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EXAMPLES OF EXOTIC MODULI IN LOCAL CLASSIFICATION OF GOURSAT FLAGS

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Abstract. A distribution D of corank $r \geq 2$ on a manifold W is Goursat when its Lie square $[D,\, D]$ is a distribution of constant corank $r-1$, the Lie square of $[D,\, D]$ is of constant corank $r-2$ and so on, the '2^{r-1}th Lie power' of D – of constant corank 1, the '2^rth Lie power' of D – the whole tangent bundle TW.
Local classification of Goursat distributions has been advanced over the past 20 years, including continuous moduli for $r \geq 8$ falling within the fifth class of a recent geometric systematization [**8**]. In the note we present examples of 'exotic' moduli corresponding to the fourth class in [**8**], thus showing that all five possibilities separated in [**8**] really happen in the Goursat world.

1. Introduction. The note deals with Goursat flags – nested sequences of $r \geq 2$ (flags of length r) distributions in the tangent bundle TW to a (C^∞, or real analytic) manifold W of dimension $n \geq r+2$, every bigger one being the Lie square of the preceding and having by one bigger rank. That is, indexing the members of a flag by their coranks, $TW = D^0 \supset D^1 \supset D^2 \supset \cdots \supset D^r$, $D^{j-1} = D^j + [D^j,\, D^j]$, $\operatorname{rk} D^j = n - j$ for $j = 1, 2, \ldots, r$.
One says also that any flag member save D^1 is a *Goursat distribution.* Such a member clearly determines all members with smaller coranks; D^r determines the whole flag.

The purpose of the note is to give examples of the most complicated local behaviour – when passing from flags of length r to those of length $r+1$ –

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emanating from a geometric systematization recently proposed in [**8**]. Montgomery and Zhitomirskii have shedded light on a series of local classification results obtained so far for Goursat distributions by separating five distinctly different cases in the *local* prolongation from $D^1 \supset D^2 \supset \cdots \supset D^r$ to $D^1 \supset D^2 \supset \cdots \supset D^r \supset D^{r+1}$. In fact, reproducing their notation and denoting by $L(D^r)$ the characteristic subdistribution of D^r that, for any Goursat distribution, is of codimension 2 in D^r, any local symmetry, say ϕ, of D^r around a point p, $\phi_* D^r = D^r$, preserves $L(D^r)$, hence induces $[d\phi(p)] : D^r(p)/L(D^r)(p) \hookleftarrow$. In turn, $[d\phi(p)]$ clearly induces a projective automorphism of $S^1(D^r)(p) = P(D^r(p)/L(D^r)(p))$ denoted by g_ϕ; automorphisms of the real projective line are termed *projectivities*.
The group of all projectivities g_ϕ obtained in this way is denoted by $\Gamma_p(D^r)$. According to Prop. 3 of [**8**], the orbits of $\Gamma_p(D^r)$ acting on $S^1(D^r)(p)$ are in 1–1 correspondence with the equivalence classes of germs of D^{r+1} at p such that the germ of D^r is *fixed*. In the sequel we write shortly (when no ambiguity) Γ and S^1.

The analysis of orbits is done through the fixed points of Γ. From the very nature of Goursat flags it follows that Γ has always a fixed point $L = L(D^{r-1})(p)/L(D^r)(p) \in S^1$; it has sometimes also a second fixed point M, of possibly various geometric character. In such a case Montgomery and Zhitomirskii denote by $\sigma \in \Gamma$ the *reflection* in the line of one of the fixed points, along the direction of the other (the order of fixed points does not matter for this definition). At some other times, when $\Gamma = \{\,\mathrm{id}\,\}$, the whole S^1 consists of fixed points. [**8**] restricts the possibilities of Goursat prolongations to the five below, *not* precising which of them occurs for any given germ (D^r, p) (for the equivalance class of D^r at p, in fact).

I. First possibility: Γ has precisely 1 fixed point L. There are two orbits of the action of Γ on S^1: L and $S^1 \setminus \{\,L\,\}$.
II. Second possibility: Γ has precisely 2 fixed points and $\Gamma \supsetneq \{\,\mathrm{id},\ \sigma\,\}$. There are three orbits: L, M, and $S^1 \setminus \{\,L,\ M\,\}$.
III. Third possibility: Γ has precisely 2 fixed points and $\sigma \notin \Gamma$. There are four orbits: L, M, and the two connected components of $S^1 \setminus \{\,L,\ M\,\}$.
IV. Fourth possibility: $\Gamma = \{\,\mathrm{id},\ \sigma\,\}$. There are infinitely many orbits: L, M, and pairs $\{\,N,\ \sigma(N)\,\}$, $N \in S^1 \setminus \{\,L,\ M\,\}$.
V. Fifth possibility: all points of S^1 are fixed and $\Gamma = \{\,\mathrm{id}\,\}$ (any projectivity with at least three fixed points is identity).

