, which is allowed to have self-adjacencies and multiple edges (1-CW-complex). With each non-oriented edge $\tilde{e}$ of H one can associate in an obvious way the two oriented edges with opposite orientations. By a local rotation at the vertex v of H we understand a cyclic permutation of oriented edges of H coming from v. The rotation system R on H is a choice of a local rotation at each vertex of H. The rotation graph is a pair (H, R), where H is a non-oriented graph and R is a rotation system on it. By the Heffter-Edmonds theorem [**8**], to each rotation graph (H, R) there corresponds a 2-cell embedding of H into a closed oriented surface M and, conversely, any 2-cell embedding of the graph H into a closed oriented surface M induces a rotation graph (H, R), so that the above correspondences are one-to-one and each other's inverse [**8**]. When defining a rotation system R on an oriented graph G, we forget the given orientation of its edges. Each rotation system R on a graph G produces a collection $\{C_i\}_{i\in I}$ of oriented walks in G corresponding to the oriented circuits of the boundaries of the faces of the 2-cell embedding of G into M induced by R. If M is a sphere, we can distinguish one of such walks

C_i. This gives an embedding of G into the plane $\mathbf{R}^2$, where C_i corresponds to the circuit of the outer face of this embedding.

Let $K \in \mathcal{L}_D$ be any coloured knot diagram respecting the chord diagram D of order n. Define the rotation coloured graph H_K associated with K as follows. The vertex set V_{H_K} of H_K is the set of all double points of K(i.e., the vertices and crossings of K). The edge set E_{H_K} of H_K is the set of all strands f of K with the endpoints in V_{H_K} such that no other double points of K lie on f. The orientations of the edges of H_K are induced by the orientations of the corresponding edges of K. An obvious incident structure is defined in H_K. Therefore, as a 1-CW-complex, H_K is an oriented plane graph which coincides with K as set in $\mathbf{R}^2$. Notice that all open faces of the natural embedding of H_K into the plane are 2-cells, except for one which is outer. The colour of each edge f of H_K is induced by the colour i of the edge e_i in K which contains f as its subset. Since H_K is defined as a plane 1-CW-complex, it may be considered as a coloured rotation graph with the distinguished walk C corresponding to the outer face of the embedding. Notice that H_K is 4-valent graph and for each $v \in V_{H_K}$ we have $\deg_+(v) = \deg_-(v) = 2$. The vertices of K will be called the distinguished vertices of H_K. Notice that K induces also in H_K an oriented Euler circuit E. Let u be any non-distinguished vertex of H_K (a crossing of K). Then for any edge f_+ of H_K coming out u there exists exactly one edge, say f_-, going in u and having the same colour as f_+. Denote by e_+ and e_- another pair of oriented edges of H_K, incident to u, and having the same colours. Then a cyclic sequence of oriented edges, incident to u, which is given by the local rotation at u, looks as e_-, f_-, e_+, f_+ or f_-, e_-, f_+, e_+. Notice that e_+ meets just after e_- and f_+ meets just after f_-, when traveling along the oriented Euler circuit E. Fix the counterclockwise orientation in the plane $\mathbf{R}^2$.

Two coloured rotation graphs H_K and H_L of the knot diagrams K and L, respectively, are called isomorphic, if there exists an isomorphism $g\colon \hat{H}_K \to \hat{H}_L$ of their underlying orgraphs $\hat{H}_K$ and $\hat{H}_L$ which preserves the colours of the edges, and preserves the local rotations at all vertices, and sends the distinguished vertices of H_K onto the distinguished vertices of H_L, and the distinguished walk of H_K onto the distinguished walk of H_L. Denote by $\bar{H}_K$ the same rotation graph as H_K only with its distinguished walk forgotten.

