UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXXVIII
2000

ON 4-DIMENSIONAL CONFORMALLY FLAT LEFT INVARIANT PARA-HERMITIAN STRUCTURES

BY ZBIGNIEW OLSZAK

Dedicated to Professor Radu Roşca on the occasion of his 90th birthday.

Abstract. The paper deals with natural left invariant para-Hermitian structures on semidirect products of Lie groups. In dimension 4 they are conformally flat if and only if they are locally conformally para-Kählerian. In higher dimensions this equivalence does not hold.

1. Introduction. In the previous papers [**7**], [**8**], we have defined and studied the so-called natural left invariant para-Hermitian structures (J, g) on semidirect products of two n-dimensional Lie groups $G_0 \times_f G_1$. Investigating curvature properties of these structures, we proved there: (i) if the structure (J, g) is additionally locally conformally para-Kählerian, then the metric g is conformally flat; and (ii) if (J, g) is para-Kählerian, then g is flat ([**8**], Th. 1). Examples illustrating these results were constructed, and it was also shown that the converse to the implication (ii) fails in general at any even dimension $2n \geqslant 4$.

In the present paper, the investigations of natural left invariant para-Hermitian structures on semidirect products are continued. The main results are related to the implication (i). We prove that the converse to (i) is true in dimension 4. Thus, any natural left invariant para-Hermitian structure on a semidirect product of two 2-dimensional Lie groups is conformally flat if and only if it is locally conformally para-Kählerian. In dimension 6, the converse to (i) does not hold. Namely, we construct a natural left invariant conformally flat para-Hermitian structure on a 6-dimensional semidirect product $\mathrm{SU}(2) \times_f \mathbb{R}^3$, which is not locally conformally para-Kählerian. In dimensions $2n \geqslant 8$, similar questions are still unsolved.

The conformally flat para-Kählerian manifolds were studied in [**5**], where we described the local structure of such manifolds.

2. Preliminaries. The basic references for this section are the survey articles [**1**] and [**2**]. However, for covenience, we recall the necessary definitions of main classes of almost para-Hermitian manifolds.

Let M be a connected C^∞-differentiable manifold of even dimension $2n$. The all objects involved on M are of class C^∞ too. By $\mathfrak{X}(M)$ we denote the Lie algebra of vector fields on M.

Assume that J is a $(1,1)$-tensor field on M such that for each point $p \in M$, $J_p^2 = \mathrm{Id}_p$ (the identity operator) and the (± 1)-eigenspaces $T_p^\pm M$ of J_p are both n-dimensional subspaces of the tangent space T_pM. J is then said to be an almost paracomplex structure on M, and the pair (M, J) is called an almost paracomplex manifold. The manifold M admits an almost paracomplex structure if and only if there is a G-structure on M with structure group $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$.

For an almost paracomplex manifold (M, J), let N be the so-called Nijenhuis torsion tensor field of J,

$$N(X,Y) = [JX, JY] - J[X, JY] - J[JX, Y] + J^2[X, Y]$$

for $X, Y \in \mathfrak{X}(M)$. The structure J and the manifold (M, J) are said to be paracomplex if N vanishes identically on M. Thus, (M, J) is paracomplex if and only if the eigen-distributions $T^\pm M : M \ni p \longmapsto T_p^\pm M$ are both completely integrable. The paracomplex manifolds can also be characterized by the existence of an atlas with coordinate maps satisfying the so-called para-Cauchy-Riemann equations.

Let (M, J) be an almost paracomplex manifold and suppose that g is a pseudo-Riemannian metric on M for which J is an antiisometry, that is,

$$g(JX, JY) = -g(X, Y) \tag{2.1}$$

for $X, Y \in \mathfrak{X}(M)$. Then the pair (J, g) is said to be an almost para-Hermitian structure on M and the triple (M, J, g) an almost para-Hermitian manifold; and in the case when J is additionally paracomplex, we say that (J, g) and (M, J, g) are para-Hermitian. Note that under assumption (2.1), the eigen-distributions $T^\pm M$ become isotropic.

If (J, g) is an almost para-Hermitian structure on M, then a conformally deformed metric $\overline{g} = e^{2f} g$ also satisfies the compatibility condition (2.1), f being a function on M. Therefore, the pair $(J, \overline{g})$ is an almost para-Hermitian structure on M too.

For an almost para-Hermitian manifold, the fundamental form Ω is defined by

$$\Omega(X,Y) = g(JX,Y)$$

for $X, Y \in \mathfrak{X}(M)$. The 2-form Ω is in fact an almost symplectic form on M.

An (almost) para-Hermitian manifold is said to be (almost) para-Kählerian if its fundamental form Ω is closed. Every 2-dimensional almost para-Hermitian manifold is necessarily para-Kählerian. An almost para-Hermitian manifold is para-Kählerian if and only if $\nabla J = 0$, where ∇ is the Levi-Civita connection of g. Another characterization of the para-Kählerian manifolds can be formulated in the following way [**5**]: A pseudo-Riemannian manifold M of dimension $2n$ is a para-Kählerian manifold if and only if there are two n-dimensional totally isotropic and parallel distributions $\mathcal{H}$ and $\mathcal{V}$ on M such that $\mathcal{H} \cap \mathcal{V} = 0$.

