(i) Show that π^* is injective.

(ii) Show that, if $\Phi \in \operatorname{Im} \pi^*$, then $i(Z)\Phi = 0$ and $\theta(Z)\Phi = 0$ for every vertical vector field. Show that if F is connected, then the converse is true.

(iii) Show that if $\mathscr{B}$ admits a cross-section, then the map, $\pi^\#: H(M) \leftarrow H(B)$, is injective.

19. Let E and F be the Lie algebras of $GL(n; \mathbb{R})$ and $U(n)$.

(i) Construct an isomorphism of graded differential algebras

$$(\wedge E^* \otimes \mathbb{C}, \delta_E \otimes \iota) \cong (\wedge F^* \otimes \mathbb{C}, \delta_F \otimes \iota).$$

(ii) Compute $H_L(GL(n; \mathbb{R}))$ and compare it with $H(SO(n))$.

(iii) Compute $H_L(O(p, q))$ (cf. problem 12, Chap. II).

20. Outer automorphisms. Construct an automorphism of $U(n)$ which is not an inner automorphism. Determine its action on $H(U(n))$. Do the same for $SO(2n)$.

Chapter VI

Principal Connections and the Weil Homomorphism

In this chapter G denotes an r-dimensional Lie group with Lie algebra E. $\mathscr{P} = (P, \pi, B, G)$ denotes a fixed principal bundle ($\dim B = n$). $T: P \times G \to P$ denotes the principal action of G on P. The fibre over $x \in B$ is denoted by G_x ; note that this is *not* an isotropy subgroup.

For every $h \in E$, Z_h denotes the fundamental vector field generated by h. The operators $i(Z_h)$, $\theta(Z_h)$ in $A(P)$ are denoted by $i(h)$ and $\theta(h)$ (cf. sec. 3.13). The Lie algebra of invariant vector fields on P is denoted by $\mathscr{X}^I(P)$.

The vertical subbundle of τ_P will be denoted by $\mathbf{V}_P$; we use the boldface notation to avoid confusion with the notation for a principal connection (cf. sec. 6.8). A cross-section of $\mathbf{V}_P$ is called a *vertical* vector field; thus a vector field, Z, on P is vertical if and only if $Z \underset{\pi}{\sim} 0$. The module of vertical vector fields is denoted by $\mathscr{X}_V(P)$.

§1. Vector fields

6.1. The vertical subbundle. Recall that the vertical subbundle is the subbundle $\mathbf{V}_P$ of the tangent bundle τ_P of P whose fibre at z is given by

$$\mathbf{V}_z(P) = \ker(d\pi)_z\,, \qquad z \in P,$$

(sec. 7.1, volume I).

Since G acts freely on P, we also have the fundamental bundle $F_P \subset T_P$ (cf. sec. 3.11).

Proposition I: The fundamental and vertical subbundles coincide.

Proof: Since $d\pi \circ dA_z = 0$, it follows that

$$F_P \subset \mathbf{V}_P\,.$$

On the other hand,

$$\text{rank}\,(F_P) = \dim G = \text{rank}(\mathbf{V}_P).$$

Hence

$$F_P = \mathbf{V}_P\,.$$

Q.E.D.

Corollary I: The map $P \times E \to T_P$ given by $(z, h) \mapsto Z_h(z)$ defines a strong bundle isomorphism

$$P \times E \xrightarrow{\cong} \mathbf{V}_P\,.$$

Proof: Apply sec. 3.9 and sec. 3.11.

Q.E.D.

Corollary II: The map $\mathscr{S}(P) \otimes E \to \mathscr{X}(P)$ given by

$$f \otimes h \mapsto f \cdot Z_h\,, \qquad f \in \mathscr{S}(P), \quad h \in E,$$

defines an isomorphism of $\mathscr{S}(P) \otimes E$ onto $\mathscr{X}_V(P)$. In particular, $\mathscr{X}_V(P)$ is a free $\mathscr{S}(P)$-module, generated by the fundamental vector fields.

Proof: Apply Corollary I.

Q.E.D.

Corollary III: An isomorphism $\mathscr{S}(P; E) \xrightarrow{\cong} \mathscr{X}_V(P)$ is given by $f \mapsto Z_f$, where

$$Z_f(z) = Z_{f(z)}(z).$$

Proof: This is the isomorphism of Corollary II.

Q.E.D.

Example: Suppose B is a single point and $P = G$. Then

$$\mathbf{V}_P = T_P = T_G$$

and the isomorphism,

$$G \times E \xrightarrow{\cong} T_G,$$

is given by

$$(a, h) \mapsto X_h(a),$$

where X_h is the left invariant vector field generated by h.

6.2. Invariant vector fields. Recall from sec. 3.10 that the action of G on P determines the action $(Z, a) \mapsto Z \cdot a$ of G on $\mathscr{X}(P)$, where $Z \cdot a = (T_a)_* Z$. If $Z \cdot a = Z$, $a \in G$, then Z is called an invariant vector field and the space of invariant vector fields is denoted by $\mathscr{X}^I(P)$.

Example: Recall from Example 3, sec. 3.10, that

$$Z_f \cdot a^{-1} = Z_{a \cdot f}, \qquad f \in \mathscr{S}(P; E), \quad a \in G,$$

where $a \cdot f$ is the E-valued function defined by

$$(a \cdot f)(z) = (\mathrm{Ad}\, a) f(z \cdot a).$$

In particular, the vector field Z_f is invariant if and only if the function f is equivariant.

Proposition II: Let Z be an invariant vector field on P. Then there is a unique vector field X on B such that $Z \underset{\pi}{\sim} X$. The correspondence $Z \mapsto X$ is a surjective Lie algebra homomorphism

$$\pi_*: \mathscr{X}^I(P) \to \mathscr{X}(B).$$

Its kernel is given by

$$\ker \pi_* = \mathscr{X}^I(P) \cap \mathscr{X}_V(P).$$

Proof: Since Z is invariant,

$$Z(z \cdot a) = (dT_a)\, Z(z), \qquad a \in G, \quad z \in P.$$

It follows that

$$(d\pi)\, Z(z \cdot a) = (d\pi)\, Z(z), \qquad a \in G, \quad z \in P.$$

This shows that, for each $x \in B$, there is a unique tangent vector $X(x)$ at x satisfying

$$(d\pi)\, Z(z) = X(x), \qquad z \in G_x.$$

The correspondence $x \mapsto X(x)$ defines a set map $X\colon B \to T_B$. To show that X is smooth, let $\sigma\colon U \to P$ be a cross-section over an open set U. Then $X = (d\pi) \circ Z \circ \sigma$, and so X is smooth in U (and hence in B). Hence it is a vector field on B. Clearly, $Z \underset{\pi}{\sim} X$. Since π is surjective, X is uniquely determined by Z.

To prove the second part, consider the map $\pi_*\colon \mathscr{X}^I(P) \to \mathscr{X}(B)$ defined by $Z \mapsto X$. It follows directly from Proposition VIII, sec. 3.13, volume I, that π_* is a homomorphism of Lie algebras. Moreover, $\pi_* Z = 0$ if and only if $(d\pi)\, Z(z) = 0$, $z \in P$; i.e., if and only if Z is vertical. This shows that

$$\ker \pi_* = \mathscr{X}^I(P) \cap \mathscr{X}_V(P).$$

It remains to show that π_* is surjective. Let $X \in \mathscr{X}(B)$ and choose a principal coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ for $\mathscr{P}$. Let $\{p_\alpha\}$ be a partition of unity for B subordinate to the covering $\{U_\alpha\}$. Define vector fields, X_α, in $U_\alpha \times G$ by

$$X_\alpha(x, a) = X(x), \qquad x \in U_\alpha, \quad a \in G.$$

Then $(\psi_\alpha)_* X_\alpha \in \mathscr{X}^I(\pi^{-1}(U_\alpha))$ and so an invariant vector field Z on P is given by

$$Z = \sum_\alpha \pi^* p_\alpha \cdot (\psi_\alpha)_* X_\alpha.$$

Evidently, $\pi_* Z = X$.

Q.E.D.