Examples of situations in the Goursat world illustrating Possibilities I, II, III, and V exist in the recent literature and are supplied in [**8**]. It is not so with a quite exotic Possibility IV. In view of the works [**3**], [**2**], [**1**], [**4**], it assuredly does not occur in flags of lengths not greater than 7. We are going to describe two fairly different examples of Possibility IV occurring on the circle $S^1(D^r)(p)$ for $r = 8$ and $r = 9$, i. e., among Goursat flags of length 9 and 10.[1]

2. Example for $r = 8$ with several singular positions in the flag. The manifold with a distinguished point is $(\mathbb{R}^{11}, 0)$. A Goursat distribution D^9 will be given in a Kumpera–Ruiz pseudo-normal form [**3**] (see also [**1**]) using coordinates $x^1, x^2, \dots, x^{11}$ on $\mathbb{R}^{11}$.

The example is a two–step prolongation of either of the couple $(***)$ of non-equivalent Goursat germs of length 7 in Main Theorem in [**4**]. In members of that pair Possibility III (four orbits) shows up; the two orbits different from the fixed points are treated in Thm. [32332] in [**4**]. We prolong through a position $D^7(0)$ chosen in either of those orbits, then take the singular position

(1) $$D^8(0) = L(D^6)(0)\,.$$

We have, therefore, D^8 under the normal form $D^8 = (\omega^1, \omega^2, \dots, \omega^8)^{\perp}$, where $\omega^1 = dx^2 - x^3dx^1$, $\omega^2 = dx^3 - x^4dx^1$, $\omega^3 = dx^1 - x^5dx^4$, $\omega^4 = dx^5 - (1 + x^6)dx^4$, $\omega^5 = dx^4 - x^7dx^6$, $\omega^6 = dx^6 - x^8dx^7$, $\omega^7 = dx^8 - (a + x^9)dx^7$, $\omega^8 = dx^7 - x^{10}dx^9$

and $a = 1$ (a representative of one orbit consisting of all objects with $a > 0$) or $a = -1$ (a representative of the other orbit – all $a < 0$). This parameter will be fixed in our arguments.

Remark 1. The fact of prolonging through the singular position (1), instead of other possible positions of $D^8(0)$, is important. It will immediately help in computing, in the next prolongation, the second fixed point M. It is not that simple with M in the other example in Sec. 3, where we prolong not through a fixed point position.

A short calculus shows that, independently of a, the circle $S^1(D^8)(0)$ is equal to $P(\text{span}(\partial_9, \partial_{10}))$.

We compute the fixed points L and M, skipping here and in the sequel writing 'span' and using the square brackets for the points of the projective line. We start with, existing in any situation, $L = [L(D^7)(0)/L(D^8)(0)] = [(\partial_{10}, \partial_{11})/(\partial_{11})] = [\partial_{10}]$. Searching for M, we guess it as in [**8**], using a standard tool in singularities. In the context of Goursat it works always after a singular prolongation like the one in (1). After a computation, the locus, say Sing, of points $q \in \mathbb{R}^{11}$ where $D^8(q) = L(D^6)(q)$ holds, is just

[1] We do not know yet whether Possibility IV happens among flags of length 8. In particular, we do not know if D^7 – a member of the couple $(***)$ discussed below admits at all a second fixed point on $S^1(D^7)(0)$. Also, both examples will be in codimension 4. In codimension 1, by [**6**], only Possibilities I and II materialize. Does Possibility IV show already up in codimension 2 or 3?

$\{x^{10} = 0\}$, and we have, defined in invariant terms, a fixed point $M = [D^8(0) \cap T_0\mathrm{Sing}/L(D^8)(0)] = [(\partial_9,\, \partial_{11})/(\partial_{11})] = [\partial_9]$ different from L.
Now our actual prolongation from D^8 to D^9 consists in adding one more Pfaffian equation $\omega^9 = dx^{10} - (b + x^{11})dx^9 = 0$ depending on a real parameter b, so that $D^9_b = (\omega^1, \ldots, \omega^8, \omega^9)^\perp$. All points of $S^1 = P(\partial_9,\, \partial_{10})$ (i.e., possible positions $D^9_b(0)/L(D^8)(0)$) **excepting** L are parametrized by $b \in \mathbb{R}$. Indeed, $[D^9_b(0)/L(D^8)(0)] = [(\partial_9 + b\partial_{10},\, \partial_{11})/(\partial_{11})] = [\partial_9 + b\partial_{10}]$. In particular, for $b = 0$ we get M.