We say that an almost para-Hermitian (M, J, g) is locally conformally para-Kählerian if for any point $p \in M$ there exist a neighborhood U of p and a function $f : U \to \mathbb{R}$ such that $(U, J, \overline{g} = e^{2f} g)$ is para-Kählerian. An almost para-Hermitian manifold (M, J, g) is locally conformally para-Kählerian if and only if it is para-Hermitian and there is a 1-form ω on M satisfying the conditions $d\Omega = 2\omega \wedge \Omega$ and $d\omega = 0$ [**3**]. [Note when $\dim M = 2n \geqslant 6$, the second condition $d\omega = 0$ follows from the first one $d\Omega = 2\omega \wedge \Omega$. When $\dim M = 4$, the condition $d\Omega = 2\omega \wedge \Omega$ is automatically fulfilled with a certain unique 1-form ω, however $d\omega \neq 0$ in general. An explicit example of a 4-dimensional para-Hermitian manifold with $d\omega \neq 0$ can be found in [**6**].]

3. Semidirect products of Lie groups. Let $G = G_0 \times_f G_1$ be a Lie group which is a semidirect product of two n-dimensional Lie groups G_0 and G_1. This means that the underlying manifold of G is just the product manifold $G_0 \times G_1$ and there exists a smooth map $f : G_0 \times G_1 \to G_1$ such that $p_0 \mapsto f(p_0, \cdot)$ is a homomorphism of G_0 into the abstract group of automorphisms of G_1 (i.e., G_0 acts on G_1 by automorphisms) such that the multiplication and inversion are given in G by

$$\begin{aligned} (p_0, p_1)(q_0, q_1) &= (p_0 q_0, f(q_0^{-1}, p_1) q_1), \\ (p_0, p_1)^{-1} &= (p_0^{-1}, f(p_0, p_1^{-1})) \end{aligned}$$

for $p_0, q_0 \in G_0$ and $p_1, q_1 \in G_1$ (see e.g. [**4**]). In the case when $f(p_0, \cdot)$ is the identity automorphism of G_1 for any $p_0 \in G_0$, G becomes just the product Lie group $G_0 \times G_1$.

Denote by $\mathfrak{g}_0$, $\mathfrak{g}_1$, $\mathfrak{g}$, the Lie algebras of G_0, G_1, G, respectively. Then $\mathfrak{g}$ is the semidirect sum of the Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}_1$. As the linear space $\mathfrak{g}$ is identified with the direct sum of the linear spaces $\mathfrak{g}_0$ and $\mathfrak{g}_1$, and an element $X \in \mathfrak{g}$ we shall write as $X = X_0 + X_1$, where $X_0 \in \mathfrak{g}_0$ and $X_1 \in \mathfrak{g}_1$. Moreover, if $[\cdot,\cdot]_0$ and $[\cdot,\cdot]_1$ denote the Lie brackets in $\mathfrak{g}_0$ and $\mathfrak{g}_1$, respectively, then the Lie bracket in $\mathfrak{g}$ is given by

$$\begin{aligned} [X_0, Y_0] &= [X_0, Y_0]_0, \\ [X_0, Y_1] &= -[Y_1, X_0] = A_{X_0} Y_1, \\ [X_1, Y_1] &= [X_1, Y_1]_1 \end{aligned} \tag{3.1}$$

for $X_0, Y_0 \in \mathfrak{g}_0$, $X_1, Y_1 \in \mathfrak{g}_1$, where $A : \mathfrak{g}_0 \to \mathrm{Der}(\mathfrak{g}_1)$ is a Lie algebra homomorphism of $\mathfrak{g}_0$ into $\mathrm{Der}(\mathfrak{g}_1)$ (the Lie algebra of derivations of $\mathfrak{g}_1$). This semidirect sum we shall denote by $\mathfrak{g} = \mathfrak{g}_0 \oplus_A \mathfrak{g}_1$. In the sequel, for simplicity, instead of $[\cdot,\cdot]_0$ and $[\cdot,\cdot]_1$ we shall write just $[\cdot,\cdot]$.

Since for each $X_0 \in \mathfrak{g}_0$, A_{X_0} is a derivation of $\mathfrak{g}_1$, we have

$$A_{X_0}[Y_1, Z_1] = [A_{X_0} Y_1, Z_1] + [Y_1, A_{X_0} Z_1] \tag{3.2}$$

for $Y_1, Z_1 \in \mathfrak{g}_1$; and since the mapping $\mathfrak{g}_0 \ni X_0 \mapsto A_{X_0} \in \mathrm{Der}(\mathfrak{g}_1)$ is a Lie algebra homomorphism,

$$A_{[X_0, Y_0]} Z_1 = A_{X_0} A_{Y_0} Z_1 - A_{Y_0} A_{X_0} Z_1 \tag{3.3}$$

for $X_0, Y_0 \in \mathfrak{g}_0$ and $Z_1 \in \mathfrak{g}_1$.