Corollary: If $Z \in \mathscr{X}^I(P)$ and $Y \in \mathscr{X}_V(P)$, then

$$[Z, Y] \in \mathscr{X}_V(P).$$

Proof: Since $Z \underset{\pi}{\sim} \pi_* Z$ and $Y \underset{\pi}{\sim} 0$, it follows that

$$[Z, Y] \underset{\pi}{\sim} [\pi_* Z, 0] = 0.$$

Q.E.D.

§2. Differential forms

6.3. The homomorphism π^*. A differential form, Φ, on P is called *invariant* if it is invariant under the right action of G. The algebra of invariant forms is denoted by $A_I(P)$.

A differential form Φ on P is called *horizontal* if $i(Y)\,\Phi = 0$, $Y \in \mathscr{X}_V(P)$. Since the fundamental bundle coincides with the vertical bundle (Proposition I, sec. 6.1), Φ is horizontal if and only if it is horizontal with respect to the action of G (cf. sec. 3.13). The algebra of horizontal forms is denoted by $A(P)_{i=0}$.

Now consider the homomorphism $\pi^*\colon A(P) \leftarrow A(B)$.

Proposition III: The homomorphism π^* is injective. The image of π^* consists precisely of the differential forms which are both invariant and horizontal.

Proof: Since the maps π and $(d\pi)_z$ $(z \in P)$ are surjective, π^* must be injective. Moreover, the relations,

$$T_a^* \circ \pi^* = \pi^*, \qquad a \in G,$$

and

$$(d\pi)\, Z_h(z) = 0, \qquad h \in E, \quad z \in P,$$

imply that the differential forms in $\operatorname{Im} \pi^*$ are invariant and horizontal.

Now assume $\Psi \in A(P)$ is invariant and horizontal. Choose a principal coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ for $\mathscr{P}$. Since ψ_α is equivariant, $\psi_\alpha^*\Psi \in A(U_\alpha \times G)$ is invariant and horizontal with respect to the action, $((x, a), b) \mapsto (x, ab)$, of G on $U_\alpha \times G$. It follows that there is a *unique* $\Phi_\alpha \in A(U_\alpha)$ such that

$$\Phi_\alpha \times 1 = \psi_\alpha^*\Psi.$$

This uniqueness implies that Φ_α and Φ_β agree in $U_\alpha \cap U_\beta$. Hence there is a unique differential form, $\Phi \in A(B)$, such that

$$\Phi(x) = \Phi_\alpha(x), \qquad x \in U_\alpha .$$

Clearly, $\pi^*\Phi = \Psi$.

Q.E.D.

Definition: The differential forms which are both invariant and horizontal are called *basic*. They form the *basic subalgebra*, $A_B(P)$, of $A(P)$.

Remark: Proposition III shows that π^* can be considered as an isomorphism

$$\pi^*: A(B) \xrightarrow{\cong} A_B(P).$$

Finally recall that $A(P)_{\theta=0}$ denotes the subalgebra of $A(P)$ consisting of differential forms, Φ, satisfying

$$\theta(h)\Phi = 0, \qquad h \in E.$$

Set $A(P)_{i=0} \cap A(P)_{\theta=0} = A(P)_{i=0,\theta=0}$ (cf. sec. 3.13). If G is connected, Proposition VI, sec. 3.13, shows that $A(P)_{i=0,\theta=0}$ is the basic subalgebra. Thus in this case we can write

$$\pi^*: A(B) \xrightarrow{\cong} A(P)_{i=0,\theta=0}\,.$$

6.4. Homomorphisms. Let $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, \hat{B}, G)$ be a second principal bundle with the same group G and let $\varphi: P \to \hat{P}$ be a homomorphism of principal bundles inducing $\psi: B \to \hat{B}$. Since φ is equivariant, the fundamental vector fields on P and $\hat{P}$ generated by the same vector, $h \in E$, are φ-related,

$$Z_h \underset{\varphi}{\sim} \hat{Z}_h\,, \qquad h \in E$$

(cf. sec. 3.9). This yields the commutation relations (cf. sec. 3.14)

$$\varphi^* \circ \hat{\theta}(h) = \theta(h) \circ \varphi^*, \qquad \varphi^* \circ \hat{i}(h) = i(h) \circ \varphi^*, \qquad h \in E,$$

where $\hat{\theta}(h) = \theta(\hat{Z}_h)$ and $\hat{i}(h) = i(\hat{Z}_h)$. Moreover,

$$\varphi^* \circ \hat{T}_a^* = T_a^* \circ \varphi^*, \qquad a \in G.$$

Hence the homomorphism $\varphi^*: A(\hat{P}) \to A(P)$ restricts to homomorphisms $A_I(\hat{P}) \to A_I(P)$ and $A_B(\hat{P}) \to A_B(P)$ and we have the commutative diagram

$$\begin{array}{ccc} A_B(P) & \xleftarrow{\varphi^*} & A_B(\hat{P}) \\ {\scriptstyle \pi^*}\uparrow{\scriptstyle\cong} & & {\scriptstyle\cong}\uparrow{\scriptstyle\hat{\pi}^*} \\ A(B) & \xleftarrow[\psi^*]{} & A(\hat{B}) \end{array}.$$

6.5. Integration over the fibre. An orientation of E (the Lie algebra of G) determines an orientation in the fibre bundle $\mathscr{P}$ (cf. sec. 7.4, volume I) as follows: Give the trivial vector bundle, $P \times E$, the induced orientation, and then use the bundle isomorphism,

$$P \times E \xrightarrow{\cong} \mathbf{V}_P ,$$

(Corollary I to Proposition I, sec. 6.1) to orient $\mathbf{V}_P$. Finally, recall from sec. 7.4, volume I, that an orientation of $\mathbf{V}_P$ determines an orientation of $\mathscr{P}$.

Example: If B is a point, $P = G$, then $\mathbf{V}_P = T_P = T_G$ and the induced orientation of $\mathscr{P}$ is simply an orientation of G. It is the *left* invariant orientation induced by that of E (cf. sec. 1.13) as follows from the example of sec. 6.1.

More generally, if $P = B \times G$, then $\mathbf{V}_P = B \times T_G$ and the orientation of $\mathbf{V}_P$ is that obtained from the orientation of T_G. Thus the orientation of $\{x\} \times G$ induced from that of $\mathscr{P}$ is simply the orientation of G just defined.

Now, let $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, \hat{B}, G)$ be a second principal bundle, and suppose $\varphi: P \to \hat{P}$ is a homomorphism of principal bundles inducing $\psi: B \to \hat{B}$. Then (since φ is equivariant) the diagram,

$$\begin{array}{ccc} P \times E & \xrightarrow{\varphi \times \iota} & \hat{P} \times E \\ \cong \downarrow & & \downarrow \cong \\ \mathbf{V}_P & \xrightarrow[d\varphi]{} & \mathbf{V}_{\hat{P}} \end{array} ,$$

commutes. It follows that φ preserves the induced bundle orientations.

In particular, if $\{(U_\alpha , \psi_\alpha)\}$ is a principal coordinate representation for $\mathscr{P}$, then

$$\psi_{\alpha,x}: (G, p, \{x\}, G) \to (P, \pi, B, G)$$

can be considered as a homomorphism of principal bundles. It follows that the maps $\psi_{\alpha,x}: G \to G_x$ are orientation preserving, where G_x is given the orientation induced from the orientation of $\mathscr{P}$.

Next, assume that G is compact and connected. Since $\mathscr{P}$ is orientable, the fibre integral (cf. sec. 7.12, volume I) is defined, depending of course, on the orientation of $\mathscr{P}$.

