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AFFINE SUBMANIFOLDS OF $\mathbb{R}^N$ OF HIGHER CODIMENSIONS

BY PAWEŁ WITOWICZ

Abstract. The definition of the type number of an immersion of any codimension is simplified. An example shows that the type number in the assumptions of an equivalence theorem cannot be lower than two.

1. Preliminaries. We consider the notion of the type number of a submanifold of $\mathbb{R}^n$ of a codimension greater than one ([**8**], [**9**]) and give its new simplified definition. We recall that the notion is a generalization of the rank of a hypersurface to the case of greater codimension. It is used to formulate some fundamental theorems in those cases (compare [**3**], [**5**], [**7**], [**4**]).

We also give some examples including an example showing that the assumptions of an equivalence theorem are all essential.

In the paper we study n-dimensional submanifolds of the standard affine space $\mathbb{R}^{n+p}$, where n is greater than one and no conditions are put on the transversal bundle. The submanifold, M^n, is equipped with an affine connection ∇. As a submanifold, it is considered together with its immersion f into $\mathbb{R}^{n+p}$, which is endowed with its standard flat affine connection D. The immersion is called an affine immersion if the connection ∇ on M^n comes from the connection D on $\mathbb{R}^{n+p}$. This means that there is a vector bundle σ over M^n with p-dimensional fibres such that the connection ∇ is the connection induced on M^n by D with respect to the splitting $R^{n+p} = T_xM \oplus \sigma_x$ for every $x \in M^n$. It is expressed by the Gauss formula

$$D_X f_* Y = f_*(\nabla_X Y) + h(X,Y), \tag{1.1}$$

where h is a symmetric σ-valued bilinear form called the second fundamental form or affine fundamental form.

If σ is locally spanned by a frame $\{\xi_1, \xi_2, ..., \xi_p\}$, we have the equality $h(X,Y) = \sum_{i=1}^{p} h^i(X,Y)\xi_i$, which defines affine fundamental forms h^i, $i =$

$1, \ldots, p$. A choice of the transversal bundle σ also induces the shape operator S as the linear mapping $\xi \longmapsto S_\xi$ and the so-called normal connection $\nabla^\perp$ in σ, according to the Weingarten formula

$$D_X \xi = -f_*(S_\xi X) + \nabla^\perp_X \xi, \tag{1.2}$$

where ξ is a local section of σ. It is easy to see that the mapping $X \longmapsto S_\xi X$, also called the shape operator connected with ξ, is an endomorphism of the tangent bundle TM^n. Let $\{\xi_1, \xi_2, \ldots, \xi_p\}$ be a transversal frame. We will use the notation S_i instead of S_{ξ_i}. We can rewrite the Weingarten formula using the decompositions $\nabla^\perp_X \xi_j = \sum_{i=1}^p \tau^i_j(X)\xi_i$, defining the normal connection forms τ^i_j for $i, j = 1, \ldots, p$. For the objects defined above and a chosen transversal frame $\{\xi_1, \xi_2, \ldots, \xi_p\}$ the following fundamental equations are satisfied (equations of Gauss (1.3), Codazzi (1.4), (1.5) and Ricci (1.6):

$$R(X,Y)Z = S_{h(Y,Z)}X - S_{h(X,Z)}Y \tag{1.3}$$

$$\nabla h(X,Y,Z) \text{ is symmetric in X,Y and Z,} \tag{1.4}$$

$$(\nabla_X A)_\xi(Y) = (\nabla_Y A)_\xi(X) \tag{1.5}$$

$$R^\perp(X,Y)\xi = h(X, S_\xi Y) - h(S_\xi X, Y). \tag{1.6}$$

We can rewrite these equations using the expression of h in a given local basis $\{\xi_1, \xi_2, \ldots, \xi_p\}$ of σ:

$$R(X,Y)Z = \sum_{i=1}^{p} \left[h^i(Y,Z)S_i X - h^i(X,Z)S_i Y\right] \tag{1.3'}$$

$$(\nabla_X h^j)(Y,Z) + \sum_{i=1}^{p} \tau^j_i(X) h^i(Y,Z) \text{ is symmetric in X,Y and Z,} \tag{1.4'}$$

$$(\nabla_X S_j)Y - (\nabla_Y S_j)X = \sum_{i=1}^{p} \left[\tau^i_j(X)S_i Y - \tau^i_j(Y)S_i X\right] \tag{1.5'}$$

(1.6')

$$h^j(X, S_k Y) - h^j(Y, S_k X) = d\tau^j_k(X,Y) + \sum_{i=1}^{p} \left[\tau^j_i(X)\tau^i_k(Y) - \tau^j_i(Y)\tau^i_k(X)\right],$$

for every $j, k = 1, \ldots, p$.

2. The type number and fundamental theorems. We define the type number of a symmetric, bilinear form on a manifold M^n, the type number of its immersion into $\mathbb{R}^{n+p}$, and we formulate an equivalence and existence theorems.

For a symmetric bilinear mapping α we will always use the same symbol α to denote the linear mapping $u \longmapsto \alpha(\cdot, u)$.

DEFINITION 2.1. (compare [**8**], [**9**]). Let V, W be vector spaces, $\dim V \geq 2$, $\dim W = p$ and $h : V \times V \longrightarrow W$ be a symmetric bilinear form. Let $\{e_1, ..., e_k\}$ be a basis of $\text{span} h = \text{span}\{h(u,v) : u, v \in V\}$ and the forms $h^i : V \times V \longrightarrow \mathbb{R}$ are defined by the equality $h = \sum h^i e_i$. The type number of h is the maximal integer r such that there exist r vectors $v_1, ..., v_r \in V$ for which one-forms $h^i(\cdot, v_j)$ are linearly independent for i=1,...,k and j=1,...,r.