Observe that D^8 has the following simple symmetry ϕ, $\phi(x^1,\, x^2, \ldots, x^{11}) = (x^1,\, -x^2,\, -x^3,\, -x^4,\, -x^5,\, x^6,\, -x^7,\, -x^8,\, x^9,\, -x^{10},\, -x^{11})$. The induced projectivity g_ϕ sends $[A\partial_9 + B\partial_{10}]$ to $[A\partial_9 - B\partial_{10}]$. One recognizes in it the reflection σ with fixed points $[\partial_9]$ and $[\partial_{10}]$. Therefore,

$$\Gamma_0(D^8) \supset \{\,\mathrm{id},\ \sigma\,\}. \tag{2}$$

The symmetry ϕ is not taken at random; preserving D^8, it conjugates D^9_b and D^9_{-b} for every $b \in \mathbb{R}$. This is a starting observation. In fact, we are going to show

THEOREM 1. *In the family of germs at $0 \in \mathbb{R}^{11}$ of KR pseudo-normal forms D^9_b with fixed value of $a \in \{-1,\, 1\}$, the value $|b|$ is a module of local smooth, or real analytic, classification.*

COROLLARY 1. *There is equality in (2) and the prolongation from D^8 to D^9_b materializes Possibility IV of Sec. 1: in view of Thm. 1 and the existence of ϕ, there is infinitely many two-point orbits of $\Gamma_0(D^8)$.*

PROOF OF THEOREM 1. We write $X^6 = 1 + x^6$ and $X^9 = a + x^9$. Suppose that a local diffeomorphism $g = (g^1,\, g^2, \ldots, g^{11}) : (\mathbb{R}^{11},\, 0) \hookleftarrow$ conjugates D^9_c and $D^9_{\tilde c}$: $g_* D^9_c = D^9_{\tilde c}$. By general considerations related to conjugating by g all respective members of the two flags as well, we know from the beginning that g^l depends only on: $x^1,\, x^2,\, x^3$ when $l \le 3$, and on $x^1,\, x^2, \ldots, x^l$ when $4 \le l \le 11$ (the preservation by g of one and the same for both flags $L(D^8) = (\partial_{11})$ means that $g^1, \ldots, g^{10}$ depend only on $x^1, \ldots, x^{10}$; then one passes to the preservation by g of $L(D^7) = (\partial_{10},\, \partial_{11})$, etc). On the other hand, it is directly verifiable that the singular phenomenon $D^j(\cdot) = L(D^{j-2})(\cdot)$ for $j \in \{3,\, 5,\, 6,\, 8\}$ happens exactly at points of $\{\,x^{j+2} = 0\,\}$. (For these j's, in the 1–forms ω^j there are transpositions of indices that are responsible for the respective phenomena: dx^1 transposed with dx^4 in ω^3, dx^4 transposed with dx^6 in ω^5, etc.) Coupling these facts, we get that

$$g^j = x^j\, G^j(x^1,\, x^2, \ldots, x^j)\,, \qquad j \in \{5,\, 7,\, 8,\, 10\} \tag{3}$$

for certain smooth functions G^j such that $G^j\,|0 \neq 0$. Here and in the sequel we denote by $|0$ the evaluation at 0 of *no matter how long* expression φ:

$\varphi(0) = \varphi\,|0$; the last inequalities are a consequence of the invertibility of $Dg(0)$.

Looking at the Pfaffian equations above, we deduce from them that $D_b^9 = (\partial_{11}, Y_b)$, where $Y_b = \partial_9 + (b + x^{11})\partial_{10} + x^{10}(\partial_7 + X^9\partial_8 + x^8(\partial_6 + x^7(\partial_4 + X^6\partial_5 + x^5(\partial_1 + x^4\partial_3 + x^3\partial_2))))$. Now the fact that $g^1, g^2, \ldots, g^{10}$ do not depend on x^{11} implies that $Dg(0)\partial_{11}$ is a non-zero multiple of ∂_{11}. Hence, writing

$$Dg(x)Y_c(x) = f(x)\,Y_{\tilde{c}}(g(x)) + h(x)\,\partial_{11} \tag{4}$$

with certain function coefficients f and h, we obtain $f\,|0 \neq 0$. This basic information and (4) in general, together with (3), are going to be used many times; the aim is to deduce that $|c| = |\tilde{c}|$.