On the other hand let $\Delta \in A^r(G)$ be the unique invariant r-form such that $\int_G \Delta = 1$ (cf. sec. 1.15). Let $\epsilon \in \wedge^r E$ be the element satisfying

$$\langle \Delta(e), \epsilon \rangle = 1.$$

Write $\epsilon = h_1 \wedge \cdots \wedge h_r$ ($h_i \in E$). The operator,

$$i(\epsilon) = i(h_r) \circ \cdots \circ i(h_1),$$

in $A(P)$ depends only on ϵ. Moreover, since

$$i(h) \circ T^*_{a^{-1}} = T^*_{a^{-1}} \circ i((\mathrm{Ad}\, a)h), \qquad a \in G, \quad h \in E$$

(cf. diagram (3.1), sec. 3.9), it follows that

$$i(\epsilon) \circ T^*_{a^{-1}} = T^*_{a^{-1}} \circ i(\det \mathrm{Ad}\, a \cdot \epsilon) = T^*_{a^{-1}} \circ i(\epsilon), \qquad a \in G.$$

Since $h \wedge \epsilon = 0$ ($h \in E$), we also have $i(h) \circ i(\epsilon) = 0$. These relations show that $i(\epsilon)$ restricts to an operator

$$i(\epsilon): A_I(P) \to A_B(P).$$

Proposition IV: The diagram,

$$\begin{array}{ccc} A_I(P) & \xrightarrow{\text{inclusion}} & A(P) \\ {\scriptstyle \omega \circ i(\epsilon)} \downarrow & & \downarrow {\scriptstyle \int_G} \\ A_B(P) & \xleftarrow[\pi^*]{\cong} & A(B), \end{array}$$

commutes, where ω is the involution defined by

$$\omega(\Phi) = (-1)^{pr}\Phi, \qquad \Phi \in A_B^p(P).$$

Proof: It is clearly sufficient to consider the case that $\mathscr{P}$ is the product bundle: $P = B \times G$. We must show that

$$\pi^* \int_G \Phi = \omega(i(\epsilon)\Phi), \qquad \Phi \in A_I(P).$$

Recall the bigradation of $A(P) = A(B \times G)$ (cf. sec. 3.20, volume I). Evidently, $A_I(P)$ is a bigraded subalgebra of $A(P)$,

$$A_I(P) = \sum_{p=0}^{n} \sum_{q=0}^{r} A_I^{p,q}(P).$$

Moreover, the operators $i(\epsilon)$ and $\pi^* \circ \oint_G$ are both homogeneous of bidegree $(0, -r)$. Hence it is sufficient to consider the case that that $\Phi \in A_I^{p,r}(P)$.

In this case a simple computation shows that

$$\Phi = (-1)^{pr}\, i(\epsilon)\, \Phi \wedge \pi_G^* \Delta$$

(where $\pi_G\colon B \times G \to G$ is the projection). Since $i(\epsilon)\Phi \in \operatorname{Im} \pi^*$, it follows that (cf. Example 2, sec. 7.12, volume I)

$$\pi^* \oint_G \Phi = (-1)^{pr}\, i(\epsilon)\Phi = \omega(i(\epsilon)\Phi),$$

as desired.

Q.E.D.

6.6. Vector-valued differential forms. We recall, for convenience, some facts from volume I, and from Chap. III. Let W be a finite-dimensional vector space. Then $A(P; W)$, the space of W-valued differential forms in P, is a graded left module over the graded algebra $A(P)$, and an isomorphism, $A(P) \otimes W \to A(P; W)$, is given by $\Phi \otimes w \mapsto \Phi \wedge w$, $w \in W$. (Here w also denotes the constant function $P \to w$.)

The operators $i(Z)$, $\theta(Z)$, T_a^*, and δ (where $Z \in \mathscr{X}(P)$ and $a \in G$) extend to operators $i(Z) \otimes \iota$, $\theta(Z) \otimes \iota$, $T_a^* \otimes \iota$, and $\delta \otimes \iota$ in $A(P; W)$, again denoted by $i(Z)$, $\theta(Z)$, T_a^* and δ. In particular, $i(h)$ and $\theta(h)$ ($h \in E$) are regarded as operators in $A(P; W)$.

A W-valued differential form, Ω, is called *horizontal* if $i(h)\Omega = 0$, $h \in E$ (cf. sec. 6.3). The horizontal forms are a graded subspace of $A(P; W)$, denoted by $A(P; W)_{i=0}$. The isomorphism,

$$A(P) \otimes W \xrightarrow{\cong} A(P; W),$$

restricts to an isomorphism

$$A(P)_{i=0} \otimes W \xrightarrow{\cong} A(P; W)_{i=0}.$$

Now suppose that R is a representation of G in W, and let R' be the derived representation of E in W. The operators $\iota \otimes R(a)$ and $\iota \otimes R'(h)$ ($a \in G$, $h \in E$) in $A(P; W)$ are denoted simply by $R(a)$ and $R'(h)$.

Thus (cf. sec. 3.15) a W-valued form Φ is equivariant if

$$T_a^*\Phi = R(a^{-1})\Phi, \qquad a \in G.$$

According to Proposition VII, sec. 3.15, if G is connected this is equivalent to

$$\theta(h)\Phi = -R'(h)\Phi, \qquad h \in E.$$

The space of equivariant forms is written $A_I(P; W)$.

On the other hand, a W-valued form, Φ, is called *invariant* if $T_a^*\Phi = \Phi$, $a \in G$. Thus Φ is invariant if and only if $\Phi \in A_I(P) \otimes W$. If the representation is trivial, then the definitions of equivariant and invariant forms coincide.

Finally, the space $A(P; W)_{i=0} \cap A_I(P; W)$ is called the space of *basic W-valued differential forms* and is denoted by $A_B(P; W)$. If $W = \mathbb{R}$ and R is the trivial representation, this reduces to the definition of sec. 6.3. A generalization of Proposition III of that section to vector-valued forms will be given in sec. 8.22.

6.7. Multilinear maps of vector-valued forms. Recall that if W_1 and W are finite-dimensional vector spaces, then a linear map $\varphi: W_1 \to W$ induces the $\mathscr{S}(P)$-linear map,

$$\varphi_* = \iota \otimes \varphi: A(P; W_1) \to A(P; W),$$

given by

$$\varphi_*\Psi(Z_1, ..., Z_p) = \varphi(\Psi(Z_1, ..., Z_p)), \qquad \Psi \in A^p(P; W_1), \quad Z_1, ..., Z_p \in \mathscr{X}(P).$$

More generally, let $\varphi: W_1 \times \cdots \times W_k \to W$ be a k-linear map of finite-dimensional vector spaces. Then φ determines the k-linear map (over $\mathscr{S}(P)$),

$$\varphi_*: A(P; W_1) \times \cdots \times A(P; W_k) \to A(P; W),$$

given by

$$\varphi_*(\Psi_1, ..., \Psi_k)(Z_1, ..., Z_m) = \frac{1}{p_1! \cdots p_k!} \sum_{\sigma \in S^m} \epsilon_\sigma \varphi(\Psi_1(Z_{\sigma(1)}, ...), ..., \Psi_k(..., Z_{\sigma(m)})),$$

where

$$\Psi_i \in A^{p_i}(P; W_i) \quad (i = 1,..., k), \qquad Z_\nu \in \mathscr{X}(P) \quad (\nu = 1,..., m),$$

$$\sum_{i=1}^k p_i = m.$$

If we identify $A(P; W_i)$ with $A(P) \otimes W_i$, we can write

$$\varphi_*(\Phi_1 \otimes w_1, ..., \Phi_k \otimes w_k) = (\Phi_1 \wedge \cdots \wedge \Phi_k) \otimes \varphi(w_1, ..., w_k),$$

$$\Phi_i \in A(P), \quad w_i \in W_i, \quad i = 1, 2, ..., k.$$

In particular, if R represents G in W, then a bilinear map, $E \times W \to W$, is given by

$$(h, w) \mapsto R'(h)w.$$

The corresponding map of differential forms is written

$$(\Phi, \Psi) \mapsto \Phi(\Psi), \qquad \Phi \in A(P; E), \quad \Psi \in A(P; W).$$

Thus if $h \in E$ and h also denotes the constant function $P \to h$, then

$$h(\Psi) = R'(h)\Psi.$$

As a special case suppose $W = E$ and R is the adjoint representation. In this case the original bilinear map is given by $(h, k) \mapsto [h, k]$ $(h, k \in E)$ and the corresponding map of differential forms is written

$$(\Phi, \Psi) \mapsto [\Phi, \Psi].$$

The relation, $R'([h, k]) = R'(h) \circ R'(k) - R'(k) \circ R'(h)$, leads to the formula

$$[\Phi_1, \Phi_2](\Psi) = \Phi_1(\Phi_2(\Psi)) - (-1)^{pq}\, \Phi_2(\Phi_1(\Psi)),$$
$$\Phi_1 \in A^p(P; E), \quad \Phi_2 \in A^q(P; E), \quad \Psi \in A(P; W).$$

In particular, if $\Phi \in A(P; E)$ has odd degree, then

$$[\Phi, \Phi](\Psi) = 2\Phi(\Phi(\Psi)), \qquad \Psi \in A(P; W).$$

§3. Principal connections

6.8. Connections in a principal bundle. The right action of G on P induces a right action, dT, of G in the tangent bundle T_P. It is given by

$$dT(\zeta, a) = (dT_a)\zeta, \qquad a \in G, \quad \zeta \in T_P$$

(cf. Example 7, sec. 3.2). The equations $\pi \circ T_a = \pi \quad (a \in G)$ yield

$$d\pi \circ dT_a = d\pi.$$

Thus the vertical subbundle $\mathbf{V}_P$ is stable under dT.

