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Endomorphisms of the Lie Algebroids $TM \times \mathfrak{g}$ up to Homotopy

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Abstract

The notion of a homotopy joining two homomorphisms of Lie algebroids comes from J.Kubarski [5] and the definition of the fundamental group $\pi(A)$ of a Lie algebroid A (which consists of homotopy classes of strong automorphisms of A), from B.Balcerzak [1]. It is given in this paper a homotopic classification of strong homomorphisms of the trivial Lie algebroid $TM \times \mathfrak{g}$ and computed the fundamental group of the algebroid $TM \times \mathbb{R}$ with the aid of $H^1_{dR}(M)$.

1 Introduction

The notion of homotopic homomorphisms between vector bundles or principal bundles and the problem of classification of homomorphisms up to homotopy is important in all classical parts of geometry and differential topology where we have to deal with problems of homotopy invariance (classification of bundles, characteristic classes). Vector bundles and principal bundles have their infinitesimal objects – transitive Lie algebroids – which contain the most of information, interesting from the topological point of view. For example, homotopy of mappings can be expressed by these infinitesimal invariants, moreover, characteristic homomorphisms of principal bundles (primary and secondary [5], [7], [8]) can also be expressed in terms of these invariants. We add that the characteristic homomorphisms can also be defined in the category of Lie algebroids or more generally, in the geometry (commutative) of Lie-Rinehart algebras, see [10], [11], [3], [Lie-Rinehart algebras are called sometimes Lie modules or Lie pseudoalgebras].

A homotopy joining two homomorphisms H_0, $H_1 : A' \to A$ of Lie algebroids is defined in J.Kubarski [6]; it is a homomorphism of Lie algebroids $H : TR \times A' \to A$ with $H(\theta_0, \cdot) = H_0$, $H_2(\theta_1, \cdot) = H_1$, where $\theta_0 \in T_0\mathbb{R}$ and $\theta_1 \in T_1\mathbb{R}$, are null vectors ($T\mathbb{R} \times A'$ denotes the product of the trivial Lie algebroid $T\mathbb{R}$ and A'). The reader can find the definition of a product of Lie algebroids and a homomorphism (not necessarily strong) in [2], [6], [5]. If A' and A are algebroids on a manifold M, H_0 and H_1 are strong homomorphisms between them, then the homotopy H is called *strong* if H is defined over projection $\mathrm{pr}_2 : \mathbb{R} \times M \to M$. This definition is well seen if we consider the following case: let h_0 and $h_1 : P' \to P$ be two homomorphisms of G-principal bundles and h_t – a familly of homomorphisms (a homotopy) joining h_0 and h_1. This family gives a homomorphism of G-principal bundles $h : \mathbb{R} \times P' \to P$. If we consider infinitesimal mappings between the algebroids of these bundles (see [9], [4]), $h_* : A(\mathbb{R} \times P') \to A(P)$, and indentify $\mathbb{R} \times P'$ with the cartesian product of the trivial $\{e\}$-principal bundle $\mathbb{R}$ and G-bundle P' we obtain (after identification $A(\mathbb{R} \times P') \cong T\mathbb{R} \times A(P')$) a homomorphism $h_* : T\mathbb{R} \times A(P') \to A(P)$ which is the homotopy joining h_{0*} and h_{1*}, in the terms above. We can interpret a homotopy joining linear homomorphisms of vector bundles (assuming isomorphisms on each fibre) in the category of algebroids (by principal bundles of frames or directly), using (1) the algebroid of a vector bundle (see [9], [5]) which is isomorphic to the algebroid of a frame bundle; (2) the homomophism of algebroids induced by a homomorphism of vector bundles [6].