THEOREM 3.1. *Any two coloured knot diagrams $K, L \in \mathcal{L}_D$ are strongly equivalent if and only if the corresponding coloured rotation graphs H_K and H_L are isomorphic.*

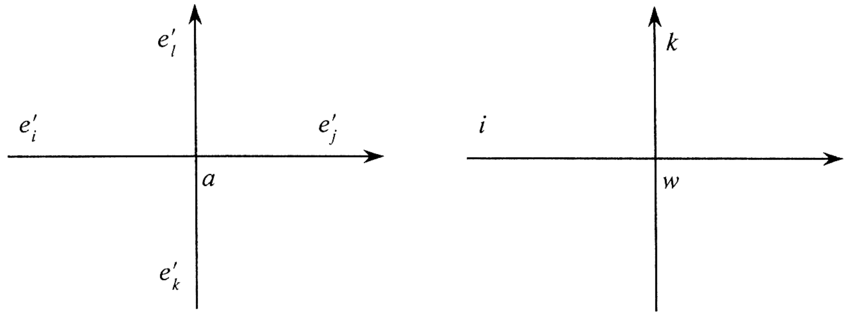
PROOF. It follows directly from the definition of a strong equivalence (strong isotopy) of knot diagrams and the definition of a coloured rotation graph associated with a coloured knot diagram that any two strongly equivalent coloured knot diagrams have the isomorphic coloured rotation graphs.

Conversely, let $K, L \in \mathcal{L}_D$ be any two coloured knot diagrams such that H_K is isomorphic to H_L and let $g\colon H_K \to H_L$ be the corresponding isomorphism of coloured rotation graphs H_K and H_L. First notice that all the faces of the embedding of H_K in $\mathbf{R}^2$ but one which is outer are 2-cells. The same is valid for the embedding of H_L in $\mathbf{R}^2$. Furthermore, g can be considered as the homeomorphism of 1-CW-complexes. It follows from the definition of g and from the above remark that g can be extended to a homeomorphism $\tilde{g}$, $\tilde{g}\colon \mathbf{R}^2 \to \mathbf{R}^2$ which preserves the counterclockwise orientation in $\mathbf{R}^2$. Therefore, there exists an isotopy of $\mathbf{R}^2$, the final homeomorphism of which is $\tilde{g}$. This gives the strong equivalence of the coloured knot diagrams K and L. □

Two perfectly coloured knot diagrams $K, L \in \mathcal{L}_D$ will be called equivalent if there exists an isotopy between the knot diagrams K and L which preserves the colouring of the diagrams. Denote by $p(D)$ the number of different equivalence classes of perfectly coloured knot diagrams of the collection $\mathcal{L}_D$ and denote by $|\mathcal{L}_D|$ the number of different knot types in $\mathcal{L}_D$.

Let $L \in \mathcal{L}_D$ be any perfectly coloured knot diagram. Denote by $u_1, \ldots, u_n$ the distinguished vertices (i.e. corresponding to the vertices of L) of L and let $w_1, \ldots, w_s$, $s \leq n(2n-1)$, be all the non-distinguished vertices of H_L, i.e. corresponding to the crossings of L. For each vertex a of H_L one can choose an open disc neighbourhood V_a of a in $\mathbf{R}^2$, so that $H_L \cap V_a$ looks as in Figure 9a, where e'_i, e'_j, e'_k and e'_l are the coloured strands of L with the colours i, j, k and l, respectively. If a is a distinguished vertex of H_L, then $j = i+1$ and $l = k+1$, otherwise (i.e. when a corresponds to a crossing of L) $k = l, i = j$ and $i \neq k$. We shall call the local picture of the graph H_L at the vertex a, as shown in Figure 9a, the coloured reper of H_L at a. Each coloured (oriented) reper R_w of H_L at the vertex w corresponding to a crossing of L can be encoded by the (ordered) pair (i, k), where i and k are the two colours of the strands of R_w, as depicted in Figure 9b. Any coloured reper R_a of H_L at the vertex a is determined uniquely by the coloured rotation graph H_L. The reper R_a is called positively oriented, if $i < k$, otherwise R_a is negatively oriented. The planar coloured orgraph $\bar{H}_L$ can be thought of as made of its coloured oriented repers R_a by joining their oriented strands of the same colour each to another in the following way. Let $i, i \leq 2n$, be any colour and let u and $\bar{u}$ be the two distinguished vertices of L such that R_u contains the output strand and $R_{\bar{u}}$ contains the input strand with the same colour i. Let $w_1, \ldots, w_p$ be the non-distinguished vertices of $\bar{H}_L$, as they appear, when traveling along the oriented edge e_i of L of the colour i. This determines a sequence q_i of repers, corresponding to the colour $i, q_i = R_u, R_{w_1}, \ldots, R_{w_p}, R_{\bar{u}}$. Notice that the collection R of all repers of H_L, taken together with the sequences $q_i, i = 1, \ldots, 2n$, determines uniquely up to isomorphism the coloured rotation graph $\bar{H}_L$.