Definition: A *principal connection* in $\mathscr{P}$ is a strong bundle map $V: T_P \to T_P$ satisfying the conditions:

(i) $V^2 = V$.
(ii) $\operatorname{Im} V_z = \mathbf{V}_z(P)$, $z \in P$.
(iii) V is equivariant; i.e..

$$dT_a \circ V = V \circ dT_a, \qquad a \in G.$$

Remark: We remind the reader of the following notation conventions:

(1) $\mathbf{V}_P$ is the vertical bundle with fibre $\mathbf{V}_z(P)$ at $z \in P$.
(2) V is a principal connection restricting to linear projections

$$V_z: T_z(P) \to \mathbf{V}_z(P), \qquad z \in P.$$

Examples: 1. For the trivial bundle $P = B \times G$, the vertical subbundle is $B \times T_G$, and a principal connection, V, is given by

$$V(\xi, \eta) = (0, \eta), \qquad \xi \in T_x(B), \quad \eta \in T_a(G).$$

2. Let $\{U_\alpha\}$ be an open cover of B and let V_α be a principal connection in the bundle $(\pi^{-1}U_\alpha, \pi, U_\alpha, G)$. Let $\{\hat{U}_\alpha\}$ be a locally finite refinement of the open cover $\{U_\alpha\}$ and suppose that $\{p_\alpha\}$ is a family of smooth functions on B such that $\operatorname{carr} p_\alpha \subset \hat{U}_\alpha$ and $\sum_\alpha p_\alpha = 1$. (Note that

$\{p_\alpha\}$ need not satisfy $p_\alpha(x) \geq 0$.) Then a principal connection, V, in $\mathscr{P}$ is given by

$$V = \sum_\alpha \pi^* p_\alpha \cdot V_\alpha .$$

Remark: Examples 1 and 2 show that every principal bundle admits a principal connection.

6.9. Horizontal subbundles. Let V be a principal connection in $\mathscr{P}$. The subspaces $\ker V_z \subset T_z(P)$ ($z \in P$) are the fibres of a subbundle, $\mathbf{H}_P$, of τ_P. Evidently

$$\tau_P = \mathbf{H}_P \oplus \mathbf{V}_P;$$

i.e., $\mathbf{H}_P$ is a horizontal bundle (cf. see 7.2, volume I). It is called *the horizontal bundle associated with the connection.* Its fibres are called the *horizontal subspaces* and are written $\mathbf{H}_z(P)$.

The bundle $\mathbf{H}_P$ is stable under the action of G. Moreover the map, $V \mapsto \mathbf{H}_P$, is a bijection between principal connections and G-stable horizontal bundles.

Examples: 1. The horizontal subbundle corresponding to the principal connection of Example 1, sec. 6.8, is given by $\mathbf{H}_P = T_B \times G$.

2. Suppose a Riemannian metric has been defined in P so that the bundle maps dT_a: $T_P \to T_P$ ($a \in G$) are all isometries. Then $\mathbf{H}_P = \mathbf{V}_P^\perp$ is a G-stable horizontal subbundle. The corresponding principal connection is simply the orthogonal projection $T_z(P) \to \mathbf{V}_z(P)$ at each point $z \in P$.

Now let V be a fixed principal connection in $\mathscr{P}$ and let $\mathbf{H}_P$ be the corresponding horizontal subbundle. Then

$$H = \iota - V \colon T_P \to \mathbf{H}_P$$

is the projection with kernel $\mathbf{V}_P$.

Since V and H are strong bundle maps, they determine module homomorphisms,

$$V_* \colon \mathscr{X}(P) \to \mathscr{X}(P) \qquad \text{and} \qquad H_* \colon \mathscr{X}(P) \to \mathscr{X}(P),$$

given by

$$(V_* Z)(z) = V(Z(z)) \qquad \text{and} \qquad (H_* Z)(z) = H(Z(z)), \quad Z \in \mathscr{X}(P), \quad z \in P.$$

The cross-sections in $\mathbf{H}_P$ are called *horizontal vector fields*, and the module of horizontal vector fields is denoted by $\mathscr{X}_H(P)$. It is, in general, not stable under the Lie bracket. The decomposition $\tau_P = \mathbf{H}_P \oplus \mathbf{V}_P$ leads to the direct decomposition,

$$\mathscr{X}(P) = \mathscr{X}_H(P) \oplus \mathscr{X}_V(P),$$

which is given explicitly by

$$Z \mapsto (H_*Z, V_*Z).$$

Since the operator V is equivariant with respect to the action of G, so is H. It follows that H_* and V_* commute with the isomorphisms,

$$(T_a)_* : \mathscr{X}(P) \xrightarrow{\cong} \mathscr{X}(P), \qquad a \in G.$$

In particular, if Z is invariant then so are H_*Z and V_*Z. Thus the direct decomposition above restricts to a direct decomposition,

$$\mathscr{X}^I(P) = \mathscr{X}_H^I(P) \oplus \mathscr{X}_V^I(P),$$

where

$$\mathscr{X}_H^I(P) = \mathscr{X}^I(P) \cap \mathscr{X}_H(P) \qquad \text{and} \qquad \mathscr{X}_V^I(P) = \mathscr{X}^I(P) \cap \mathscr{X}_V(P).$$

Now consider the surjective homomorphism,

$$\pi_*: \mathscr{X}^I(P) \to \mathscr{X}(B),$$

of $\mathscr{S}(B)$-modules (cf. sec. 6.2). Since $\ker \pi_* = \mathscr{X}_V^I(P)$, it follows that π_* restricts to an isomorphism

$$\pi_*: \mathscr{X}_H^I(P) \xrightarrow{\cong} \mathscr{X}(B).$$

The inverse isomorphism, $\lambda: \mathscr{X}(B) \xrightarrow{\cong} \mathscr{X}_H^I(P)$, is called the *horizontal lifting isomorphism* for the principal connection V.

Proposition V: The lifting isomorphism satisfies

$$\lambda([X_1, X_2]) = H_*([\lambda X_1, \lambda X_2]), \qquad X_1, X_2 \in \mathscr{X}(B).$$

Proof: In fact,

$$\pi_*\lambda([X_1, X_2]) = [X_1, X_2] = [\pi_*\lambda X_1, \pi_*\lambda X_2] = \pi_*([\lambda X_1, \lambda X_2]),$$

whence $\pi_*(\lambda([X_1, X_2]) - [\lambda X_1, \lambda X_2]) = 0.$

Thus $\lambda([X_1, X_2]) - [\lambda X_1, \lambda X_2]$ is vertical. It follows that

$$\lambda([X_1, X_2]) = H_*\lambda([X_1, X_2]) = H_*([\lambda X_1, \lambda X_2]).$$

Q.E.D.

6.10. The connection form. Let $V\colon T_P \to T_P$ be a principal connection in $\mathscr{P}$. In sec. 6.1 we obtained a strong bundle isomorphism

$$P \times E \xrightarrow{\cong} \mathbf{V}_P .$$

Composing V with the inverse of this isomorphism gives a strong bundle map

$$\alpha\colon T_P \to P \times E.$$

The isomorphism, $P \times E \xrightarrow{\cong} \mathbf{V}_P$, is given by

$$(z, h) \mapsto Z_h(z) = (dA_z)_e\,(h), \qquad z \in P, \quad h \in E.$$

It follows that, for $\zeta \in T_z(P)$, $\alpha(\zeta) = (z, (dA_z)_e^{-1} V_z\zeta)$. Thus an E-valued 1-form, ω, on P is given by

$$\omega(z; \zeta) = (dA_z)_e^{-1}\,(V_z\zeta).$$

Definition: ω is called the *connection form* associated with V.