In the sequel we shall write simply $g^l_{\ k}$ for $\frac{\partial g^l}{\partial x^k}$. For instance, the inequality $\frac{\partial g^l}{\partial x^l}\,|0 \neq 0$ will henceforth be denoted $g^l_{\ l}\,|0 \neq 0$.

Taking (4) at 0, we get $Dg(0)(\partial_9 + c\partial_{10}) = f(0)(\partial_9 + \tilde{c}\partial_{10}) + h(0)\partial_{11}$. Comparing the coefficients at ∂_9 and ∂_{10}, we obtain

$$g^9_{\ 9}\,|0 = f\,|0 \qquad \text{and} \qquad c\,G^{10}\,|0 = \tilde{c}f\,|0\,. \tag{5}$$

Comparing in (4) the coefficients at ∂_7 and ∂_8, one sees that they can be divided sidewise by x^{10} (not a zero divisor): thanks to (3) for $j = 10$, the coefficients at $\partial_1, \partial_2, \ldots, \partial_8$ on the RHS of (4) are multiplicities of x^{10}. After removing x^{10}, we get

$$fG^{10} = \text{a function of } x^1, x^2, \ldots, x^8\,, \tag{6}$$

$$(fG^{10})^{-1}\left(x^8(*) + g^8_{\ 8}(a + x^9)\right) = a + g^9\,; \tag{7}$$

$(*)$ in (7) means a certain function whose explicit formula could be written using (3) for $j = 8$. Formula (7) evaluated at 0 reads

$$(fG^{10})^{-1}g^8_{\ 8}\,a\,|0 \;=\; a\,. \tag{8}$$

Now (6) and (7) imply $g^9_{\ 9}\,|0 = (fG^{10})^{-1}g^8_{\ 8}\,|0 = 1$ (by (8)). Hence $f\,|0 = 1$ by (5). As a byproduct, with $g^8_{\ 8}\,|0 = G^8\,|0$ ((3) again), we get

$$G^8\,|0 \;=\; G^{10}\,|0\,. \tag{9}$$

Comparing in (4) the coefficients at ∂_6, and then dividing sidewise by x^8x^{10} (it is explicit on the RHS thanks to (3) for $j = 8$ and 10), one obtains

$$g^6_{\ 6}\,|0 \;=\; fG^{10}G^8\,|0 \;=\; G^{10}G^8\,|0\,. \tag{10}$$

At last, the coefficients at ∂_4 and ∂_5 in (4), divided by $x^7x^8x^{10}$ (with (3) taken into account on the RHS's for $j = 7, 8, 10$), yield

$$fG^{10}G^8G^7 = \text{a function of } x^1, x^2, \ldots, x^5\,, \tag{11}$$

$$(12) \qquad (fG^{10}G^8G^7)^{-1}\left(x^5(*) + g_5^5(1+x^6)\right) = 1 + g^6\,,$$

with $(*)$ – a certain function expressable by G^5 (see (3)). The previous trick with g^9 can now be repeated. (12) evaluated at 0 is

$$(13) \qquad (fG^{10}G^8G^7)^{-1}g_5^5\,|0 = 1$$

while (11) and (12) imply $g_6^6\,|0 = (fG^{10}G^8G^7)^{-1}g_5^5\,|0 = 1$ (by (13)). Now (10) takes the form

$$(14) \qquad G^8G^{10}\,|0 = 1\,.$$

Finally, (9) and (14) imply $(G^{10}\,|0)^2 = 1$, and the second equation in (5) is reduced to $c\,(\pm1) = \tilde{c}$. The proof is finished.

3. Example for $r = 9$ with just one singular position in the flag. In the second example $r = 9$, the underlying manifold is $(\mathbb{R}^{12}, 0)$, $D^9 = (\omega^1, \omega^2, \ldots, \omega^9)^\perp$, where
$\omega^1 = dx^2 - x^3dx^1$, $\omega^2 = dx^3 - x^4dx^1$, $\omega^3 = dx^1 - x^5dx^4$, $\omega^4 = dx^5 - x^6dx^4$, $\omega^5 = dx^6 - x^7dx^4$, $\omega^6 = dx^7 - x^8dx^4$, $\omega^7 = dx^8 - (1+x^9)dx^4$, $\omega^8 = dx^9 - x^{10}dx^4$, $\omega^9 = dx^{10} - (a + x^{11})dx^4$