The problem of classification of endomorphisms for a given Lie algebroid up to homotopy had been initiated by B.Balcerzak in [1], where it was introduced the notion of the fundamental group $\pi(A)$ of a Lie algebroid A ($\pi(A)$ is a set of homotopy classes of strong automorphisms in the Lie algebroid A). In [1] the author gave an effective classification of endomorphisms of the trivial Lie algebroid $T\mathbb{R}^n \times \mathbb{R}$ (on $\mathbb{R}^n$ with the 1-dimensional abelian isotropy Lie algebra $\mathbb{R}$) and proved that $\pi(T\mathbb{R}^n \times \mathbb{R}) \cong \mathrm{GL}(\mathbb{R})$. The aim of this paper is homotopic classification of endomorphisms of the trivial Lie algebroid $TM \times \mathfrak{g}$ on a manifold M with an isotropy algebra $\mathfrak{g}$ and computation $\pi(TM \times \mathbb{R})$ using the first de Rham cohomology group $H^1_{dR}(M)$. The problem of computation $\pi(TM \times \mathfrak{g})$ and more generally, $\pi(A)$ for any Lie algebroid A, is still open.

2 Endomorphisms of trivial algebroids

2.1 The case of $TM \times \mathfrak{g}$

Let M be a connected smooth manifold and $\mathfrak{g}$ a finite dimensional real Lie algebra. Then $(TM \times \mathfrak{g}, [\![\cdot,\cdot]\!], pr_1)$ is a transitive Lie algebroid with the anchor $pr_1 : TM \times \mathfrak{g} \to TM$ and the Lie algebra structure $[\![\cdot,\cdot]\!]$ in the module of cross-sections $\mathrm{Sec}\,(TM \times \mathfrak{g}) = \{(X,\sigma)\,; X \in \mathfrak{X}(M), \sigma \in C^{\infty}(M;\mathfrak{g})\}$ defined by

$$[\![(X,\sigma),(Y,\tau)]\!] = ([X,Y], \mathfrak{L}_X(\eta) - \mathfrak{L}_Y(\sigma) + [\sigma,\tau]),$$

where $X, Y \in \mathfrak{X}(M)$ and $\sigma, \tau \in C^{\infty}(M;\mathfrak{g})$. An endomorphism of vector bundles $H : TM \times \mathfrak{g} \to TM \times \mathfrak{g}$ is a homomorphism of Lie algebroids if $pr_1 \circ H = pr_1$ and $\mathrm{Sec}\,H : \mathrm{Sec}\,(TM \times \mathfrak{g}) \to \mathrm{Sec}\,(TM \times \mathfrak{g})$ is a homomorphism of Lie algebras.

Let

$$\pi_{\mathbb{R}} : \mathbb{R} \times M \longrightarrow \mathbb{R}, \;\; \pi_M : \mathbb{R} \times M \longrightarrow M, \;\; \pi : TM \longrightarrow M$$

be (in this paper) canonical projections and

$$j_t : M \longrightarrow \mathbb{R} \times M, \; x \longmapsto (t,x)\,, \quad j_x : \mathbb{R} \longrightarrow \mathbb{R} \times M, \; t \longmapsto (t,x)\,,$$

injections.

Theorem 2.1 *Each endomorphism $H : TM \times \mathfrak{g} \to TM \times \mathfrak{g}$ of the Lie algebroid $TM \times \mathfrak{g}$ is given by*

$$H(u,r) = \left(u, \omega_{\pi(u)}(u) + \phi_{\pi(u)}(r)\right)$$

for some smooth function $\phi : M \to \mathrm{End}\,(\mathfrak{g})$ and 1-form $\omega \in \Omega^1(M;\mathfrak{g})$ such that

- ***W1*** $\delta\omega(X,Y) + [\omega(X), \omega(Y)] = 0,$
- ***W2*** $((d\phi)X)(\eta) = [\phi(\eta), \omega(X)]\,, \quad X,\, Y \in \mathfrak{X}(M)\,,\; \eta \in C^{\infty}(M;\mathfrak{g})\,,\; \phi(\eta) : M \to \mathfrak{g},\, x \mapsto \phi_x(\eta_x).$

ϕ and ω are uniquely determined. H is an automorphism if and only if ϕ_x is a linear automorphism of $\mathfrak{g}$ for all $x \in M$. Under this condition there is H^{-1} given by $H^{-1}(u,r) = \left(u, \phi^{-1}(\omega(u)) + \phi^{-1}(r)\right)$.