Recall from Corollary III to Proposition I, sec. 6.1, that every E-valued function f on P determines a vertical vector field Z_f. In particular, suppose $Y \in \mathscr{X}(P)$ and consider the function $\omega(Y)$. It follows from the definition of ω that

$$Z_{\omega(Y)} = V_*Y.$$

Thus, $\omega(Y) = 0$ if and only if Y is horizontal.

Proposition VI: The connection form has the following properties:

(1) $i(h)\omega = h$, $h \in E$.
(2) $T_a^*\omega = (\operatorname{Ad} a^{-1})\omega$, $a \in G$.

Conversely, if $\sigma \in A^1(P; E)$ satisfies these conditions, there is a unique principal connection in $\mathscr{P}$ for which it is the connection form.

Remark: Note that (2) asserts that ω is equivariant with respect to the adjoint representation of G.

Proof: Suppose first that $\omega \in A^1(P; E)$ is derived from a principal connection V as described above. Then

$$(i(h)\omega)(z) = \omega(z; Z_h(z)) = (dA_z)^{-1}(dA_z)h = h, \qquad z \in P, \quad h \in E,$$

whence (1).

Moreover, according to sec. 3.1, $T_a \circ A_z = A_{z \cdot a} \circ \tau_{a^{-1}}$. Hence

$$dT_a \circ (dA_z)_e = (dA_{z \cdot a})_e \circ \operatorname{Ad} a^{-1}, \qquad a \in G.$$

Since V is equivariant, it follows that for $a \in G$, $z \in P$, $\zeta \in T_z(P)$,

$$\omega(z \cdot a; (dT_a)\zeta) = (\operatorname{Ad} a^{-1})\,\omega(z; \zeta),$$

whence (2).

Conversely, assume that σ is an E-valued 1-form on P which satisfies (1) and (2). Thus each $\sigma(z)$ is a linear map $T_z(P) \to E$. Define $V: T_P \to T_P$ by setting

$$V(z) = (dA_z)_e \circ \sigma(z), \qquad z \in P.$$

Then V is the unique principal connection inducing σ.

Q.E.D.

Corollary I: The connection form satisfies the relations

$$i(h)\omega = h \qquad \text{and} \qquad \theta(h)\omega = -(\operatorname{ad} h)\omega, \quad h \in E.$$

Conversely, let σ be an E-valued 1-form on P which satisfies these relations. Assume that G is connected. Then σ is a connection form on P.

Proof: This is an immediate consequence of the proposition and Proposition VII, sec. 3.15.

Q.E.D.

Recall from Proposition V, sec. 3.10, that the Lie product of a fundamental field and an invariant field is zero. On the other hand, we have

Corollary II: The Lie product of a fundamental field and a horizontal field is horizontal.

Proof: We must show that $\omega([Z_h, Y]) = 0$, where Z_h is a fundamental field and Y is horizontal. Since Y is horizontal,

$$i(Y)\omega = \omega(Y) = 0.$$

Thus, by Corollary I,

$$i(Y)\,\theta(h)\omega = -i(Y)(\operatorname{ad} h)\omega = -(\operatorname{ad} h)\,i(Y)\omega = 0$$

and so

$$\omega([Z_h, Y]) = i([Z_h, Y])\omega = \theta(h)\,i(Y)\omega - i(Y)\,\theta(h)\omega = 0.$$

Q.E.D.

§4. The covariant exterior derivative

6.11. The operator H^*. Fix a principal connection, V, in $\mathscr{P}$ and set $H = \iota - V$. Consider the space $A(P; W)$, where W is a finite-dimensional vector space. The operator, $H^*: A(P; W) \to A(P; W)$, defined by

$$(H^*\Omega)(z; \zeta_1, ..., \zeta_p) = \Omega(z; H\zeta_1, ..., H\zeta_p), \quad z \in P, \quad \zeta_i \in T_z(P), \quad \Omega \in A^p(P; W),$$

is called the *horizontal projection associated with V*.

Lemma I: The operator H^* has the following properties:

(1) $H^*(\Phi \wedge \Omega) = H^*\Phi \wedge H^*\Omega, \quad \Phi \in A(P), \Omega \in A(P; W)$.
(2) H^* is a projection on the subspace of horizontal forms:

$$(H^*)^2 = H^* \qquad \text{and} \qquad \operatorname{Im} H^* = A(P; W)_{i=0}.$$

(3) $H^* \circ T_a^* = T_a^* \circ H^*, \quad a \in G$.
(4) $H^* \circ \theta(h) = \theta(h) \circ H^*, \quad h \in E$.
(5) $H^*\omega = 0$ (ω, the connection form).

Proof: Property (1) is obvious. Properties (2) and (3) follow from the relations

$$H^2 = H, \qquad H \circ V = V \circ H = 0, \qquad H \circ dT_a = dT_a \circ H.$$

(4) is a consequence of (3) and Proposition X, sec. 4.11, volume I, and (5) is obvious.

Q.E.D.

6.12. Covariant exterior derivative. The *covariant exterior derivative* associated with a principal connection, V, is the linear map, $\nabla: A(P; W) \to A(P; W)$, given by

$$\nabla = H^* \circ \delta.$$

Proposition VII: The covariant exterior derivative has the following properties:

(1) $\nabla(\Phi \wedge \Omega) = \nabla\Phi \wedge H^*\Omega + (-1)^p H^*\Phi \wedge \nabla\Omega,$
$$\Phi \in A^p(P), \quad \Omega \in A(P; W).$$

(2) $i(h) \circ \nabla = 0, \quad h \in E.$
(3) $\nabla \circ T_a^* = T_a^* \circ \nabla, \quad a \in G.$
(4) $\nabla \circ \theta(h) = \theta(h) \circ \nabla, \quad h \in E.$
(5) $\nabla \circ \pi^* = \delta \circ \pi^*.$

Proof: (1): Apply H^* to the formula

$$\delta(\Phi \wedge \Omega) = \delta\Phi \wedge \Omega + (-1)^p \Phi \wedge \delta\Omega.$$

(2), (3), and (4) follow from Lemma I, and (5) is a consequence of the relation $H^* \circ \pi^* = \pi^*$.

Q.E.D.

Corollary: ∇ restricts to a map ∇_H: $A(P; W)_{i=0} \to A(P; W)_{i=0}$.

Remark: In general, $\nabla^2 \neq 0$.

Proposition VIII: Let φ: $W_1 \times \cdots \times W_k \to W$ be a k-linear map and let Φ_i be a W_i-valued differential form of degree p_i ($i = 1, ..., k$). Then

$$\nabla[\varphi_*(\Phi_1, ..., \Phi_k)] = \sum_{i=1}^{k} (-1)^{p_1+\cdots+p_{i-1}} \varphi_*(H^*\Phi_1, ..., \nabla\Phi_i, ..., H^*\Phi_k).$$

Proof: It is sufficient to consider the case $\Phi_i = \Psi_i \otimes w_i$ with $\Psi_i \in A^{p_i}(P)$ and $w_i \in W_i$. Then

$$\varphi_*(\Phi_1, ..., \Phi_k) = (\Psi_1 \wedge \cdots \wedge \Psi_k \otimes) \varphi(w_1, ..., w_k)$$

and so the proposition follows from Proposition VII, (1).

Q.E.D.