and $a = 1$, or else $a = -1$, is a parameter fixed in the whole section. This is another KR normal form whose real meaning will gradually become clear. This time $D^9(0)$ is not a fixed point position (*cf.* Rem. 1); $a = 0$ in ω^9 would give such a position; another such position would be given by a different ninth Pfaffian equation $dx^4 - x^{11}dx^{10} = 0$ added to the preceding eight equations. Prolonging not through a fixed point, it is all the more surprising that at the next prolongation we are bound to have two fixed points, and even more – IV possibility. The outcome of that next prolongation depends, as in the previous example in Sec. 2, on a real parameter b: $D_b^{10} = (\omega^1, \ldots, \omega^9, \omega^{10})^\perp$, $\omega^{10} = dx^{11} - (b + x^{12})dx^4$. We write explicitly its vector field generators, $D_b^{10} = (\partial_{12}, Y_b)$, using T for transpose:

$$Y_b^{\mathrm{T}} = \left[x^5,\, x^3x^5,\, x^4x^5,\, 1,\, x^6,\, x^7,\, x^8,\, 1+x^9,\, x^{10},\, a+x^{11},\, b+x^{12},\, 0\right].$$

THEOREM 2. *In the family of germs at $0 \in \mathbb{R}^{12}$ of Goursat distributions D_b^{10} with fixed value of $a \in \{-1, 1\}$, the value $|b|$ is a module of local smooth, or real analytic, classification.*

In the proof, we will try to conjugate D_c^{10} to $D_{\tilde{c}}^{10}$ by a diffeomorphism $g : (\mathbf{R}^{12}, 0) \hookleftarrow$, $g_*D_c^{10} = D_{\tilde{c}}^{10}$; $g = (g^1, g^2, \ldots, g^{12})$. Let us concentrate for a while on the one before last member of the flags, D^9 (common for both flags). It does not depend on x^{12}, $L(D^9) = (\partial_{12})$, and $(g^1, \ldots, g^{11})$ is a symmetry of D^9 considered on $\mathbb{R}^{11}(x^1, \ldots, x^{11})$ $(g^1, \ldots, g^{11}$ do not depend on x^{12}, similarly as in Sec. 2). Such a symmetry has much in common with the conjugacies of $(\omega^1, \ldots, \omega^5, dx^7 - (1+x^8)dx^4, dx^8 - x^9 dx^4, dx^9 - (a+x^{10})dx^4)$ and $(\omega^1, \ldots, \omega^5, dx^7 - (1+x^8)dx^4, dx^8 - x^9 dx^4, dx^9 - (\tilde{a}+x^{10})dx^4)$ in $\mathbb{R}^{10}$ analyzed in detail in [**5**], Thm. 3. Presently there is a longer sequence of Pfaffian equations after ω^3 with no constants in them (now ω^4, ω^5, ω^6; then ω^4, ω^5), but apart from that the differences are only secondary. The conclusion in [**5**] was that a and $\tilde{a}$ were necessarily of the same sign, and equal to 0 only simultaneously. Putting $a = \tilde{a}$ there, the conclusions concerning *any* conjugating diffeomorphism (in fact – symmetry then) keep holding. In the present work we take from the beginning $a = 1$ (or $a = -1$) in both flags, and draw conclusions concerning $g^1, \ldots, g^{11}$ very much similar to those in [**5**]. (Saying differently, we could prove a theorem generalizing [**5**], but our present objective is different – we are doing one Goursat prolongation 'beyond' Thm. 3 in [**5**]. In view of the remark terminating [**5**], this cannot be understood literally. A longer sequence of Pfaffian equations without constants is put forward to make all this work, *cf.* [**6**], Obs. 4.2.) It is not all – there is also g^{12} – but it is a lot. We will give all steps important in calculations, underlining technical differences with [**5**], but skipping the calculations themselves.

PROOF OF THEOREM 2. We start as in the proof of Thm. 1, noting the existence of certain functions f and h, $f\,|0 \neq 0$, such that

$$Dg(x)Y_c(x) = f(x)\,Y_{\tilde{c}}(g(x)) + h(x)\,\partial_{12}\,, \tag{15}$$

and knowing beforehand that the function coordinates g^1, g^2, g^3 depend only on x^1, x^2, x^3, and that, for $4 \leq l \leq 12$, the function g^l depends on $x^1, x^2, .., x^l$. Comparing coefficients at ∂_l in (15), we will say: taking the scalar equation "l" of (15).
Evaluating at 0 the scalar equation "11" of (15), we obtain

$$g^{11}_4 + g^{11}_8 + a\,g^{11}_{10} + c\,g^{11}_{11}\,|0 \;=\; \tilde{c}\,f\,|0\,. \tag{16}$$