Proof. Let H be an endomorphism of the Lie algebroid $TM \times \mathfrak{g}$. Since $pr_1 \circ H = pr_1$, we obtain $H(u,r) = (u, \lambda(u,r))\,,\; u \in TM,\; r \in \mathfrak{g}$ for a smooth function $\lambda : TM \times \mathfrak{g} \to \mathfrak{g}$. Each function $\lambda_x : T_xM \times \mathfrak{g} \to \mathfrak{g}$ is linear and hence $\lambda_x(u,r) = \omega_x(u) + \phi_x(r)$ for $\omega_x \in \mathrm{Hom}\,(T_xM;\mathfrak{g})$ and $\phi_x \in \mathrm{Hom}\,(\mathfrak{g};\mathfrak{g})$. Thus

$$H(u,r) = \left(u, \omega_{\pi(u)}(u) + \phi_{\pi(u)}(r)\right)$$

for some 1-form $\omega \in \Omega^1(M;\mathfrak{g})$ and smooth function $\phi : M \to \mathrm{Hom}\,(\mathfrak{g};\mathfrak{g})$.

$\mathrm{Sec}\,H : \mathrm{Sec}\,(TM \times \mathfrak{g}) \to \mathrm{Sec}\,(TM \times \mathfrak{g}),\; \xi \mapsto H \circ \xi$ is a homomorphism of Lie algebras, so it satisfies $H \circ [\![(X,\sigma),(Y,\eta)]\!] = [\![H \circ (X,\sigma), H \circ (Y,\eta)]\!]$ for all $X,Y \in \mathfrak{X}(M)$, $\sigma,\eta \in C^{\infty}(M;\mathfrak{g})$. If $\sigma = \eta = 0$, then we easily obtain $\omega([X,Y]) = X(\omega(Y)) - Y(\omega(X)) + [\omega(X),\omega(Y)]$, i.e. W1. If $X = Y = 0$, we have $[\phi(\sigma),\phi(\eta)] = \phi([\sigma,\eta])$, so linear endomorphisms ϕ_x are endomorphisms of the Lie algebra $\mathfrak{g}$ for all $x \in M$. Finally, for any $X \in \mathfrak{X}(M)$, $\eta \in C^{\infty}(M,\mathfrak{g})$ and $Y = 0$, $\sigma = 0$ we obtain $X(\phi(\eta)) + [\omega(X),\phi(\eta)] = \phi(X(\eta))$, hence W2 holds.

Conversely, define a linear endomorphism $H : TM \times \mathfrak{g} \to TM \times \mathfrak{g}$ by

$$H(u,r) = \left(u, \omega_{\pi(u)}(u) + \phi_{\pi(u)}(r)\right)$$

for given $\phi : M \to \mathrm{End}(\mathfrak{g})$ and $\omega \in \Omega^1(M;\mathfrak{g})$ which satisfy W1 and W2. This endomorphism is a homomorphism of Lie algebroids because (a) condition $pr_1 \circ H = pr_1$ holds evidently, (b) $\mathrm{Sec}\, H$ is a homomorphism of Lie algebras. In fact, for $X,\, Y \in \mathfrak{X}(M)$, $\sigma,\, \eta \in C^{\infty}(M;\mathfrak{g})$ we compute

$$\begin{aligned}
& [\![H \circ (X,\sigma), H \circ (Y,\eta)]\!] \\
= \; & [\![(X, \omega(X) + \phi(\sigma)), (Y, \omega(Y) + \phi(\eta))]\!] \\
= \; & ([X,Y], \delta\omega(X,Y) + [\omega(X),\omega(Y)] \\
& + ((d\phi)X)(\eta) - [\phi(\eta),\omega(X)] \\
& - ((d\phi)Y)(\sigma) + [\phi(\sigma),\omega(Y)]) \\
& + \phi((X(\sigma)) - \phi(Y(\eta)) + \phi[\sigma,\eta] + \omega([X,Y])) \\
= \; & ([X,Y], \omega([X,Y]) + \phi(X(\eta)) - \phi(Y(\sigma)) + \phi([\sigma,\eta])) \\
= \; & H \circ ([X,Y], X(\eta) - Y(\sigma) + [\sigma,\eta]) \\
= \; & H \circ [\![(X,\sigma),(Y,\eta)]\!].
\end{aligned}$$

The rest is obvious. ■

We can easily generalize the theorem above in the case of strong homomorphisms of trivial Lie algebroids with the different isotropy algebras.