Applying the covariant exterior derivative to functions on P we obtain an operator

$$\nabla: \mathscr{S}(P) \to A^1(P)$$

which satisfies the relations

(1) $\nabla(f \cdot g) = \nabla f \cdot g + f \cdot \nabla g, \quad f, g \in \mathscr{S}(P).$
(2) $i(h) \circ \nabla = 0, \quad h \in E.$
(3) $T_a^* \circ \nabla = \nabla \circ T_a^*, \quad a \in G.$
(4) $\nabla f = \delta f, f \in \mathscr{S}_I(P).$

Conversely, assume that an operator ∇: $\mathscr{S}(P) \to A^1(P)$ which satisfies these equations is given. Then there is a unique principal connection on $\mathscr{P}$ such that ∇ is the corresponding covariant exterior derivative. In fact, with each vector field Z on P associate the map

$$Q_Z: \mathscr{S}(P) \to \mathscr{S}(P)$$

given by

$$Q_Z(f) = i(Z)(\delta f - \nabla f).$$

In view of (1), Q_Z is a derivation in the algebra $\mathscr{S}(P)$. Hence there is a unique vector field, Y_Z, on P such that $Q_Z(f) = Y_Z(f)$. The operator $Z \mapsto Y_Z$ in $\mathscr{X}(P)$ is $\mathscr{S}(P)$-linear, and hence it determines a bundle map, $V: T_P \to T_P$, such that

$$V_*(Z) = Y_Z, \qquad Z \in \mathscr{X}(P).$$

Condition (4) implies that each Y_Z is vertical and so V maps T_P into $\mathbf{V}_P$. On the other hand, if Z is vertical, condition (2) implies that

$$Y_Z = Z, \qquad Z \in \mathscr{X}_V(P);$$

thus V restricts to the identity on $\mathbf{V}_P$. Finally, (3) shows that

$$dT_a \circ V = V \circ dT_a, \qquad a \in G.$$

Hence the bundle map, V, is a principal connection in $\mathscr{P}$.

Now set $H = \iota - V$. Then

$$\begin{aligned} H_*(Z)(f) &= Z(f) - V_*(Z)(f) \\ &= i(Z)\nabla f, \qquad Z \in \mathscr{X}(P), \quad f \in \mathscr{S}(P). \end{aligned}$$

Hence $H^*\delta f = \nabla f$. It follows that ∇ is the covariant exterior derivative of f with respect to this connection.

Finally, if V_1 is any connection on $\mathscr{P}$ such that the corresponding covariant exterior derivative coincides with ∇, then we have

$$\nabla f(z; \zeta) = \delta f(z; H_1\zeta), \qquad z \in P, \quad \zeta \in T_z(P), \quad f \in \mathscr{S}(P).$$

This relation shows that the operator H_1 (and hence the connection) is uniquely determined by ∇.

6.13. Basic forms. Let R be a representation of G in W. It follows from Proposition VII, sec. 6.12, that the space $A_B(P; W)$ of basic forms (cf. sec. 6.6) is stable under the covariant exterior derivative of a principal connection.

Proposition IX: Let ∇ and ω be the covariant exterior derivative and connection form of a principal connection. Then (cf. sec. 6.7)

$$\nabla\Phi = \delta\Phi + \omega(\Phi), \qquad \Phi \in A_B(P; W).$$

Proof: Since Φ is horizontal,

$$i(h)\,\delta\Phi = \theta(h)\Phi \qquad \text{and} \qquad i(h)\,(\omega(\Phi)) = (i(h)\omega)\,(\Phi), \quad h \in E.$$

Moreover, according to Proposition VI, sec. 6.10, $i(h)\omega$ is the constant function $P \mapsto h$. Thus (cf. sec. 6.7)

$$(i(h)\,\omega)\,(\Phi) = R'(h)\,\Phi.$$

Since Φ is equivariant, these relations yield (cf. Proposition VII, sec. 3.15)

$$i(h)(\delta\Phi + \omega(\Phi)) = \theta(h)\Phi + R'(h)\Phi = 0, \qquad h \in E,$$

and so $\delta\Phi + \omega(\Phi)$ is horizontal. It follows that

$$\delta\Phi + \omega(\Phi) = H^*(\delta\Phi + \omega(\Phi)) = \nabla\Phi + (H^*\omega)(H^*\Phi) = \nabla\Phi,$$

(because $H^*\omega = 0$).

Q.E.D.

Corollary: If $W = E$ and R is the adjoint representation, then

$$\nabla\Phi = \delta\Phi + [\omega, \Phi], \qquad \Phi \in A_B(P; E).$$

§5. Curvature

In this article V denotes a principal connection in the principal bundle $\mathscr{P}$. The corresponding connection form, horizontal projection, and covariant exterior derivative are denoted by ω, H^*, and ∇, respectively.

6.14. Curvature. The *curvature form* of the connection V is the E-valued 2-form, Ω, on P given by

$$\Omega = \nabla\omega.$$

Proposition X: The curvature form has the following properties:

(1) Ω is horizontal: $i(h)\Omega = 0$, $h \in E$.

(2) Ω is equivariant: $T_a^*\Omega = (\mathrm{Ad}\, a^{-1})\Omega$, $a \in G$. In particular, $\theta(h)\Omega = -(\mathrm{ad}\, h)\Omega$, $h \in E$.

(3) Let $Y_1, Y_2 \in \mathscr{X}_H(P)$ be horizontal vector fields. Then

$$V_*([Y_1, Y_2]) = -Z_{\Omega(Y_1, Y_2)}\,.$$

(Recall, from sec. 6.1, that Z_f denotes the vertical vector field generated by $f \in \mathscr{S}(P; E)$.)

Proof: (1) is obvious. (2) follows from the equivariance of ω (cf. Proposition VI, sec. 6.10). To prove (3) observe that, since Y_1 and Y_2 are horizontal, $\omega(Y_1) = \omega(Y_2) = 0$. Thus

$$\Omega(Y_1, Y_2) = \delta\omega(Y_1, Y_2) = -\omega([Y_1, Y_2]).$$

According to sec. 6.10, $V_*Y = Z_{\omega(Y)}$, $Y \in \mathscr{X}(P)$. Now (3) follows.

Q.E.D.

Recall that $\lambda\colon \mathscr{X}(B) \xrightarrow{\cong} \mathscr{X}_H^I(P)$ denotes the horizontal lift (cf. sec. 6.9).

Corollary I: If $X_1, X_2 \in \mathscr{X}(B)$, then the decomposition of $[\lambda X_1, \lambda X_2]$ into horizontal and vertical parts is given by

$$[\lambda X_1, \lambda X_2] = \lambda([X_1, X_2]) - Z_{\Omega(\lambda X_1, \lambda X_2)}\,.$$

Proof: Apply Proposition V, sec. 6.9, and part (3) of the proposition above.

Q.E.D.

Corollary II: The curvature is zero if and only if the Lie product of any two horizontal fields is horizontal.

Next, consider the real bilinear map,

$$[\ ,\ ]: A(P; E) \times A(P; E) \to A(P; E),$$

induced by the Lie multiplication in E (cf. sec. 6.7). The differential form, $[\omega, \omega] \in A^2(P; E)$, is given by

$$[\omega, \omega](z; \zeta_1, \zeta_2) = 2[\omega(z; \zeta_1), \omega(z; \zeta_2)], \qquad \zeta_1, \zeta_2 \in T_z(P).$$

Proposition XI: The curvature form satisfies the *structure equation of Maurer–Cartan*

(1) $\Omega = \delta\omega + \frac{1}{2}[\omega, \omega]$

and the *Bianchi identity*

(2) $\nabla\Omega = 0.$

Proof: To verify (1) it is sufficient to check that

$$i(h)\Omega = i(h)(\delta\omega + \tfrac{1}{2}[\omega, \omega]), \qquad h \in E, \qquad \text{and} \qquad H^*\Omega = H^*(\delta\omega + \tfrac{1}{2}[\omega, \omega]).$$

Proposition VI, sec. 6.10, implies that $i(h)\omega$ is the constant function $P \to h$. Hence

$$i(h)\,\delta\omega = \theta(h)\omega = -\mathrm{ad}(h)\omega, \qquad h \in E.$$

On the other hand,

$$i(h)[\omega, \omega] = 2[i(h)\omega, \omega] = 2[h, \omega] = 2\,(\mathrm{ad}\,h)\omega.$$

Thus $i(h)(\delta\omega + \frac{1}{2}[\omega, \omega]) = 0 = i(h)\Omega$.

Since, clearly, $H^*([\omega, \omega]) = [H^*\omega, H^*\omega] = 0$, we have

$$H^*\Omega = \Omega = H^*\delta\omega = H^*(\delta\omega + \tfrac{1}{2}[\omega, \omega])$$

and so (1) is proved.