PROPOSITION 1. $g^{11}_4 + g^{11}_8 + a\,g^{11}_{10}\,|0 \;=\; 0\,.$

A sketch of the proof of this proposition is given in Sec. 5.
How to compute $g^{11}_{11}\,|0$? g^{11} can be expressed from the scalar equation "10" of (15) in terms of g^{10} and f, and f – obtainable from the equation "4" of (15) – depends only on $x^1, \ldots, x^5$. Jointly, $g^{11}_{11}\,|0 = f^{-1}g^{10}_{10}\,|0$. $g^{10}_{10}\,|0$ is easily computable and equals $f^{-1}\,|0$ (the exact analogue of Claim in [**5**]). So, assuming Prop. 1, (16) boils down to

$$c\,f^{-2}\,|0 \;=\; \tilde{c}\,f\,|0\,. \tag{17}$$

How to grasp the value $f\,|0$? With a fixed in Thm. 2, there is not much choice for that value. Let us evaluate at 0 the equation "10" of (15):

$$g_4^{10} + g_8^{10} + a\,g_{10}^{10}\,|0 \;=\; a\,f\,|0\,. \tag{18}$$

PROPOSITION 2. $g_4^{10} + g_8^{10}\,|0 \;=\; 0\,.$

This is a direct analogue of Basic Lemma in [**5**]. A sketch of the proof of Prop. 2 is given in Sec. 4.

Assuming Prop. 2, (18) becomes $f^{-1}\,|0 \;=\; f\,|0$, meaning $f\,|0 = \pm 1$. Now (17) says that $c = \pm\,\tilde{c}$. The proof is finished.

Thm. 2 is the last word for the family D_b^{10}, $b \in \mathbb{R}$, because there exists a symmetry of D^9, Φ, sending D_b^{10} to D_{-b}^{10} for every b. Indeed, $\Phi(x^1,\, x^2, \dots,\, x^{12})$ $= (\,-x^1,\, -x^2,\, x^3,\, -x^4,\, x^5,\, -x^6,\, x^7,\, -x^8,\, x^9,\, -x^{10},\, x^{11},\, -x^{12}\,)$ does all that. How does the induced projectivity $g_\Phi \in \Gamma_0(D^9)$ act on $S^1(D^9)(0) = P\,(\partial_4 + \partial_8 + a\partial_{10},\ \partial_{11})$? It sends $[A\partial_{11} + B(\partial_4 + \partial_8 + a\partial_{10})]$ to $[A\partial_{11} - B(\partial_4 + \partial_8 + a\partial_{10})]$, hence it is the reflection σ with fixed points $[\partial_{11}]$ and $[\partial_4 + \partial_8 + a\partial_{10}]$. The first of these points is the well known $L = [L(D^8)(0)/L(D^9)(0)] = [(\partial_{11},\, \partial_{12})/(\partial_{12})] = [\partial_{11}]$. The second is $M = [D_0^{10}(0)/L(D^9)(0)]$ obtained for the value $b = 0$. And indeed, for this value of b, in view of Thm. 2, the respective orbit in S^1 is just one point. This is clear only after Thm. 2; before it the second fixed point is *not* visible, although it exists. By the same theorem, and thanks to Φ, for all non-zero values of b (i. e., in the remaining part of S^1 save L) the orbits are two-point. One meets again Possibility IV. We repeat: this time the second fixed point M is poorly explained geometrically – just the vanishing of a certain parameter in a family of KR pseudo-normal forms.

REMARK 2. The example in this section contributes to the question put forward at the end of [**6**]. The family D_b^{10}, it is – in the language of [**6**] – the situation $k = 0$, $j = 3$, and $i = 3$. From a theory developed there it follows that the *interesting* distances (from the first non-zero constant) are then 2, 3, and 8. $c^{11} = a$ is at distance 2 from the first non-zero $c^9 = 1$, $c^{12} = b$ is at distance 3 from c^9. It appears that not entire c^{12}, but only $|c^{12}|$, is an invariant of the local classification of Goursat.