Define a left invariant almost paracomplex structure J on G by assuming

$$J|_{\mathfrak{g}_0} = -\mathrm{Id}|_{\mathfrak{g}_0}, \qquad J|_{\mathfrak{g}_1} = \mathrm{Id}|_{\mathfrak{g}_1}.$$

By (3.1), the eigen-distributions $T^- = \mathfrak{g}_0$ and $T^+ = \mathfrak{g}_1$ are both completely integrable, so that J is in fact paracomplex. Now take an arbitrary left invariant pseudo-Riemannian metric g of signature (n, n) on G for which $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are both isotropic, i.e., $g(X_0, Y_0) = 0$ for $X_0, Y_0 \in \mathfrak{g}_0$, and $g(X_1, Y_1) = 0$ for $X_1, Y_1 \in g_1$. Under these assumptions, for $X = X_0 + X_1, Y = Y_0 + Y_1 \in \mathfrak{g} = \mathfrak{g}_0 \oplus_A \mathfrak{g}_1$, we have

$$g(JX, JY) = -g(X_0, Y_1) - g(X_1, Y_0) = -g(X, Y).$$

Consequently, the pair (J, g) is a left invariant para-Hermitian structure on G (cf. [**7**], [**8**]).

In the sequel, we will consider only the structures (J, g) defined as above, and we will call such a structure a natural left invariant para-Hermitian structure on the semidirect product $G = G_0 \times_f G_1$. Moreover, if it is not otherwise stated, $X_0, Y_0, \ldots$ and $X_1, Y_1, \ldots$ will always denote arbitrary elements of $\mathfrak{g}_0$ and $\mathfrak{g}_1$, respectively. Meanwhile, $X, Y, \ldots$ will be arbitrary elements of $\mathfrak{g}$.

By the above definition, the fundamental 2-form Ω of (J, g) is given by

$$\begin{aligned} \Omega(X_0, Y_0) &= 0, \\ \Omega(X_0, Y_1) &= -\Omega(Y_1, X_0) = -g(X_0, Y_1), \\ \Omega(X_1, Y_1) &= 0. \end{aligned} \tag{3.4}$$

The para-Hermitian metric g enables us to define, for any $X_0 \in \mathfrak{g}_0$, in some sense "conjugate" linear endomorphism $A^c_{X_0}$ of the linear space $\mathfrak{g}_0$ by assuming

$$g(A^c_{X_0}Y_0, Z_1) = g(A_{X_0}Z_1, Y_0). \tag{3.5}$$

Of course, the mapping $X_0 \mapsto A^c_{X_0}$ is a linear mapping from $\mathfrak{g}_0$ into the space of the linear endomorphisms of $\mathfrak{g}_0$.

With the help of A^c, we can formulate the following theorems:

THEOREM 1 ([7], Th. 3.2). *A natural para-Hermitian structure (J, g) on a semidirect product of Lie groups $G = G_0 \times_f G_1$ is para-Kählerian if and only if $\mathfrak{g}_1$ is Abelian and*

$$[X_0, Y_0] = -A^c_{X_0}Y_0 + A^c_{Y_0}X_0. \tag{3.6}$$

THEOREM 2 ([7], Th. 3.3). *A natural para-Hermitian structure (J, g) on a semidirect product of Lie groups $G = G_0 \times_f G_1$ is locally conformally para-Kählerian if and only if there exists a left invariant 1-form ω on G such that*

$$[X_0, Y_0] = -A^c_{X_0}Y_0 + A^c_{Y_0}X_0 - 2\omega(X_0)Y_0 + 2\omega(Y_0)X_0, \tag{3.7}$$

$$[X_1, Y_1] = -2\omega(X_1)Y_1 + 2\omega(Y_1)X_1, \tag{3.8}$$

$$\omega([X_0, Y_0]) = 0, \tag{3.9}$$

$$\omega(A_{X_0}Y_1) = 0. \tag{3.10}$$

In Theorem 2, form ω is just the form fulfilling the conditions $d\Omega = 2\omega \wedge \Omega$ and $d\omega = 0$. Moreover, relations (3.7), (3.8) correspond to the condition $d\Omega = 2\omega \wedge \Omega$; and relations (3.9), (3.10) correspond to the condition $d\omega = 0$. Therefore, (i) in the case when $\dim G \geqslant 6$, conditions (3.9), (3.10) follow from (3.7), (3.8); and (ii) in the case when $\dim G = 4$, conditions (3.7), (3.8) are fulfilled automatically with some unique left invariant 1-form ω, so we have only to assume that this form satisfies (3.9), (3.10).

4. Main results. Let (J, g) be a natural para-Hermitian structure on a semidirect product of Lie groups $G = G_0 \times_f G_1$. As in the previous section, by $\mathfrak{g}_0$, $\mathfrak{g}_1$, $\mathfrak{g} = \mathfrak{g}_0 \times_A \mathfrak{g}_1$ we denote the Lie algebras of G_0, G_1, G, respectively.

Moreover, we assume that $\dim G_0 = \dim G_1 = 2$. Thus, there are left invariant 1-forms σ_0 and σ_1 on $\mathfrak{g}_0$ and $\mathfrak{g}_1$, respectively, such that

$$[X_0, Y_0] = -\sigma_0(X_0)Y_0 + \sigma_0(Y_0)X_0 \quad \text{for} \quad X_0, Y_0 \in \mathfrak{g}_0, \tag{4.1}$$

$$[X_1, Y_1] = -\sigma_1(X_1)Y_1 + \sigma_1(Y_1)X_1 \quad \text{for} \quad X_1, Y_1 \in \mathfrak{g}_1. \tag{4.2}$$

Let ω be the left invariant 1-form in $\mathfrak{g}$ satisfying the condition $d\Omega = 2\omega \wedge \Omega$. Thus, the form ω fulfils (3.7) and (3.8). As we have already mentioned, such a form always exists and it is unique. However, the form ω is not closed in general. Essentially, in this section we shall prove that if the metric g is conformally flat, then ω is closed and consequently the structure (J, g) is locally conformally para-Kählerian.

