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LOCAL AND NICE STRUCTURES OF THE GROUPOID OF AN EQUIVALENCE RELATION

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ABSTRACT. This paper is devoted to compare local and nice structures of the groupoid of an equivalence relation. The first was given by R.Brown and O.Mucuk (1996) due to J.Pradines (1966) but the second was given by the author in 1987.

Let $R \subset X \times X$ be any equivalence relation on a connected paracompact C^∞-manifold X of dimension n. R becomes a topological groupoid on X with the topology as a subspace of $X \times X$

$$(R, \alpha, \beta, \gamma)$$

where $\alpha = (\mathrm{pr}_1)_{|R}$, $\beta = (\mathrm{pr}_2)_{|R}$, $\gamma : R_\alpha \times_\beta R \to R$, $((x,y),(z,x)) \mapsto (z,y)$. Let $\delta : R \times_\alpha R \to R$, $((x,z),(x,y)) \mapsto (y,z)$. According to the definition of Brown and Mucuk [A-B], [B-M1, 1996], [B-M2, 2000] (due to the original definition of J.Pradines [P, 1966]) R is called *locally Lie groupoid* if there is given a subset $W \subset R$ equipped with a structure of a manifold such that

1) $O_R \subset W \subset R$, ($O_R = \{(x,x)\,;\ x \in X\}$),
2) $W = W^{-1}$,
3) $W_\delta = (W \times_\alpha W) \cap \delta^{-1}[W]$ is open in $W \times_\alpha W$ and the restriction of δ to W_δ is smooth,
4) the restriction to W of the source and the target maps α and β are smooth and the triple (α, β, W) is smoothly locally sectionable,
5) W generates R as a groupoid.

W satisfying 1)-5) will be called a *local smooth structure* of R.

Remark 1. *The second part of 4) implies that $\alpha_{|W} : W \to X$ is a coregular mapping therefore $W \times_\alpha W = \left(\alpha_{|W}, \alpha_{|W}\right)^{-1}[O_R]$ is a proper (i.e. embedded) submanifold of $W \times W$.*

Let F be any k-dimensional foliation on X and $\Lambda = \{(U_\lambda, \phi_\lambda)\}$ any atlas of foliated charts. The equivalence relation on X determined by the leaves of F is write R_F. For a chart $(U_\lambda, \phi_\lambda)$ we write R_λ for the equivalence relation on U_λ whose classes are the plaques of U_λ

$$R_\lambda = \{(x,y) \in X \times X;\ x \in U_\lambda,\, y \in Q_\lambda^x\},$$

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Q_λ^x denotes the plaque of U_λ passing through x. We write

$$W(\Lambda) = \bigcup_\lambda R_\lambda.$$

Brown and Mucuk in the work [B-M2] proved the following

Theorem 1. *If X is a paracompact foliated manifold, then a foliated atlas Λ may be chosen so that the pair $(R_F, W(\Lambda))$ is a locally Lie groupoid.*

Clearly $W(\Lambda)$ is a $n+k$-dimensional proper submanifold of $X \times X$ and $W(\Lambda)_x = (\alpha_{|W(\Lambda)})^{-1}(x)$ is a connected k-dimensional submanifold of $W(\Lambda)$. The crucial role in the proof is played by Lemma 4.4 from [T, 1992].

On the other hand, in my work [K1, 1987] there is a second approach to the groupoid of an equivalence relation giving a necessary and sufficient condition to that the family of all equivalence classes is a k-dimensional foliation.

Theorem 2 (J.Kubarski [K1, Th. 3])**.** *If X is a paracompact manifold, then the following conditions are equivalent:*

(1) *the family of all equivalence classes of R is a k-dimensional foliation,*
(2) *there exists a subset $\Omega \subset R$ such that*
  (i) $O_R \subset W \subset R$,
  (ii) *W is a proper $n+k$-dimensional C^∞-submanifold of $X \times X$,*
  (iii) *$\alpha_{|W} : W \to X$ is a submersion,*
  (iv) *For $(x, y) \in R$, the set*
  $$D_{(x,y)}[W_y] \cap W_x$$
  is open in the manifold $W_x = (\alpha_{|W})^{-1}(x)$, where $D_{(x,y)} : R_y \to R_x$, $(y, z) \mapsto (x, z)$,
  (v) *W generates R as a groupoid,*
  (vi) *the manifolds W_x are connected.*

We can chose W to be symmetric $W = W^{-1}$. The crucial role in the proof of the theorem is played by the *nice covering* [H-H, 1981] and the Frobenius theorem in the version [D, p.86]. The manifold W satisfying the above conditions (i)-(vi) is called to be a *nice structure of R* (thanks to the nice covering used in the proof, see also [K2]). In the above theorem it is proved that $W(\Lambda)$ is a nice structure if Λ is any nice covering.