FIGURE 9



a *b*

THEOREM 3.2. *Let D be any chord diagram of order $n > 0$. Then* $|\mathcal{L}_D| \leq 2n \cdot 2^n \cdot \sum_{s=0}^{n(2n-1)} C_{n(2n-1)}^s \cdot 2^s \cdot s!$.

PROOF. By the definition of $p(D)$, we have $|\mathcal{L}_D| \leq p(D)$. By the definition of the strong equivalence of the coloured knot diagrams, the number $p(D)$ does not exceed the number of equivalence classes of perfectly coloured knot diagrams via the strong equivalence of such diagrams. Let $K, L \in \mathcal{L}_D$ be any two coloured knot diagrams and let H_K and H_L be the coloured rotation graphs associated with them. Suppose $\bar{H}_K = \bar{H}_L$. Recall that the certain equivalence relation on the set of coloured knot diagrams is defined by local moves R1–R5 and that the strong equivalence of coloured knot diagrams implies the certain equivalence of them. Therefore, the equality $\bar{H}_K = \bar{H}_L$ implies the (certain) equivalence of the coloured knot diagrams K and L. It follows now from Theorem 3.1 and the above reasoning that the number $p(D)$ does not exceed the number of different up to isomorphism (non-distinguished)rotation coloured graphs of the collection $\{\bar{H}_L\}_{L \in D}$. Let $L \in \mathcal{L}_D$ be any perfectly coloured knot diagram. It follows from Remark 2.2 that the corresponding orgraph $\hat{H}_L$ has no more than $n + n(2n-1) = 2n^2$ vertices. Let us consider $\bar{H}_L$ as made of its coloured repers R_a by gluing them one to another according to the colours of the strands involved. Let $u_1, \ldots, u_n$ be the distinguished vertices of L and let $w_1, \ldots, w_s, s \leq n(2n-1)$, be all the non-distinguished vertices of H_L. Denote by q_i the sequence of coloured repers of H_L corresponding to the colour $i, i = 1, \ldots, 2n$. For a given chord diagram D of order n , there are no more than $2n$ possibilities (corresponding to the colourings of L) to choose the different collections $\{R'_{u_i},\dots,R'_{u_n}\}$ of (non-oriented) coloured repers at the distinguished vertices of H_L. For each vertex $u_i, i=1,\dots,n$, we have the two possible orientations at it. Taking into account the above observations, we conclude that there are no more than $2n\cdot 2^n$ different collections $\{R_{u_1},\dots,R_{u_n}\}$ of oriented coloured repers of H_L at all distinguished vertices of H_L. Define $\mathcal{M}_D$ to be the set of all $(2n\times 2n)$-matrices $M=(m_{ij})$ satisfying the following properties:

1): each entry m_{ij} of M is 0 or 1;
2): if $m_{ij}=1$, then $m_{ji}=0,\ i,j=1,\dots,2n$;
3): $m_{ii}=0$ for each $i=1,\dots.2n$.