Theorem 2.2 *Each homomorphism $H : TM \times \mathfrak{g} \to TM \times \mathfrak{g}^{'}$ of Lie algebroids is given by*

$$H(u,r) = \left(u, \omega_{\pi(u)}(u) + \phi_{\pi(u)}(r)\right)$$

for some smooth function $\phi : M \to \mathrm{Hom}\left(\mathfrak{g};\mathfrak{g}^{'}\right)$ and 1-form $\omega \in \Omega^1\left(M;\mathfrak{g}^{'}\right)$ such that

W1' $\delta\omega(X,Y) + [\omega(X),\omega(Y)] = 0$,

W2' $((d\phi)X)(\eta) = [\phi(\eta),\omega(X)]$, $\quad X,\, Y \in \mathfrak{X}(M)$, $\eta \in C^{\infty}(M;\mathfrak{g})$, $\phi(\eta) : M \to \mathfrak{g}$, $\phi(\eta)(x) = \phi_x(\eta_x)$.

2.2 The case of $TM \times \mathbb{R}$

Consider the 1-dimensional abelian Lie algebra $\mathfrak{g} = \mathbb{R}$. It follows from Theorem 2.1 that each strong endomorphism H of the Lie algebroid $TM \times \mathbb{R}$ is given by

$$H(u,r) = \left(u, \omega_{\pi(u)}(u) + \phi_{\pi(u)}(r)\right), \quad (u,r) \in TM \times \mathbb{R}$$

for an uniquely determined function $\phi : M \to \mathrm{End}(\mathbb{R})$ and closed 1-form $\omega \in \Omega^1(M)$ such that $((d\phi)X)(\eta) = 0$ for $X \in \mathfrak{X}(M)$, $\eta \in C^\infty(M)$. Since $\phi_x(r) = B(x) \cdot r$, $r \in \mathbb{R}$, $B \in C^\infty(M)$, we see that $((d\phi)X)(\eta) = X(B) \cdot \eta = 0$. In particular, if we take a non-zero η, then $X(B) = 0$,which means that B is constant. Hence we have

Corollary 2.3 *Each strong endomorphism H of the Lie algebroid $TM \times \mathbb{R}$ is given by*

$$H(u,r) = \left(u, \omega_{\pi(u)}(u) + B \cdot r\right),$$

where B is a real number and ω is a closed 1-form on M. H is an automorphism if and only if $B \neq 0$; then $H^{-1}(u,r) = \left(u, \frac{1}{B}\omega(u) + \frac{1}{B}r\right)$.

Example 2.4 [1, Theorem. 2.1]Consider the special case $M = \mathbb{R}^n$. Since the fact that the 1-form $\omega = \sum\limits_{i=1}^{n} A^i dx^i \in \Omega^1(\mathbb{R}^n)$, $A^i \in C^\infty(\mathbb{R}^n)$, is closed is equivalent to

$$\frac{\partial A^i}{\partial x^j} = \frac{\partial A^j}{\partial x^i},\ i, j = 1, ..., n, \tag{2.1}$$

an endomorphism H of the trivial Lie algebroid $T\mathbb{R}^n \times \mathbb{R}$ has the following form

$$H(u,r) = \left(u, \sum_{i=1}^{n} A^i(x) \cdot u^i + B \cdot r\right)$$

(u^i are coordinates of the vector u with respect to the basis $\frac{\partial}{\partial x^i}$), where $B \in \mathbb{R}$, and $A^i \in C^\infty(\mathbb{R}^n)$ satisfy condition (2.1).

3 Homotopic classification of endomorphisms of trivial algebroids

3.1 The case of $TM \times \mathfrak{g}$

Consider the trivial Lie algebroids $TM \times \mathfrak{g}$, $T(\mathbb{R} \times M) \times \mathfrak{g}$ and the product of algebroids $T\mathbb{R} \times (TM \times \mathfrak{g})$. Mappings

$$\begin{aligned} K &: T(\mathbb{R} \times M) \times \mathfrak{g} \longrightarrow T\mathbb{R} \times (TM \times \mathfrak{g}),\ (u,r) \longmapsto (d\pi_{\mathbb{R}}(u), (d\pi_M(u), r)), \\ L &: T(\mathbb{R} \times M) \times \mathfrak{g} \longrightarrow \pi_M^{\wedge}(TM \times \mathfrak{g}),\ (u,r) \longmapsto (u, (d\pi_M(u), r)), \end{aligned}$$

are isomorphisms of Lie algebroids, where $\pi_M^{\wedge}(TM \times \mathfrak{g})$ denotes the inverse-image of $TM \times \mathfrak{g}$ by π_M, see.[5].