To verify (2) apply $H^* \circ \delta$ to the structure equation just established. This gives

$$\nabla\Omega = H^*\delta\tfrac{1}{2}[\omega, \omega] = H^*[\delta\omega, \omega] = [H^*\delta\omega, H^*\omega].$$

But $H^*\omega = 0$ and so we obtain (2).

Q.E.D.

Proposition XII: If R is a representation of G in a vector space W and $\Phi \in A_B(P; W)$, then

$$\nabla^2\Phi = \Omega(\Phi).$$

Proof: In view of Proposition IX, sec. 6.13,

$$\nabla\Phi = \delta\Phi + \omega(\Phi).$$

Since $\nabla\Phi$ is again basic, the proposition can be applied a second time to yield

$$\begin{aligned}\nabla^2\Phi &= \delta(\omega(\Phi)) + \omega(\delta\Phi) + \omega(\omega(\Phi))\\ &= \delta\omega(\Phi) + \tfrac{1}{2}[\omega, \omega](\Phi) = \Omega(\Phi)\end{aligned}$$

(cf. sec. 6.7).

Q.E.D.

Corollary: If $f \in \mathscr{S}_I(P; W)$, then $\nabla^2 f = \Omega(f)$.

6.15. Induced connection. Let $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, \hat{B}, G)$ be a second principal bundle with the same group G and let $\varphi: P \to \hat{P}$ be a homomorphism of principal bundles. Then a principal connection $\hat{V}$ in $\hat{\mathscr{P}}$ induces a principal connection in $\mathscr{P}$.

In fact, if $\hat{\omega}$ is the connection form in $\hat{\mathscr{P}}$ corresponding to $\hat{V}$, then $\varphi^*\hat{\omega}$ is a connection form in $\mathscr{P}$. The principal connection V determined by ω is called the *connection induced by* φ. It makes the diagram

$$\begin{array}{ccc} T_P & \xrightarrow{d\varphi} & T_{\hat{P}} \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle \hat{V}} \\ T_P & \xrightarrow[d\varphi]{} & T_{\hat{P}} \end{array}$$

commute. (These results follow easily from sec. 6.4 and sec. 6.10.)

The following relations are immediate:

$$H^* \circ \varphi^* = \varphi^* \circ \hat{H}^*, \qquad \nabla \circ \varphi^* = \varphi^* \circ \hat{\nabla}, \qquad \Omega = \varphi^*\hat{\Omega}.$$

§6. The Weil homomorphism

V continues to denote a principal connection in the principal bundle $\mathscr{P}$. Its connection and curvature forms are denoted by ω and Ω, while H^* and ∇ denote, respectively, the horizontal projection and covariant derivative.

6.16. Multilinear functions. Recall that we may regard an element $\Gamma \in \bigotimes^k E^*$ as the real-valued k-linear function in E given by

$$\Gamma(h_1, ..., h_k) = \langle \Gamma, h_1 \otimes \cdots \otimes h_k \rangle, \qquad h_1, ..., h_k \in E.$$

Thus Γ determines a map

$$\Gamma_*: \underbrace{A(P; E) \times \cdots \times A(P; E)}_{(k \text{ terms})} \to A(P).$$

We shall denote Γ_* simply by Γ, and write

$$\Gamma(\Psi_1, ..., \Psi_k) = \Gamma_*(\Psi_1, ..., \Psi_k), \qquad \Psi_1, ..., \Psi_k \in A(P; E).$$

As an immediate consequence of the definitions of sec. 6.7, we have

Lemma II: Let $\Gamma_1 \in \bigotimes^p E^*$, $\Gamma_2 \in \bigotimes^q E^*$ and form $\Gamma_1 \otimes \Gamma_2 \in \bigotimes^{p+q} E^*$. Then

$$(\Gamma_1 \otimes \Gamma_2)(\Psi_1, ..., \Psi_{p+q}) = \Gamma_1(\Psi_1, ..., \Psi_p) \wedge \Gamma_2(\Psi_{p+1}, ..., \Psi_{p+q}),$$
$$\Psi_i \in A(P; E), \quad i = 1, ..., p + q.$$

6.17. The homomorphism γ. Recall that $\vee E^*$ is the symmetric algebra over E^*. The purpose of this section is to construct a homomorphism

$$\gamma: \vee E^* \to A(P).$$

Recall that the curvature form is a 2-form on P with values in E. Define a linear map,

$$\beta: \bigotimes E^* \to A(P),$$

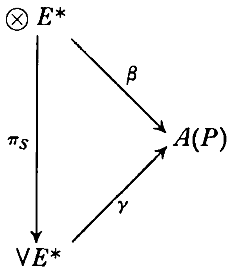
by

$$\beta(\Gamma) = \Gamma \underbrace{(\Omega, ..., \Omega)}_{(p \text{ arguments})}, \qquad \Gamma \in \bigotimes^p E^*.$$

Lemma III: (1) β is a homomorphism of algebras.
(2) $\beta(\otimes^p E^*) \subset A^{2p}(P)$.
(3) Let $\pi_S\colon \otimes E^* \to \vee E^*$ be the canonical projection given by

$$\pi_S(h_1^* \otimes \cdots \otimes h_p^*) = h_1^* \vee \cdots \vee h_p^* .$$

Then β factors over π_S to yield a homomorphism $\gamma\colon \vee E^* \to A(P)$ making the diagram,



,

commute.

Proof: (1) follows from Lemma II (set $\Psi_1 = \cdots = \Psi_{p+q} = \Omega$). (2) is a consequence of the fact that Ω is a 2-form. To prove (3), simply observe (via (2)) that

$$\operatorname{Im} \beta \subset \sum_p A^{2p}(P)$$

and that this is a commutative algebra.

Q.E.D.

The adjoint representation of G in E determines the representation, $\mathrm{Ad}^\vee$, of G in $\vee E^*$ given by

$$\mathrm{Ad}^\vee(a)(h_1^* \vee \cdots \vee h_p^*) = (\mathrm{Ad}\, a^{-1})^* h_1^* \vee \cdots \vee (\mathrm{Ad}\, a^{-1})^* h_p^*$$

$$a \in G, \quad h_i^* \in E^*, \quad i = 1, \ldots, p,$$

cf. sec. 1.9. Since G acts via homomorphisms in the graded algebra $\vee E^*$, it follows that the invariant subspace $(\vee E^*)_I$ is a graded subalgebra of $\vee E^*$; $(\vee E^*)_I = \sum_{k=0}^{\infty} (\vee^k E^*)_I$.

Proposition XIII: The homomorphism γ defined in Lemma III has the properties:

(1) $\operatorname{Im} \gamma \subset A(P)_{i=0}$.
(2) $T_a^* \circ \gamma = \gamma \circ \mathrm{Ad}^\vee(a)$, $a \in G$.
(3) $\nabla \circ \gamma = 0$.

Proof: (1) Since γ is a homomorphism of algebras and since $\vee E^*$ is generated by E^*, it is sufficient to show that

$$\gamma(h^*) \in A(P)_{i=0}, \qquad h^* \in E^*.$$

But for $h \in E$, $i(h)(\gamma(h^*)) = i(h)(h^*(\Omega)) = h^*(i(h)\Omega) = 0$ (cf. Proposition X, (1), sec. 6.14).

(2) Since both sides of (2) are algebra homomorphisms we need only verify that

$$(T_a^* \circ \gamma)(h^*) = (\gamma \circ (\mathrm{Ad}\, a^{-1})^*)(h^*), \qquad a \in G, \quad h^* \in E^*.$$

But since Ω is equivariant (cf. Proposition X, (2), sec. 6.14),

$$\begin{aligned}(T_a^* \circ \gamma)(h^*) &= h^*(T_a^*\Omega) = h^*(\mathrm{Ad}(a^{-1})\Omega) \\ &= (\mathrm{Ad}(a^{-1})^* h^*)(\Omega) = (\gamma \circ \mathrm{Ad}(a^{-1})^*)(h^*).\end{aligned}$$

(3) Every element $\Gamma \in \vee^p E^*$ can be written in the form $\pi_S(\Gamma_1)$, where $\Gamma_1 \in \otimes^p E^*$. Then

$$\nabla(\gamma(\Gamma)) = \nabla(\Gamma_1(\Omega, \ldots, \Omega)) = \sum_{i=1}^{p} \Gamma_1(\Omega, \ldots, \underset{(i\text{th position})}{\nabla\Omega}, \ldots, \Omega)$$

(cf. Proposition VIII, sec. 6.12).