QUESTION. What is the relation between local and nice structures of R ?Are the notions equivalent ?

Proposition 1. *If $W \subset R$ is a local smooth structure such that W is a proper $n+k$-dimensional C^∞-submanifold of $X \times X$ and $W_x = (\alpha_{|W})^{-1}(x)$ are connected then W is a nice structure, therefore the family of equivalence classes of R is a k-dimensional foliation.*

Proof. We only must prove condition (iv). In this purpose take $(x_0, y_0) \in R$ and the smooth mapping

$$i_{x_0,y_0} : W_{x_0} \longrightarrow W \times_\alpha W, \ \ (x_0, z) \longmapsto ((x_0, z), (x_0, y_0)).$$

It is easy to see that

$$(i_{x_0,y_0})^{-1}[W_\delta] = D_{(x_0,y_0)}[W_{y_0}] \cap W_{x_0}. \tag{0.1}$$

"$\subset$" Let $(x_0, z) \in (i_{x_0,y_0})^{-1}[W_\delta]$, then $(x_0, z) \in W_{x_0}$ and $((x_0, z), (x_0, y_0)) \in W_\delta$ which means that $\delta((x_0, z), (x_0, y_0)) = (y_0, z) \in W \cap R_{y_0} = W_{y_0}$ and $(x_0, z) = D_{(x_0,y_0)}(y_0, z) \in D_{(x_0,y_0)}[W_{y_0}] \cap W_{x_0}$.

"$\supset$" Let $(y_0, z) \in W_{y_0}$ and $(x_0, z) \in W_{x_0}$, then $\delta(i_{x_0,y_0}(x_0, z)) = \delta((x_0, z), (x_0, y_0)) = (y_0, z) \in W_{y_0} \subset W$.

0.1 implies (iv). $\square$

The problem which is still open is: does every nice structure is a local structure? . This problem means the same, what means the problem of a reduction of the axioms of a local smooth structure. The first step in this direction give the following

Proposition 2. *If $W \subset R$ fulfils (i)-(iii) and $W = W^{-1}$ then the triple (α, β, W) is smoothly locally sectionable.*

Proof. Since $\beta = \alpha \circ \iota$, and $\iota : W \to W$ is a diffeomorphism, β is a submersion, too ($\iota(x, y) = (y, x)$ is the inverse map). For $(x, y) \in W$, $\ker \alpha_{*(x,y)}$ and $\ker \beta_{*(x,y)}$ are k-dimensional and

$$\ker \alpha_{*(x,y)} \cap \ker \beta_{*(x,y)} = 0.$$

The last follows from the inclusions: $\ker \alpha_{*(x,y)} \subset \{\theta_x\} \times T_y X$ and $\ker \beta_{*(x,y)} \subset T_x X \times \{\theta_y\}$ (θ_x and θ_y are the null vectors). Find a k-dimensional vector subspace $V \subset T_{(x,y)} W$ such that

$$T_{(x,y)} W = V \bigoplus \ker \alpha_{*(x,y)} \bigoplus \ker \beta_{*(x,y)}$$

and find a k-dimensional vector subspace $V_0 \subset \ker \alpha_{*(x,y)} \bigoplus \ker \beta_{*(x,y)}$ transversal to both $\ker \alpha_{*(x,y)}$ and $\ker \beta_{*(x,y)}$. For example if $(u_1, ..., u_k)$ and $(v_1, ..., v_k)$ are basis of $\ker \alpha_{*(x,y)}$ and $\ker \beta_{*(x,y)}$, respectively, we can take $V_0 = \mathrm{Lin}\{u_1 + v_1, ..., u_k + v_k\}$ (clearly, $V_0 \cap \ker \alpha_{*(x,y)} = V_0 \cap \ker \beta_{*(x,y)} = 0$). Put

$$U = V \bigoplus V_0.$$

It is easy to see that

$$U \cap \ker \beta_{*(x,y)} = 0.$$

This implies that $\alpha_* : U \to T_x X$ is an isomorphism. Since $\alpha : W \to X$ is a submersion there exists a local cross-section s of α near x such that $s_* T_x X = U$. It is clear, that

$$(\beta \circ s)_* : T_x X \to T_y X$$

is an isomorphism: if $(\beta \circ s)_*(v) = \theta_y$ then $s_*(v) \in \ker \beta_{*(x,y)} \cap U = 0$ which yields $v = 0$. From this $\beta \circ s$ is a diffeomorphism near a point x. $\square$

To finish a problem of reduction of axioms of local smooth structure we need to proof that assumptions (i)-(iv) together with the symmetry 2) ($W = W^{-1}$) imply condition 3). Up to now this is an open problem.

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