Let H_0, H_1 be strong endomorpisms of the Lie algebroid $TM \times \mathfrak{g}$. They are given by

$$H_i(u,r) = \left(u, \omega^i_{\pi(u)}(u) + \phi^i_{\pi(u)}(r)\right),\ i = 0, 1, \tag{3.1}$$

where $\phi^i : M \to \mathrm{End}(\mathfrak{g})$, $\omega^i \in \Omega^1(M; \mathfrak{g})$ and

(a) $\delta\omega^{i}(X,Y)+\left[\omega^{i}(X),\omega^{i}(Y)\right]=0,$

(b) $\left(\left(d\phi^{i}\right)X\right)(\eta)=\left[\phi^{i}(\eta),\omega^{i}(X)\right],\quad X,\,Y\in\mathfrak{X}(M),\,\eta\in C^{\infty}(M;\mathfrak{g}).$

The theorem below gives characterization of the homotpy joining these two endomorphisms.

Theorem 3.1 *Endomorphisms H_0 and H_1 of the trivial Lie algebroid $TM\times\mathfrak{g}$ given by (3.1), are homotopic if and only if there exists a mapping $\phi:\mathbb{R}\times M\rightarrow$* End$(\mathfrak{g})$ *and a 1-form $\omega\in\Omega^{1}(\mathbb{R}\times M;\mathfrak{g})$ such that*

$$\phi(0,\cdot)=\phi^{0},\quad\phi(1,\cdot)=\phi^{1},\quad\omega^{0}=j_{0}^{*}\omega\,,\quad\omega^{1}=j_{1}^{*}\omega,$$

and

$$\delta\omega(X,Y)+[\omega(X),\omega(Y)]=0,\ \ ((d\phi)X)(\eta)=[\phi(\eta),\omega(X)]\,,\tag{3.2}$$

for $X,\,Y\in\mathfrak{X}(\mathbb{R}\times M)\,,\ \eta\in C^{\infty}(\mathbb{R}\times M;\mathfrak{g})$.

Proof. Suppose that $H:T\mathbb{R}\times(TM\times\mathfrak{g})\longrightarrow TM\times\mathfrak{g}$ is a homotopy (over π_M) joining endomorphisms H_0 i H_1, that is

$$H(\theta_0,\cdot)=H_0,\ H(\theta_1,\cdot)=H_1,\tag{3.3}$$

($\theta_0\in T_0\mathbb{R}$ and $\theta_1\in T_1\mathbb{R}$ are null vectors). H is canonically represented as the superposition $H=\chi\circ\overline{H}$ of the homomorphisms $\overline{H}:T\mathbb{R}\times(TM\times\mathfrak{g})\rightarrow\pi_M^{\wedge}(TM\times\mathfrak{g})$ and $\chi:\pi_M^{\wedge}(TM\times\mathfrak{g})\rightarrow TM\times\mathfrak{g}$, [5], see [6]. Hence the mapping $L^{-1}\circ\overline{H}\circ K$ is an endomorphism of the Lie algebroid $T(\mathbb{R}\times M)\times\mathfrak{g}$. It follows from Theorem 2.1 that there exists a mapping $\phi:\mathbb{R}\times M\rightarrow\mathrm{End}(\mathfrak{g})$ and a 1-form $\omega\in\Omega^{1}(\mathbb{R}\times M;\mathfrak{g})$ which satisfy (3.2) and

$$\left(L^{-1}\circ\overline{H}\circ K\right)(w,r)=\left(w,\omega_{\pi(w)}(w)+\phi_{\pi(w)}(r)\right),\ \ w\in T(\mathbb{R}\times M)\,,\ r\in\mathbb{R}.$$