Let (E_1, E_2, E_3, E_4) denote an arbitrary basis in $\mathfrak{g}$ which is adapted to the structure (J, g). This means that (E_1, E_2) is a basis of $\mathfrak{g}_0$ and (E_3, E_4) is a basis of $\mathfrak{g}_1$ such that

$$g(E_\alpha, E_\beta) = g(E_{\alpha'}, E_{\beta'}) = 0, \quad g(E_\alpha, E_{\beta'}) = \delta_{\alpha\beta'},$$
$$JE_\alpha = -E_\alpha, \quad JE_{\alpha'} = E_{\alpha'}.$$

In this section, the Greek indices take on values 1, 2; and $\alpha' = \alpha+2$ if $1 \leqslant \alpha \leqslant 2$. Suppose

$$\sigma_\alpha = \sigma_0(E_\alpha), \qquad \sigma_{\alpha'} = \sigma_1(E_{\alpha'}),$$
$$\omega_\alpha = \omega(E_\alpha), \qquad \omega_{\alpha'} = \omega(E_{\alpha'}),$$
$$A_{E_\alpha} E_{\beta'} = \sum_\lambda A^{\lambda'}_{\alpha\beta'} E_{\lambda'},$$
$$A^c_{E_\alpha} E_\beta = \sum_\lambda A^{\lambda}_{\alpha\beta} E_\lambda.$$

By virtue of (3.5), the components of the operators A^c and A are related by $A^\gamma_{\alpha\beta} = A^{\beta'}_{\alpha\gamma'}$. Therefore, we can write

$$A_{E_\alpha} E_{\beta'} = \sum_\lambda A^{\beta}_{\alpha\lambda} E_{\lambda'}. \tag{4.3}$$

Lemma 1. *With respect to an adapted basis, the components of A and σ satisfy the following relations*

$$\sum_\lambda A^{\beta}_{\alpha\lambda}\sigma_{\lambda'} = 0, \tag{4.4}$$

$$\sigma_\alpha A^\varepsilon_{\beta\gamma} - \sigma_\beta A^\varepsilon_{\alpha\gamma} = \sum_\lambda \left(A^\varepsilon_{\alpha\lambda} A^\lambda_{\beta\gamma} - A^\varepsilon_{\beta\lambda} A^\lambda_{\alpha\gamma} \right). \tag{4.5}$$

Proof. Using (4.3) and (4.2), from (3.2) with $X_0 = E_\alpha$, $Y_1 = E_{\beta'}$, $Z_1 = E_{\gamma'}$, we find that

$$\Big(\sum_\lambda A^{\beta}_{\alpha\lambda}\sigma_{\lambda'}\Big) E_{\gamma'} - \Big(\sum_\lambda A^{\gamma}_{\alpha\lambda}\sigma_{\lambda'}\Big) E_{\beta'} = 0,$$

which is implies (4.4). In a similar way, using (4.3) and (4.1), from (3.3) with $X_0 = E_\alpha$, $Y_0 = E_\beta$, $Z_1 = E_{\gamma'}$, we obtain (4.5). □

LEMMA 2. *With respect to an adapted basis, the components of the form* ω *can be expressed as follows*

$$\omega_\alpha = \frac{1}{2}\sigma_\alpha + \frac{1}{2}\sum_\lambda \left(A^\lambda_{\lambda\alpha} - A^\lambda_{\alpha\lambda}\right), \tag{4.6}$$

$$\omega_{\alpha'} = \frac{1}{2}\sigma_{\alpha'}. \tag{4.7}$$

PROOF. With respect to an adapted basis, conditions (3.7) and (3.8) can be rewritten in the following form

$$-\sigma_\alpha\delta^\gamma_\beta + \sigma_\beta\delta^\gamma_\alpha = -A^\gamma_{\alpha\beta} + A^\gamma_{\beta\alpha} - 2\omega_\alpha\delta^\gamma_\beta + 2\omega_\beta\delta^\gamma_\alpha$$
$$-\sigma_{\alpha'}\delta^{\gamma'}_{\beta'} + \sigma_{\beta'}\delta^{\gamma'}_{\alpha'} = -2\omega_{\alpha'}\delta^{\gamma'}_{\beta'} + 2\omega_{\beta'}\delta^{\gamma'}_{\alpha'}.$$

By applying suitable contractions, the above relations enable us to express the components of ω with the help of the structure constants, as it is given in (4.6) and (4.7). □

LEMMA 3. *The natural left invariant para-Hermitian structure* (J, g) *is locally conformally para-Kählerian if and only if*

$$\sigma_\alpha \sum_\lambda \left(A^\lambda_{\lambda\beta} - A^\lambda_{\beta\lambda}\right) - \sigma_\beta \sum_\lambda \left(A^\lambda_{\lambda\alpha} - A^\lambda_{\alpha\lambda}\right) = 0. \tag{4.8}$$