4. Sketch of proof of Proposition 2. Mimicking the proof of Basic Lemma in [**5**], we note in the first turn that now g, being a symmetry of D^9, preserves $c^9 = 1$ and $c^{10} = 0$, that is

$$g_4^8 + g_8^8\,|0 = f\,|0 \tag{19}$$

(the scalar equation "8" of (15) taken at 0), and

$$g_4^9 + g_8^9\,|0 = 0 \tag{20}$$

(the equation "9" of (15) taken at 0). Constantly using (15), we gradually express relations (19) and (20) in terms of f and: g^8, then g^7, then g^6, then g^5 which already is special. Those reductions are a particular advantage of KR pseudo-normal forms, regarding coordinate functions (of any conjugating diffeomorphism) corresponding to non-singular positions in the flags being conjugated.
As for g^5, from geometric considerations $g^5 = x^5\, G(x^1, \dots, x^5)$ for certain function G (*cf.* (3); presently in the flag only $D^3(0)$ is at the singular position). In what concerns (19), it quickly reduces to

(21) $$G\,|0 = f^4\,|0$$

(*cf.* [**5**], (f)). Concerning (20), it assumes, gradually, the forms:

$$-f_4 + g^8_7 + 2g^8_{48}\,|0 = 0\,,$$
$$-3f_4 + f^{-1}g^7_6 + 3f^{-1}g^7_{47}\,|0 = 0\,,$$
$$-6f_4 + f^{-2}g^6_5 + 4f^{-2}g^6_{46}\,|0 = 0\,,$$
(22) $$-2f_4 + f^{-3}G_4\,|0 = 0$$

(an analogue of (d) in [**5**]). Passing to the main quantity $g^{10}_4 + g^{10}_8\,|0$, we express it gradually in terms of f and: g^9, then g^8, then g^7, then g^6, then G, using on way the intermediate identities leading from (20) to (22):

$$g^{10}_4 + g^{10}_8\,|0 = f^{-1}(g^9_7 + g^9_{44} + 2g^9_{48} + g^9_{88})\,|0\,,$$
$$g^{10}_4 + g^{10}_8\,|0 = f^{-2}(3g^8_{47} + g^8_6 + 3g^8_{78} - f_{44} + 3g^8_{448})\,|0\,,$$
$$g^{10}_4 + g^{10}_8\,|0 = f^{-3}(4g^7_{46} + g^7_5 + 6g^7_{447} - 4f\,f_{44} - 3(f_4)^2)\,|0\,,$$
$$g^{10}_4 + g^{10}_8\,|0 = 5f^{-4}(2g^6_{446} + g^6_{45} - 2f^2 f_{44} - 3f(f_4)^2)\,|0\,,$$
$$g^{10}_4 + g^{10}_8\,|0 = 5f^{-4}(3f^{-1}G_{44} - 4f^2 f_{44} - 9f(f_4)^2)\,|0\,.$$

$G_{44}\,|0$ can be eliminated from this expression, because the equation "1" of (15), after dividing it sidewise by x^5, says that

(23) fG is an affine function of x^4 ,

implying that $(fG)_{44}\,|0 = 0$. The result is

$$g^{10}_4 + g^{10}_8\,|0 = -\,35f^{-3}(f\,f_{44} + 3(f_4)^2)\,|0$$

(an analogue of [**5**], (13)). Now, exactly as in [**5**], $f\,f_{44}\,|0 = \frac{3}{2}(f_4)^2\,|0$, and eventually

(24) $$g^{10}_4 + g^{10}_8\,|0 = -\,\frac{315}{2}f^{-3}(f_4)^2\,|0\,.$$

(*cf.* [**5**], (14)). Writing $g^4 = \frac{fG\,g^4}{fG} = \frac{A(x^1,x^2,x^3)+B(x^1,x^2,x^3)\,x^4}{C(x^1,x^2,x^3)+D(x^1,x^2,x^3)\,x^4}$ (the equations "1" and "3" of (15) after getting rid of x^5), $f\,|0 = \frac{B}{C}\,|0$, [2] $f_4\,|0 = -2\frac{BD}{C^2}\,|0$, $G_4\,|0 = 3\frac{CD}{B}\,|0$, still exactly as in [**5**], Rem.2.

The gist is to obtain a formula for $G_4\,|0$ similar to (22), yet with another coefficient. Remembering that $fG\,|0 = C\,|0$, (21) means $B^5\,|0 = C^6\,|0$ (*cf.* [**5**], (g)) and the computation in [**5**] can be mimicked with the exponent at $f\,|0$ raised from 2 to 3:

$3\frac{CD}{B}\,|0 \;=\; -\frac{3}{2}\left(\frac{B}{C}\right)^3\left(-2\frac{BD}{C^2}\right)\,|0$. That is to say,

$$G_4\,|0 = -\frac{3}{2}f^3 f_4\,|0\,. \tag{25}$$

(22) and (25) taken together imply $f_4\,|0 = 0$. Now Prop. 2 follows from (24).