Therefore $\left(\overline{H}\circ K\right)(w,r)=\left(w,\left(d\pi_M(w),\omega_{\pi(w)}(w)+\phi_{\pi(w)}(r)\right)\right)$, whence we have $\overline{H}_{(t,x)}(v,(u,r))=\left(\overline{H}\circ K\right)(w,r)$ for $v\in T_t\mathbb{R}$, $u\in T_xM$ and $w=dj_x(v)+dj_t(u)$. Since $\pi_M^{\wedge}(TM\times\mathfrak{g})\subset T(\mathbb{R}\times M)\bigoplus\pi_M^{*}(TM\times\mathfrak{g})$ and χ is defined as the projection on the second term,

$$H_{(t,x)}(v,(u,r))=\left(u,\omega_{(t,x)}(dj_x(v)+dj_t(u))+\phi_{(t,x)}(r)\right),\tag{3.4}$$

because $\overline{H}(v,(u,r))=(u,H(v,(u,r)))$. By (3.3),

$$\begin{aligned}\left(u,\omega_{(0,x)}(dj_0(u))+\phi_{(0,x)}(r)\right)&=\left(u,\omega_x^{0}(u)+\phi_x^{0}(r)\right),\\ \left(u,\omega_{(1,x)}(dj_1(u))+\phi_{(1,x)}(r)\right)&=\left(u,\omega_x^{1}(u)+\phi_x^{1}(r)\right),\end{aligned}$$

hence we have

$$\begin{aligned}\phi_x^{0}(r)&=\phi_{(0,x)}(r)\,,\\ \phi_x^{1}(r)&=\phi_{(1,x)}(r)\,,\\ \omega_x^{0}(u)&=\omega_{(0,x)}(dj_0(u))=\left(j_0^{*}\omega\right)_x(u)\,,\\ \omega_x^{1}(u)&=\omega_{(1,x)}(dj_1(u))=\left(j_1^{*}\omega\right)_x(u)\,.\end{aligned}$$

Conversely, let $\phi : \mathbb{R} \times M \to \mathrm{End}(\mathfrak{g})$ and $\omega \in \Omega^1(\mathbb{R} \times M; \mathfrak{g})$ satisfy (3.2). Then we see that $H : T\mathbb{R} \times (TM \times \mathfrak{g}) \longrightarrow TM \times \mathfrak{g}$ defined by (3.4) is a homotopy joining endomorphisms H_0 and H_1. ■

3.2 The case of $TM \times \mathbb{R}$

The general form of strong endomorphisms of the trivial Lie algebroid $TM \times \mathbb{R}$ gives Corollary (2.3).

Theorem 3.2 *Let H_0, H_1 be two endomorphisms of the Lie algebroid $TM \times \mathbb{R}$ given by*

$$H_i(u, r) = \left(u, \omega^i_{\pi(u)}(u) + B_i \cdot r\right), \tag{3.5}$$

where B_i are real numbers and ω^i are closed 1-forms on M. The endomorphisms H_0 and H_1 are homotopic if and only if $B_0 = B_1$ and the forms ω^0 and ω^1 are homologous, $[\omega^0] = [\omega^1] \in H^1_{dR}(M)$.

Proof. Suppose that endomorphisms H_0 and H_1 are homotopic. By Theorem 3.1, there is some closed 1-form $\omega \in \Omega^1(\mathbb{R} \times M)$ and mapping $\phi : \mathbb{R} \times M \to \mathrm{End}(\mathbb{R})$ such that $\omega^0 = j_0^*\omega$, $\omega^1 = j_1^*\omega$ and $d\phi = 0$ with $\phi_{(0,\cdot)} = \phi^0$ and $\phi_{(1,\cdot)} = \phi^1$, where $\phi^0_x(r) = B_0 \cdot r$, $\phi^1_x(r) = B_1 \cdot r$ for all $x \in M$. Whence $\phi_{(t,x)}(r) = B \cdot r$ for some $B \in \mathbb{R}$ and thus $B_0 = B = B_1$. Since j_0 and j_1 are homotopic, $[\omega^0] = j_0^{\#}[\omega] = j_1^{\#}[\omega] = [\omega^1]$.