The above definition is independent of a choice of a basis of W ([**8**]). We also can see that the forms h^i are always linearly independent. We state a modified definition of the type number of an immersion from [**8**].

DEFINITION 2.2. Let $f : M^n \longrightarrow \mathbb{R}^{n+p}$ be an immersion, σ - an arbitrary transversal bundle, and h - induced affine fundamental form according to (1.1). Then the type number of f at the point x is the type number of h as the mapping $T_xM^n \times T_xM^n \longrightarrow \sigma_x$.

The type number of an immersion is independent of a choice of σ ([**8**]). It is a generalization of the rank of a hypersurface in affine geometry ([**8**]).

We recall the following equivalence theorem ([**8**]):

THEOREM 2.3. *Let $f, \widetilde{f} : (M^n, \nabla) \longrightarrow (\mathbb{R}^{n+p}, D)$ be affine immersions where D is the standard flat connection in $\mathbb{R}^{n+p}$. Suppose that the following conditions hold:*

1) the type number of f is greater than one at every point x in M^n.

2) There exists an isomorphism $F : \sigma \longrightarrow \widetilde{\sigma}$ of vector bundles over M^n such that $F \circ h = \widetilde{h}$.

3) $\dim O_x$ is constant on M^n.

Then there exists a unique $B \in A(n+p, \mathbb{R})$ such that $\widetilde{f} = B \circ f$, where $A(n+p, \mathbb{R})$ denotes the group of all affine transformations of $\mathbb{R}^{n+p}$.

THEOREM 2.4. *Let M^n be an n-dimensional simply connected manifold with a torsion-free connection ∇. Let σ be a vector bundle over M^n, h - a (0,2)-symmetric σ-valued tensor field on M^n and S_ξ - a (1,1)-tensor field on M^n for any section ξ of σ. Let $\nabla^\perp$ be a linear connection in σ. Assume that the following conditions are satisfied:*

1) the type number of h is greater than 2 for every $x \in M^n$.

2) $(\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z)$ for any vector fields X, Y, Z.

3) the Gauss equation (1.3) holds;

Then there exists an affine immersion $f : M^n \longrightarrow \mathbb{R}^{n+k}$ (where k is the fibre *dimension of σ) such that h is its second fundamental form, S_ξ - the shape operator and $\nabla^\perp$ - the normal connection. Moreover, the immersion is unique up to affine transformations.*

We give an example showing that there exist immersions of type number greater than two.

EXAMPLE 2.5. Let $x(u_1,...,u_6) = (u_1,...,u_6, \frac{1}{2}u_1^2+\frac{1}{2}u_2^2+\frac{1}{2}u_3^2, u_1u_4+u_2u_5+u_3u_6)$, where each $u_j > 0$, be a local immersion $\mathbb{R}^6 \longrightarrow \mathbb{R}^8$. Let a transversal bundle σ be the bundle spanned by $e_7, e_8 \in \mathbb{R}^8$. Let $X_j = x_{u_j}$ form a basis of the tangent bundle for $j = 1,...,6$. Computing $D_{X_j}X_k = x_{u_ju_k}$ for $j,k = 1,...,6$ we obtain the affine fundamental forms h^1, h^2 which we write in the matrix form:

$$h^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$h^2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

We can easily see that the one-forms $h^i(\cdot, X_j)$ for $i = 1,2$ and $j = 1,2,3$ are linearly independent which is described by the three first columns of the matrices. Thus the type number of the immersion x is maximal and equal to three.

The following example shows that the assumed restriction on type number in Theorem 2.3 cannot be decreased.

EXAMPLE 2.6. We have two immersions x and $\widetilde{x}$ of surfaces into $\mathbb{R}^4$ given as follows:

$$x(u,v) = (u, v, \frac{1}{2}u^2, \frac{1}{2}v^2),$$

$$\widetilde{x}(u,v) = (\sinh(u), \cosh(u), \sinh(v), \cosh(v)).$$

We take the transversal bundles to these surfaces generated by the fields $\xi_1 = (0,0,1,0)$, $\xi_2 = (0,0,0,1)$ and $\widetilde{\xi}_1 = (\sinh(u), \cosh(u), 0, 0)$, $\widetilde{\xi}_2 = (0,0,\sinh(v),\cosh(v))$, respectively. We take the bases of the tangent bundles given by $X_1 = x_u, X_2 = x_v$ and $\widetilde{X}_1 = \widetilde{x}_u, \widetilde{X}_2 = \widetilde{x}_v$, respectively. According to these bases, the affine fundamental forms have the following matrices:

$$h^1 = \widetilde{h}^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$h^2 = \widetilde{h}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

which induces the natural isomorphism F of the transversal bundles. We can now see that the type numbers of both immersions are equal to one. Computing the fundamental forms we obtain that both connections induced on the surfaces are flat. Now we consider the shape operators, after differentiating $\xi_j, \widetilde{\xi}_j$, $i, j = 1, 2$. We can see that S vanishes identically. The components of $\widetilde{S}$ have the following matrix forms in the basis $\widetilde{X}_1, \widetilde{X}_2$.

$$\widetilde{S}_{\widetilde{\xi}_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\widetilde{S}_{\widetilde{\xi}_2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus we can see that the immersions are not equivalent in the sense of Theorem 2.3 because the shape operators are not isomorphic.

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