REMARK 3. $f_4\,|0 = 0$ means $D\,|0 = 0$. Because $f = g^4_4 + x^5(*)$ (the equation "4" of (15)), the higher derivatives of f with respect to x^4 at 0 are the respective derivatives with respect to x^4 at 0 of $g^4_4 = \frac{BC-AD}{(C+Dx^4)^2}$. They all vanish since $D\,|0$ always enters them as a factor; for instance, $f_{44}\,|0 = g^4_{444}\,|0 = 6\frac{BD^2}{C^3}\,|0$ ([**5**], Rem. 2 vi)).

On the other hand, $G_4\,|0$ vanishes by (25). Inductively, in view of (23), all higher derivatives of G with respect to x^4 also vanish at 0. These facts will be useful in the next section.

5. Sketch of proof of Proposition 1. It will appear in the outcome that all summands $g^{11}_4\,|0$, $g^{11}_8\,|0$, and $g^{11}_{10}\,|0$ vanish.

When – constantly using (15) – expressing these summands by f and: g^{10}, $g^9, \ldots, g^6$, G, we keep remember that the same powerful tool (15) implies that g^l ($l = 6, \ldots, 10$) is affine with respect to x^l, hence $g^l_{\,ll}$ vanishes identically. This kills many terms in sometimes long expressions obtained on way. Yet not all of them; certain *essential* terms, for a time being, persist.

Take the simplest summand $g^{11}_{10}\,|0$. When expressed by g^{10} and f, speaking only about essential terms, terms $g^{10}_{4,10}\,|0$ and $g^{10}_9\,|0$ show up. These expressed by g^9 cause the apparition of $g^9_{49}\,|0$ and $g^9_8\,|0$, whereas $f_4\,|0 = 0$ (Sec. 4) and $f^2\,|0 = 1$ (the known consequence of Prop. 2, see Sec. 3) simplify the overall expression. So reducing down, we get $g^{11}_{10}\,|0$ expressed by $g^6_{46}\,|0$ and $g^6_5\,|0$, hence by $g^5_{45}\,|0$ and $G_4\,|0$, hence by $G_4\,|0$ which is zero (Rem. 3).

As for the more involved summands in Prop. 1, we give the intermediate results of the similar reduction made possible by (15). Skipping also $g^{10}_{8,10}\,|0$ that will have been reduced (in two more steps) to $g^8_{88} \equiv 0$, we obtain

$$g^{11}_4 + g^{11}_8\,|0 = f^{-1}\left(g^{10}_{44} + 2g^{10}_{48} + g^{10}_{88} + g^{10}_7\right)\,|0\,.$$

[2] At this moment $f\,|0$ is *not* known. Only having Prop. 2 proved we know that $f\,|0 = \pm 1$.

At the next step we skip $g^9_{888}\,|0$ being a derivative of g^8_{88} :

$$g^{11}_4 + g^{11}_8\,|0 = g^9_{444} + 3g^9_{448} + 3g^9_{488} + 3g^9_{47} + 3g^9_{78} + g^9_6\,|0\,.$$

Keeping skipping the derivatives of g^8_{88} , we get

$$g^{11}_4 + g^{11}_8\,|0 =$$
$$f^{-1}\left(g^8_{4444} + 4g^8_{4448} + 6g^8_{447} + 12g^8_{478} + 4g^8_{46} + 4g^8_{68} + 3g^8_{77} + g^8_5\right)\,|0\,.$$

Taking into account that g^7 is a combination of x^5, x^6 and x^7 (the equation "6" of (15)) and that the derivatives of g^7_{77} obviously vanish, we obtain

$$g^{11}_4 + g^{11}_8\,|0 = g^7_{44444} + 10g^7_{4447} + 10g^7_{446} + 10g^7_{67} + 5g^7_{45}\,|0\,.$$

During the next reduction we will use the fact that g^6 is a combination of x^5 and x^6. In fact, the equation "5" of (15) says that

(26) $$g^6 = f^{-1}\left((x^5)^2(*) + x^5G_4 + x^6(G + x^5G_5)\right)\,.$$

By (26), $g^6_4\,|0$, $g^6_{44}\,|0$, and $g^6_{444444}\,|0$ vanish, and eventually

$$g^{11}_4 + g^{11}_8\,|0 \;=\; f^{-1}(20g^6_{4446} + 15g^6_{445})\,|0\,.$$

Approaching the end, observe that, by (26), $g^6_{4446}\,|0 \;=\; (f^{-1}G)_{444}\,|0$ and $g^6_{445}\,|0 \;=\; (f^{-1}G_4)_{44}\,|0$. These quantities vanish in view of Rem. 3. Prop. 1 is proved.

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