PROOF. As we already know, the structure (J, g) is locally conformally para-Kählerian if and only if $d\omega = 0$; cf. Theorem 2. Therefore, we compute the components $d\omega(E_i, E_j)$. At first, by (4.1) and (4.6), we have

$$\begin{aligned} 2d\omega(E_\alpha, E_\beta) &= -\omega([E_\alpha, E_\beta]) = -\omega(-\sigma_\alpha E_\beta + \sigma_\beta E_\alpha) = \sigma_\alpha\omega_\beta - \sigma_\beta\omega_\alpha \\ &= \frac{1}{2}\left(\sigma_\alpha \sum_\lambda \left(A^\lambda_{\lambda\beta} - A^\lambda_{\beta\lambda}\right) - \sigma_\beta \sum_\lambda \left(A^\lambda_{\lambda\alpha} - A^\lambda_{\alpha\lambda}\right)\right). \end{aligned}$$

Next, using relations $[E_\alpha, E_{\beta'}] = A_{E_\alpha}E_{\beta'}$, (4.3), (4.7) and (4.4), we find

$$\begin{aligned} 2d\omega(E_\alpha, E_{\beta'}) &= -\omega([E_\alpha, E_{\beta'}]) = -\omega(A_{E_\alpha}E_{\beta'}) \\ &= -\sum_\lambda A^\beta_{\alpha\lambda}\omega_{\lambda'} = -\frac{1}{2}\sum_\lambda \sigma_{\lambda'}A^\beta_{\alpha\lambda} = 0. \end{aligned}$$

Finally, in view of (4.2) and (4.7), we obtain

$$\begin{aligned} 2d\omega(E_{\alpha'}, E_{\beta'}) &= -\omega([E_{\alpha'}, E_{\beta'}]) = -\omega(-\sigma_{\alpha'}E_{\beta'} + \sigma_{\beta'}E_{\alpha'}) \\ &= \sigma_{\alpha'}\omega_{\beta'} - \sigma_{\beta'}\omega_{\alpha'} = 0. \end{aligned}$$

By virtue of the above, the assertion of our lemma follows. □

Now, we prove the main result of the present paper. As it was mentioned in the introduction, if a natural left invariant para-Hermitian structure is locally conformally para-Kählerian, then it is conformally flat. So we need only to prove the converse to the above. However, as one can easily remark, our proof is self-contained.

THEOREM 3. *Let (J, g) be a natural left invariant para-Hermitian structure on a 4-dimensional semidirect product of Lie groups $G_0 \times_f G_1$. Then the para-Hermitian metric g is conformally flat if and only if the structure (J, g) is locally conformally para-Kählerian.*

PROOF. The proof is completely technical. Namely, we compute the components of the Weyl conformal curvature tensor and show that they vanish if and only if the structure (J, g) is locally conformally para-Kählerian.

For the Riemann curvature tensor R, the Ricci curvature tensor ϱ, and the scalar curvature τ, we apply the following conventions

$$\begin{aligned} R(X,Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \\ R(X,Y,Z,W) &= g(R(X,Y)Z, W), \\ \varrho(Y,Z) &= \operatorname{Trace}\{X \mapsto R(X,Y)Z\}, \\ \tau &= \operatorname{Trace}_g \varrho, \end{aligned}$$

∇ being the Levi-Civita connection of the Riemannian metric. Denote by R_{hijk} and ϱ_{ij}, respectively, the components of tensors R and ϱ with respect to an adapted basis.

At first, we derive the Levi-Civita connection

$$\begin{aligned} \nabla_{E_\alpha} E_\beta &= \frac{1}{2}\sum_\lambda \big(- \sigma_\alpha \delta_\beta^\lambda + \sigma_\beta \delta_\alpha^\lambda - A_{\alpha\beta}^\lambda - A_{\beta\alpha}^\lambda \big) E_\lambda, \\ \nabla_{E_\alpha} E_{\beta'} &= \frac{1}{2}\sum_\lambda \big(\sigma_{\beta'} \delta_\alpha^\lambda - \sigma_{\lambda'} \delta_\alpha^\beta \big) E_\lambda \\ &\quad + \frac{1}{2}\sum_\lambda \big(A_{\alpha\lambda}^\beta + A_{\lambda\alpha}^\beta - \sigma_\lambda \delta_\alpha^\beta + \sigma_\alpha \delta_\beta^\lambda \big) E_{\lambda'}, \\ \nabla_{E_{\alpha'}} E_\beta &= \frac{1}{2}\sum_\lambda \big(\sigma_{\alpha'} \delta_\beta^\lambda - \sigma_{\lambda'} \delta_\alpha^\beta \big) E_\lambda \\ &\quad + \frac{1}{2}\sum_\lambda \big(- A_{\beta\lambda}^\alpha + A_{\lambda\beta}^\alpha - \sigma_\lambda \delta_\alpha^\beta + \sigma_\beta \delta_\alpha^\lambda \big) E_{\lambda'}, \\ \nabla_{E_{\alpha'}} E_{\beta'} &= \frac{1}{2}\sum_\lambda \big(- \sigma_{\alpha'} \delta_\beta^\lambda + \sigma_{\beta'} \delta_\alpha^\lambda \big) E_{\lambda'}. \end{aligned}$$