The set C_L of the crossings of any perfectly coloured knot diagram $\mathcal{L}_D$ (and so the set of the coloured repers of H_L at the non-distinguished vertices) is determined uniquely by the matrix $M_L\in\mathcal{M}_D$ in an obvios way. Simply recall that each reper R_w of H_L at a non-distinguished vertex w of H_L is encoded by a pair $(i,j), i\neq j$. Therefore the equality $m_{ij}=1$ in M_L corresponds to the fact that H_L contains the reper $R_w=(i,j)$ for some non-distingiuished vertex w of H_L. Keeping in mind the above correspondence between all possible collections C_L of coloured repers and the matrices of the collection $\mathcal{M}_D$, we can evaluate the number of distinct collections C_L which one can obtain within the collection $\mathcal{L}_D$ of coloured knot diagrams.

Let $\mathcal{M}^u_D$ denote the subset of $\mathcal{M}_D$ consisting of upper-triangle matrices. For each $s, s\leq 0\leq n(2n-1)$, set $\mathcal{M}^s_D=\{M\in\mathcal{M}_D|M$ has precisely s non-zero entries $\}$. It is easy to see that the set $\mathcal{M}^u_D\cap\mathcal{M}^s_D$ consists of $C^s_{n(2n-1)}$ elements. It follows that the set $\mathcal{M}^s_D$ consists of $C^s_{n(2n-1)}\cdot 2^s$ elements. Any matrix $M\in\mathcal{M}^s_D$ determines uniquely in an obvious way a $2n$-partition $s=k_1+\dots+k_{2n}$ of the number s, where k_i is the total number of 1 in ith row and ith column of $M, 0\leq k_i\leq s, i=1,\dots,2n$. Recall that k_i is just the number of crossings in L lying on the edge e_i with the colour i. Let $s=k_1+\dots+k_{2n}$ be the $2n$-partition of s, determined by a matrix $M\in\mathcal{M}^s_D$. For a given knot diagram $L\in\mathcal{L}_D$ there are $k_1!\dots k_{2n}!$(we set $0!=1$) different ways to arrange the orders on k_1 crossings on the edge e_1, on k_2 crossings on the edge e_2, ... , and on k_{2n} crossings on the edge e_{2n}. Notice that for each $i, i=1,\dots,2n$, the choice of any such order determines $2n$ sequences $q_1,\dots,q_{2n}$ of repers, as defined above. Taking into accout the above reasonings and estimates, we conclude that there are totally no more than

$$2n\cdot 2^n\cdot\sum_{s=0}^{n(2n-1)}\sum_{M\in\mathcal{M}^s_D}k_1!\dots k_{2n}!$$

different coloured rotation graphs in the collection $\{\bar{H}_L | L \in \mathcal{L}_D\}$, where the second sum in the above expression is taken over all matrices M of the collection $\mathcal{M}_D^s$, each M inducing a $2n$-partition $s = k_1 + \cdots + k_{2n}$ of the number $s, 0 \leq s \leq n(2n-1)$. Let $s = k_1 + \cdots + k_{2n}$ be any $2n$-partition of s. Since $k_1! \ldots k_{2n}! \leq s!$, we get the following upper bound for the number $|\mathcal{L}_D|$:

$$|\mathcal{L}_D| \leq p(D) \leq 2n \cdot 2^n \cdot \sum_{s=0}^{n(2n-1)} C_{n(2n-1)}^s \cdot 2^s \cdot s!.$$

This completes the proof of the theorem. □

REMARK 3.1. Define $\mathcal{L}'_D$ to be the subset of $\mathcal{L}_D$ consisting of all coloured knot diagrams with only the positively oriented vertices. Reasoning similarly as in the proof of Theorem 3.2, we can get an upper bound for the number of $|\mathcal{L}'_D|$ of different knot types in $\mathcal{L}'_D$. In this case we take into account only the positive orientation at each vertex of a coloured knot diagram. This leads to a slightly better estimate for the number $|\mathcal{L}'_D|$, but we do not give it here.