Conversely, suppose that $B_0 = B_1$ and there is a function $f \in C^\infty(M)$, with $\omega^1 - \omega^0 = \delta f$. Define a closed 1-form on $\mathbb{R} \times M$ by $\omega = \pi_M^*\omega_0 + \delta(\pi_{\mathbb{R}} \cdot (f \circ \pi_M))$. Since $(\pi_{\mathbb{R}} \cdot (f \circ \pi_M)) \circ j_0 \equiv 0$ and $(\pi_{\mathbb{R}} \cdot (f \circ \pi_M)) \circ j_1 = f$, we obtain $j_0^*\omega = \omega_0$ and $j_1^*\omega = \omega^1$.

Next, let $\phi : \mathbb{R} \times M \to \mathrm{End}(\mathbb{R})$ be defined by $\phi_{(t,x)}(r) = B \cdot r$, where $B = B_0 = B_1$. Then $\phi_{(0,\cdot)} = \phi^0 = \phi^1 = \phi_{(1,\cdot)}$ and $d\phi = 0$ which means that H_0 and H_1 are homotopic (by Theorem 3.1). ■

Corollary 3.3 *If $H^1_{dR}(M) = 0$ (e.g. M is contractible, in particular if $M = \mathbb{R}^n$ – see [1]), any two endomorphisms H^0 and H^1 of the Lie algebroid $TM \times \mathbb{R}$ given by (3.5) are homotopic if and only if $B_0 = B_1$.*

4 The fundamental group of $TM \times \mathbb{R}$

Let A be a transitive Lie algebroid on a manifold M. Consider the set $\pi(A)$ of homotopy classes $[H]$, where $H : A \to A$ is a strong automorphism of A. $\pi(A)$ under the binary operation $[H] \cdot [G] = [H \circ G]$ forms a group (the unity element is the class of the identity on A and the inverse of $[H]$ is $[H^{-1}]$, see [1]). The group $\pi(A)$ is called the *fundamental group of A*.

Theorem 4.1 *The fundamental group $\pi(TM \times \mathbb{R})$ is isomorphic to the group*

$$\left(H^1_{dR}(M) \times (\mathbb{R} \setminus \{0\}), \circ\right),$$

where

$$([\Phi], a) \circ ([\Psi], b) = ([\Phi] + a[\Psi], ab),$$

Φ, $\Psi \in \Omega^1(M)$, a, $b \in \mathbb{R} \setminus \{0\}$.

Proof. Let G and H be automorphisms of the algebroid $TM \times \mathbb{R}$,

$$G(u,r) = \left(u, \omega^G_{\pi(u)}(u) + B_G \cdot r\right), \quad H(u,r) = \left(u, \omega^H_{\pi(u)}(u) + B_H \cdot r\right),$$

where B_G, B_H are non-zero real numbers and ω^G, ω^H are closed 1-forms on M. Then

$$(G \circ H)(u,r) = \left(u, \omega^G_{\pi(u)}(u) + B_G \cdot \omega^H_{\pi(u)}(u), B_G \cdot B_H \cdot r\right),$$

$(u,r) \in TM \times \mathbb{R}$. Hence we see that the mapping

$$\Delta : \pi(TM \times \mathbb{R}) \longrightarrow H^1_{dR}(M) \times (\mathbb{R} \setminus \{0\}), \quad [H] \longmapsto \left(\left[\omega^H\right], B_H\right),$$

is an isomorphism of the groups $\pi(TM \times \mathbb{R})$ and $\left(H^1_{dR}(M) \times (\mathbb{R} \setminus \{0\}), \circ\right)$. $\blacksquare$

Elements of the group $H^1_{dR}(M) \times (\mathbb{R} \setminus \{0\})$ can be represented by the matrices $\begin{vmatrix} a & \varphi \\ 0 & 1 \end{vmatrix}$, $a \in \mathbb{R} \setminus \{0\}$, $\varphi \in H^1_{dR}(M)$, with the standard multiplication of matrices.

Corollary 4.2 *If* $H^1_{dR}(M) = 0$ *then*

$$\pi(TM \times \mathbb{R}) \cong (\mathbb{R} \setminus \{0\}, \cdot) \cong \mathrm{GL}(\mathbb{R}).$$

It is a generalization of the result from [1].

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