Next, we find the components of the curvature tensor

$$
\begin{aligned}
R_{\alpha\beta\gamma\varepsilon} &= 0,\\
R_{\alpha\beta\gamma\varepsilon'} &= \frac{1}{4}\sum_{\lambda}\left(A^{\lambda}_{\beta\gamma}+A^{\lambda}_{\gamma\beta}\right)\left(A^{\varepsilon}_{\lambda\alpha}+A^{\varepsilon}_{\alpha\lambda}\right)-\frac{1}{4}\sum_{\lambda}\left(A^{\lambda}_{\alpha\gamma}+A^{\lambda}_{\gamma\alpha}\right)\left(A^{\varepsilon}_{\lambda\beta}+A^{\varepsilon}_{\beta\lambda}\right)\\
&\quad -\frac{1}{4}\delta^{\varepsilon}_{\alpha}\sum_{\lambda}\left(A^{\lambda}_{\beta\gamma}+A^{\lambda}_{\gamma\beta}\right)\sigma_{\lambda}+\frac{1}{4}\delta^{\varepsilon}_{\beta}\sum_{\lambda}\left(A^{\lambda}_{\alpha\gamma}+A^{\lambda}_{\gamma\alpha}\right)\sigma_{\lambda}\\
&\quad -\frac{1}{2}\left(A^{\varepsilon}_{\beta\gamma}+A^{\varepsilon}_{\gamma\beta}\right)\sigma_{\alpha}+\frac{1}{2}\left(A^{\varepsilon}_{\alpha\gamma}+A^{\varepsilon}_{\gamma\alpha}\right)\sigma_{\beta}+\frac{1}{4}\delta^{\varepsilon}_{\beta}\sigma_{\alpha}\sigma_{\gamma}-\frac{1}{4}\delta^{\varepsilon}_{\alpha}\sigma_{\beta}\sigma_{\gamma},\\
R_{\alpha\beta\gamma'\varepsilon'} &= \frac{1}{4}\delta^{\varepsilon}_{\alpha}\sum_{\lambda}A^{\gamma}_{\lambda\beta}\sigma_{\lambda'}-\frac{1}{4}\delta^{\gamma}_{\alpha}\sum_{\lambda}A^{\varepsilon}_{\lambda\beta}\sigma_{\lambda'}+\frac{1}{4}\delta^{\gamma}_{\beta}\sum_{\lambda}A^{\varepsilon}_{\lambda\alpha}\sigma_{\lambda'}\\
&\quad -\frac{1}{4}\delta^{\varepsilon}_{\beta}\sum_{\lambda}A^{\gamma}_{\lambda\alpha}\sigma_{\lambda'}+\frac{1}{2}\left(\delta^{\gamma}_{\alpha}\delta^{\varepsilon}_{\beta}-\delta^{\varepsilon}_{\alpha}\delta^{\gamma}_{\beta}\right)\sum_{\lambda}\sigma_{\lambda}\sigma_{\lambda'}\\
&\quad -\frac{1}{4}\delta^{\varepsilon}_{\beta}\sigma_{\alpha}\sigma_{\gamma'}-\frac{1}{4}\delta^{\gamma}_{\alpha}\sigma_{\beta}\sigma_{\varepsilon'}+\frac{1}{4}\delta^{\varepsilon}_{\alpha}\sigma_{\beta}\sigma_{\gamma'}+\frac{1}{4}\delta^{\gamma}_{\beta}\sigma_{\alpha}\sigma_{\varepsilon'},\\
R_{\alpha\beta'\gamma\varepsilon'} &= \frac{1}{4}\delta^{\varepsilon}_{\alpha}\sum_{\lambda}A^{\beta}_{\lambda\gamma}\sigma_{\lambda'}+\frac{1}{4}\delta^{\beta}_{\gamma}\sum_{\lambda}A^{\varepsilon}_{\lambda\alpha}\sigma_{\lambda'}-\frac{1}{2}\delta^{\varepsilon}_{\alpha}\delta^{\beta}_{\gamma}\sum_{\lambda}\sigma_{\lambda}\sigma_{\lambda'}\\
&\quad +\frac{1}{4}\delta^{\varepsilon}_{\alpha}\sigma_{\gamma}\sigma_{\beta'}+\frac{1}{4}\delta^{\beta}_{\gamma}\sigma_{\alpha}\sigma_{\varepsilon'},\\
R_{\alpha\beta'\gamma'\varepsilon'} &= \frac{1}{4}\delta^{\gamma}_{\alpha}\sigma_{\beta'}\sigma_{\varepsilon'}-\frac{1}{4}\delta^{\varepsilon}_{\alpha}\sigma_{\beta'}\sigma_{\gamma'},\\
R_{\alpha'\beta'\gamma'\varepsilon'} &= 0.
\end{aligned}
$$

In the above, we have also applied relation (4.4).