REMARK 3.2. The above estimate for the number $|\mathcal{L}_D|$ is certainly far from being satisfactory. First we should note that some coloured rotation graphs obtained by gluing the corresponding collections of repers, as in the proof of Theorem 3.2, may represent not planar graphs, i.e. they can determine the surfaces of genus > 0. Let G be a coloured graph respecting the chord diagram D with the choice of an orientation at each of its vertices $u_i, i = 1, \ldots, n$. Let $R = \{R_{u_1}, \ldots, R_{u_n}\}$ be the corresponding collection of oriented coloured repers. Choose any matrix $M \in \mathcal{M}_D^s$. The latter determines a $2n$-partition of the number $s, s = k_1 + \cdots + k_{2n}$ and the corresponding collections $\mathcal{R}_1, \ldots, \mathcal{R}_{2n}$ of oriented repers, where $|\mathcal{R}_i| = k_i, i = 1, \ldots, 2n$. For each i fix an order ρ_i in $\mathcal{R}_i$. The tuple $< G; M; \rho_1, \ldots \rho_{2n} >$ determines, as in the proof of Theorem 3.1, a coloured rotation graph H. Call a tuple $< G; M; \rho_1, \ldots, \rho_{2n} >$ admissible, if it determines a planar coloured rotation graph H. Clearly, if such a rotation coloured graph has genus > 0, it cannot belong to the collection $\mathcal{L}_D$. Finally, among the different (non-isomorphic) rotation graphs of the collection $\{\bar{H}_L | L \in \mathcal{L}_D\}$ there are such which represent the equivalent coloured knots $L \in \mathcal{L}_D$.

In [**6**] we describe briefly a possible modified algorithm for computing Vassiliev's invariants of knots, which is based on reducing of knot diagrams with the chord diagram D to the ones of the family $\mathcal{L}_D$. As the estimate for the number $|\mathcal{L}_D|$ we have obtained is far from being good, the possible practical applications of this algorithm depends strongly on solving the following problems:

Question 3.1. Can one obtain from $\mathcal{L}_D$, by applying to the knot diagrams $L \in \mathcal{L}_D$ the moves under the conditions of Theorems 2.1 and 2.2, the subset $\overline{\mathcal{L}}_D \subset \mathcal{L}_D$, which represents the class of knots considerably smaller than $\mathcal{L}_D$?

Problem 3.2. [**6**] Find within the class $\mathcal{L}_D$ a smaller subclass $\mathcal{L}'_D$ of knot diagrams which represent the same knot types as the ones of $\mathcal{L}_D$.

Problem 3.3. Find a criterion for a given tuple $< G; M; \rho_1, \dots, \rho_{2n} >$ to be admissible.

Problem 3.4. Find an effective criterion for checking whether any two different rotation graphs of the collection $\{\bar{H}_L \in \mathcal{L}_D\}$ represent the equivalent coloured knots.

Example 3.1. To see on how much the upper bound for the number $|\mathcal{L}_D|$, given by Theorem 3.2, differ from the real value of $|\mathcal{L}_D|$, let us consider a chord diagram D of order 2 with the two non-parallel chords. Keeping in mind the notices of Remark 3.2, it is not difficult to check directly that $|\mathcal{L}_D| \leq 52$. Notice also that for $n = 2$ the number s of crossings in any knot diagram $K \in \mathcal{L}_D$ can never reach the value $n(2n-1) = 6$. We suggest that the same is true for all $n > 2$.

Finally, we shall discuss the possibilities of extending some of the above results on singular knots to the case of spatial graphs in $\mathbf{R}^3$. Our definition of a spatial graph is due to J. Murakami [**5**].