Consequently, the components of the Ricci tensor are

$$
\begin{aligned}
\varrho_{\beta\gamma} &= \frac{1}{2}\sum_{\lambda,\mu}\left(A^{\lambda}_{\beta\gamma}+A^{\lambda}_{\gamma\beta}\right)\left(A^{\mu}_{\lambda\mu}+A^{\mu}_{\mu\lambda}\right)-\frac{1}{2}\sum_{\lambda,\mu}\left(A^{\lambda}_{\mu\gamma}+A^{\lambda}_{\gamma\mu}\right)\left(A^{\mu}_{\lambda\beta}+A^{\mu}_{\beta\lambda}\right)\\
&\quad -\frac{3}{2}\sum_{\lambda}\left(A^{\lambda}_{\beta\gamma}+A^{\lambda}_{\gamma\beta}\right)\sigma_{\lambda}+\frac{1}{2}\sigma_{\beta}\sum_{\lambda}\left(A^{\lambda}_{\lambda\gamma}+A^{\lambda}_{\gamma\lambda}\right)\\
&\quad +\frac{1}{2}\sigma_{\gamma}\sum_{\lambda}\left(A^{\lambda}_{\lambda\beta}+A^{\lambda}_{\beta\lambda}\right)-\frac{1}{2}\sigma_{\beta}\sigma_{\gamma},\\
\varrho_{\beta\gamma'} &= \frac{1}{2}\sum_{\lambda}A^{\gamma}_{\lambda\beta}\sigma_{\lambda'}+\frac{1}{2}\delta^{\gamma}_{\beta}\sum_{\lambda,\mu}A^{\mu}_{\lambda\mu}\sigma_{\lambda'}-\delta^{\gamma}_{\beta}\sum_{\lambda}\sigma_{\lambda}\sigma_{\lambda'}+\frac{1}{2}\sigma_{\beta}\sigma_{\gamma'},\\
\varrho_{\beta'\gamma'} &= -\frac{1}{2}\sigma_{\beta'}\sigma_{\gamma'},
\end{aligned}
$$

and the scalar curvature is

$$\tau = 3\Big(\sum_{\lambda,\mu} A^{\mu}_{\lambda\mu}\sigma_{\lambda'} - \sum_{\lambda}\sigma_\lambda\sigma_{\lambda'}\Big).$$

Finally, having the above formulas and knowing that the coefficients A^k_{ij} and σ_i must satisfy relations (4.4) and (4.5), we compute the components of the Weyl conformal curvature tensor W, which are given by

$$W_{hijk} = R_{hijk} - \frac{1}{2}(\varrho_{hk}g_{ij} - \varrho_{hj}g_{ik} + g_{hk}\varrho_{ij} - g_{hj}\varrho_{ik}) + \frac{\tau}{6}(g_{hk}g_{ij} - g_{hj}g_{ik}).$$

After certain long but standard calculations, we obtain

$$W_{1213} = W_{1224} = \frac{1}{4}((A^1_{21} - A^1_{12})\sigma_1 + (A^2_{21} - A^2_{12})\sigma_2),$$

and $W_{hijk} = 0$ in other cases. This fact and Lemma 3 enable as to conclude that $W = 0$ if and only if (J, g) is locally conformally para-Kählerian. □

5. A 6-dimensional example. In this section, we construct an example of a 6-dimensional left invariant conformally flat para-Hermitian structure, which is not locally conformally para-Kählerian. This will be a natural left invariant para-Hermitian structure on a semidirect product $\mathrm{SU}(2) \times_f \mathbb{R}^3$.

Let (E_1, E_2, E_3) be a basis in $\mathfrak{g}_0 = \mathfrak{su}(2)$ and (E_4, E_5, E_6) an arbitrary basis in $\mathfrak{g}_1 = \mathbb{R}^3$ such that

$$(5.1)\qquad [E_1, E_2] = E_3, \qquad [E_2, E_3] = E_1, \qquad [E_3, E_1] = E_2,$$

$$(5.2)\qquad [E_4, E_5] = [E_5, E_6] = [E_6, E_4] = 0.$$

Define a Lie algebra homomorphism $A : \mathfrak{su}(2) \to \mathrm{Der}(\mathbb{R}^3)$ by assuming

$$(5.3)\qquad \begin{array}{lll} A_{E_1}E_4 = 0, & A_{E_1}E_5 = E_6, & A_{E_1}E_6 = -E_5, \\ A_{E_2}E_4 = -E_6, & A_{E_2}E_5 = 0, & A_{E_2}E_6 = E_4, \\ A_{E_3}E_4 = E_5, & A_{E_3}E_5 = -E_4, & A_{E_3}E_6 = 0. \end{array}$$

(one can easily verify that A in fact satisfies conditions (3.2) and (3.3)).

Let $\mathfrak{g} = \mathfrak{su} \oplus_A \mathbb{R}^3$ be the semidirect sum of the Lie algebras $\mathfrak{su}(2)$ and $\mathbb{R}^3$; and let $G = \mathrm{SU}(2) \times_f \mathbb{R}^3$ be the semidirect product of the Lie groups corresponding to that algebra. In virtue of (5.1), (5.2) and (5.3), the Lie brackets in $\mathfrak{g}$ are given by

$$(5.4)\qquad \begin{array}{lll} [E_1, E_2] = E_3, & [E_2, E_3] = E_1, & [E_3, E_1] = E_2, \\ [E_1, E_4] = 0, & [E_1, E_5] = E_6, & [E_1, E_6] = -E_5, \\ [E_2, E_4] = -E_6, & [E_2, E_5] = 0, & [E_2, E_6] = E_4, \\ [E_3, E_4] = E_5, & [E_3, E_5] = -E_4, & [E_3, E_6] = 0, \\ [E_4, E_5] = 0, & [E_5, E_6] = 0, & [E_6, E_4] = 0. \end{array}$$

Let (J, g) be the natural para-Hermitian structure on G for which

$$JE_\alpha = -E_\alpha, \qquad JE_{\alpha'} = E_{\alpha'},$$
$$g(E_\alpha, E_\beta) = g(E_{\alpha'}, E_{\beta'}) = 0, \quad g(E_\alpha, E_{\beta'}) = \delta_{\alpha\beta}, \tag{5.5}$$

where $\alpha' = \alpha + 3$, $\beta' = \beta + 3$, $1 \leqslant \alpha, \beta \leqslant 3$.

Using Theorem 2, we shall see that (J, g) is not locally conformally para-Kählerian. Indeed, with the help of (3.5) and (5.3), we can see that the conjugate operator A^c must be given by

$$\begin{array}{lll} A^c_{E_1} E_1 = 0, & A^c_{E_1} E_2 = -E_3, & A^c_{E_1} E_3 = E_2, \\ A^c_{E_2} E_1 = E_3, & A^c_{E_2} E_2 = 0, & A^c_{E_2} E_3 = -E_1, \\ A^c_{E_3} E_1 = -E_2, & A^c_{E_3} E_2 = E_1, & A^c_{E_3} E_3 = 0. \end{array} \tag{5.6}$$

Suppose that our structure fulfils (3.7) with a certain 1-form ω. This for $X_0 = E_1$, $Y_0 = E_2$ gives

$$[E_1, E_2] = -A^c_{E_1} E_2 + A^c_{E_2} E_1 - 2\omega(E_1)E_2 + 2\omega(E_2)E_1,$$

or by (5.6)

$$E_3 = 2E_3 - 2\omega(E_1)E_2 + 2\omega(E_2)E_1.$$

But this is an obvious contradiction.

Now, we compute the conformal curvature tensor. At first, in view of (5.4) and (5.5), the Levi-Civita connection of g is given by

$$\begin{aligned}
\nabla_{E_2} E_3 = -\nabla_{E_3} E_2 &= \frac{1}{2} E_1, \\
\nabla_{E_1} E_3 = -\nabla_{E_3} E_1 &= -\frac{1}{2} E_2, \\
\nabla_{E_1} E_2 = -\nabla_{E_2} E_1 &= \frac{1}{2} E_3, \\
\nabla_{E_2} E_6 = -\nabla_{E_3} E_5 &= \nabla_{E_5} E_3 = -\nabla_{E_6} E_2 = \frac{1}{2} E_4, \\
\nabla_{E_1} E_6 = -\nabla_{E_3} E_4 &= \nabla_{E_4} E_3 = -\nabla_{E_6} E_1 = -\frac{1}{2} E_5, \\
\nabla_{E_1} E_5 = -\nabla_{E_2} E_4 &= \nabla_{E_4} E_2 = -\nabla_{E_5} E_1 = \frac{1}{2} E_6,
\end{aligned}$$

otherwise $\nabla_{E_i} E_j = 0$. Therefore, the nonzero components of the Riemann curvature tensor are related to

$$R_{1215} = -R_{1224} = R_{1316} = -R_{1334} = R_{2326} = -R_{2335} = -\frac{1}{4}.$$

Further, for the Ricci curvature tensor, we have

$$\varrho_{11} = \varrho_{22} = \varrho_{33} = 1,$$

otherwise $\rho_{ij} = 0$; and the scalar curvature $\tau = 0$. In view of the above, it is a straighforward verification that the components W_{hijk} of the Weyl conformal curvature tensor vanish identically. Thus, g is conformally flat.

References

1. Cruceanu V., Fortuny P., Gadea P. M., *A survey on paracomplex geometry*, Rocky Mountain J. Math. **26** (1996), 83–115.
2. Cruceanu V., Gadea P. M., Muñoz Masqué J., *Para-Hermitian and para-Kähler manifolds*, Quaderni Inst. Mat., Fac. Econom., Univ. Messina **1** (1995), 1–72.
3. Gadea P. M., Muñoz Masqué J., *Classification of almost para-Hermitian manifolds*, Rend. Mat. Roma (7) **11** (1991), 377–396.
4. Knapp A. W., *Lie groups beyond an introduction*, Progress in Mathematics, **140**, Birkhäuser, Boston–Basel–Berlin, 1996.
5. Olszak Z., *On conformally flat parakählerian manifolds*, Math. Balkanica N. S. **5** (1991), 302–307.
6. Olszak Z., *Four-dimensional para-Hermitian manifolds*, Tensor N. S. **56** (1995), 215–226.
7. Olszak Z., *Left invariant para-Hermitian structures on semidirect products of Lie groups*, Tensor N. S., in print.
8. Olszak Z., *On the curvature of left invariant locally conformally para-Kählerian metrics*, Geometry and Topology of Submanifolds, IX, World Sci. Publishing, Singapore, (1999), 198–207.

Received December 17, 1999

Instytut Matematyki
Politechnika Wrocławska
Wybrzeże Wyspiańskiego 27
50-370 Wrocław
Poland
e-mail: `olszak@im.pwr.wroc.pl`