Let $\mathcal{V}$ be a set of 2-discs and ε be a set of edges homeomorphic to $[0, 1]$ in S^3. Each edge has an orientation induced by the orientation of $[0, 1]$. The endpoints of an edge corresponding to 0 and 1 are called the initial point and the terminal point of the edge, respectively. The pair $\Gamma = (\mathcal{V}, \varepsilon)$ is called an oriented spatial graph if it satisfies the following conditions. The discs in $\mathcal{V}$ are mutually disjoint and the edges in ε are mutually disjoint. Moreover, the discs in $\mathcal{V}$ and the interiors of the edges in ε are also mutually disjoint. All the endpoints of edges in ε lie in the boundaries of discs in $\mathcal{V}$. Spatial graphs are considered up to equivalence. Two spatial graphs Γ and Γ' are called equivalent if there is an isotopy of S^3 which sends Γ to Γ'. We deal rather with diagrams of spatial graphs. A diagram of a spatial graph is defined in the same way as in the case of (singular)knots and links. By a colouring of the diagram Γ of a spatial graph we shall mean any enumeration of the edges of its underlying graph, so that the different edges have different colours from 1 to m (in opposite to the colourings of the edges of knot diagrams), where m is the number of edges of G. A colouring c of the diagram of a spatial graph is called perfect provided it satisfies the following condition: if $c(e) > c(f)$ then the edge e is over the edge f at each crossing of the edges e and f. To each isotopy of spatial graphs in S^3 there corresponds the equivalence relation on the set of diagrams of graphs generated by the local moves R1–R5, defined in the same way as for the diagrams of singular knots (see [**5**]).

With each diagram G of a spatial graph Γ we associate two rotation graphs, H_G and $\mathcal{H}_G$. The rotation graph $\mathcal{H}_G$ has its underlying graph the same as G. The rotation system at each vertex u of it is determined by the cyclic order of the edges of G, incident to u, with respect to the counterclockwise orientation in the plane. The rotation graph $\mathcal{H}_G$ of a diagram G of spatial graph is invariant under the moves R1–R4, while R5 move changes the orientation at the corresponding vertex of $\mathcal{H}_G$ to the opposite one. The coloured rotation graph H_G associated with a coloured diagram G of spatial graph is defined in the same way as the coloured rotation graph of a coloured knot diagram (see above). The vertices of G are the distinguished vertices of H_G. A rotation graph H_G is actually an invariant of the strong isotopy of a diagram G of spatial graph. Let $\mathcal{H}$ be any rotation graph and let $\mathcal{P}_{\mathcal{H}}$ be the set of diagrams of spatial graphs with underlying rotation graph $\mathcal{H}$. Denote by $\mathcal{L}_{\mathcal{H}}$ the set of diagrams G of spatial graphs which satisfy the following conditions:

1): no edge of G has inner loops;
2): for any pair of edges e and f, e can intersect the interior of f and f can intersect the interior of e at least at one (internal) point;
3): the rotation graph $\mathcal{H}_G$ is isomorphic to $\mathcal{H}$;
4): G admits a perfect colouring.

REMARK 3.3. Looking carefuly through the proof of Theorem 2.1, it is not difficult to see that we can apply to any diagram $G \in \mathcal{P}_{\mathcal{H}}$ the same reduction procedure as for the diagrams of singular knots. The resulting diagram of a spatial graph will be a coloured diagram of the collection $\mathcal{L}_{\mathcal{H}}$. This procedure consists actually in applying to G some sequence of combinations of local moves R1–R4 and crossing changes. In this case we define the notions of covering region, the component of a covering region, the defect of the component of index 1 and so on, in the same fashion as in the case of knot diagrams. This provides that our reduction procedure (which is of inductive character, as before) does not lead to an increase of the number of the crossings in all intermediate diagrams of spatial graphs. Therefore, an analogue of Theorem 2.1 for spatial graphs can be obtained in this way. If we allow, in addition, the application of R5 move to the diagrams of spatial graphs, we can reduce each input diagram to the one of a little smaller subset of $\mathcal{L}_H$. To get an estimate for $|\mathcal{L}_{\mathcal{H}}|$, the number of different up to isotopy spatial graphs represented by the diagrams of the collection $\mathcal{L}_{\mathcal{H}}$, we can follow the method of the proof of Theorem 3.2, slightly changing it in the part, where we consider the distinguished vertices of the rotation graph H_G of a diagram G of spatial graph, but we shall not develop this point here.

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