The Bianchi identity (Proposition XI, sec. 6.14) states that $\nabla\Omega = 0$. Thus

$$\nabla(\gamma(\Gamma)) = 0, \qquad \Gamma \in \vee^p E^*;$$

i.e., $\nabla \circ \gamma = 0$.

Q.E.D.

Corollary: γ restricts to a homomorphism,

$$\gamma_I\colon (\vee E^*)_I \to A_B(P),$$

and the differential forms in Im γ_I are closed:

$$\delta \circ \gamma_I = 0.$$

Proof: Clearly $\gamma((\vee E^*)_I) \subset A_I(P) \cap A(P)_{i=0} = A_B(P)$ (cf. sec. 6.3). Moreover, Proposition VII, (5), sec. 6.12, shows that ∇ reduces to δ in the basic subalgebra. Thus, $\delta \circ \gamma_I = \nabla \circ \gamma_I = 0$.

Q.E.D.

6.18. Explicit formulae for β and γ. Identify $\otimes^p E^*$ with the space $T^p(E)$ of p-linear functions in E (cf. sec. 6.16). Then, if $\Gamma \in T^p(E)$ and $Z_i \in \mathscr{X}(P)$, we have

$$\beta(\Gamma)(Z_1, \ldots, Z_{2p}) = \frac{1}{2^p} \sum_{\sigma \in S^{2p}} \epsilon_\sigma \Gamma(\Omega(Z_{\sigma(1)}, Z_{\sigma(2)}), \ldots, \Omega(Z_{\sigma(2p-1)}, Z_{\sigma(2p)})).$$

Moreover, Lemma III, sec. 6.17, shows that $\beta(\Gamma)$ depends only on the symmetric part of Γ.

Next, identify $\vee^p E^*$ with the space $S^p(E)$ of p-linear symmetric functions in E by writing

$$(h_1^* \vee \cdots \vee h_p^*)(h_1, \ldots, h_p) = \operatorname{perm}(\langle h_i^*, h_j \rangle).$$

Then the projection $\otimes^p E^* \xrightarrow{\pi_S} \vee^p E^*$, interpreted as a map $T^p(E) \to S^p(E)$, is given by

$$(\pi_S \Gamma)(h_1, \ldots, h_p) = \sum_{\sigma \in S^p} \Gamma(h_{\sigma(1)}, \ldots, h_{\sigma(p)}).$$

On the other hand, the inclusion $i_S: S^p(E) \to T^p(E)$, interpreted as a map $\vee^p E^* \to \otimes^p E^*$, is given by

$$i_S(h_1^* \vee \cdots \vee h_p^*) = \sum_{\sigma \in S^p} h_{\sigma(1)}^* \otimes \cdots \otimes h_{\sigma(p)}^* .$$

Hence, for $\Gamma \in \vee^p E^*$,

$$\pi_S i_S \Gamma = p!\, \Gamma.$$

It follows that, for $\Gamma \in \vee^p E^*$,

$$\gamma(\Gamma) = \left(\frac{1}{p!}\right) \gamma(\pi_S i_S(\Gamma)) = \left(\frac{1}{p!}\right) \beta(i_S \Gamma)$$

$$= \left(\frac{1}{p!}\right) (i_S \Gamma)(\Omega, \ldots, \Omega).$$

Interpret Γ as a symmetric p-linear function; this equation then yields

$$\gamma\Gamma(Z_1, \ldots, Z_{2p}) = \frac{1}{p!\, 2^p} \sum_{\sigma \in S^{2p}} \epsilon_\sigma \Gamma(\Omega(Z_{\sigma(1)}, Z_{\sigma(2)}), \ldots, \Omega(Z_{\sigma(2p-1)}, Z_{\sigma(2p)})),$$
$$Z_i \in \mathscr{X}(P).$$

6.19. The Weil homomorphism. Recall from sec. 6.3 that $\pi^*: A(B) \to A(P)$ may be considered as an isomorphism

$$\pi^*: A(B) \xrightarrow{\cong} A_B(P).$$

Hence the corollary to Proposition XIII, sec. 6.17, shows that there is a unique homomorphism,

$$\gamma_B : (\vee E^*)_I \to A(B),$$

such that $\pi^* \circ \gamma_B = \gamma_I$. It satisfies $\delta \circ \gamma_B = 0$.

Thus, composing γ_B with the projection $Z(B) \to H(B)$, ($Z(B) = \ker \delta$), we obtain an algebra homomorphism

$$h_{\mathscr{P}} : (\vee E^*)_I \to H(B).$$

Observe that $h_{\mathscr{P}}((\vee^p E^*)_I) \subset H^{2p}(B)$.

Note that we needed only the principal bundle, together with the principal connection, V, in order to define $h_{\mathscr{P}}$.

Theorem I: $h_{\mathscr{P}}$ is independent of the choice of connection. Thus it is an invariant of the bundle $\mathscr{P}$.

Proof: Assume that two principal connections are defined in $\mathscr{P}$ and let ω_0, ω_1 be the corresponding connection forms. Consider the principal bundle $\mathscr{P} \times \mathbb{R} = (P \times \mathbb{R}, \pi \times \iota, B \times \mathbb{R}, G)$. Let $f \in \mathscr{S}(\mathbb{R})$ be the function given by $f(t) = t$. Then the E-valued 1-form, ω, on $P \times \mathbb{R}$, given by

$$\omega = \omega_0 \times (1 - f) + \omega_1 \times f$$

is a connection form (cf. Example 2, sec. 6.8, and Proposition VI, sec. 6.10).

Next consider the injections,

$$j_\nu : P \to P \times \mathbb{R} \qquad \text{and} \qquad i_\nu : B \to B \times \mathbb{R} \quad (\nu = 0, 1),$$

given by

$$j_0(z) = (z, 0) \qquad j_1(z) = (z, 1) \qquad z \in P,$$

and

$$i_0(x) = (x, 0) \qquad i_1(x) = (x, 1), \qquad x \in B.$$

Then j_0 and j_1 are homomorphisms of principal bundles. Evidently,

$$j_0^* \omega = \omega_0 \qquad \text{and} \qquad j_1^* \omega = \omega_1,$$

whence (cf. sec. 6.15)

$$j_0^* \Omega = \Omega_0 \qquad \text{and} \qquad j_1^* \Omega = \Omega_1$$

(Ω, Ω_0, Ω_1 denote the curvatures corresponding to ω, ω_0, and ω_1).

Now let $(\gamma_0)_I$, $(\gamma_1)_I$, γ_I denote the homomorphisms defined via ω_0, ω_1, and ω. Clearly

$$(\gamma_0)_I = j_0^* \circ \gamma_I \qquad \text{and} \qquad (\gamma_1)_I = j_1^* \circ \gamma_I.$$

It follows that $(\gamma_0)_B = i_0^* \circ \gamma_B$ and $(\gamma_1)_B = i_1^* \circ \gamma_B$. Hence $h_0 = i_0^\# h$ and $h_1 = i_1^\# h$. But i_1 and i_0 are homotopic and so (cf. sec. 5.2, volume I or sec. 0.14) $i_0^\# = i_1^\#$. It follows that $h_0 = h_1$.

Q.E.D.

Definition: $h_{\mathscr{P}}$ is called the *Weil homomorphism* for the principal bundle $\mathscr{P}$. The subalgebra Im $h_{\mathscr{P}}$ is called the *characteristic subalgebra* of $H(B)$ and its elements are called the *characteristic classes* for $\mathscr{P}$.

Remarks: 1. Im $h_{\mathscr{P}}$ is a graded subalgebra of the commutative algebra $\sum_p H^{2p}(B)$.

2. If the bundle $\mathscr{P}$ admits a connection with curvature zero, then the Weil homomorphism is trivial and the characteristic subalgebra is zero in positive degrees. In particular, the Weil homomorphism of a product bundle is trivial (cf. Corollary II to Proposition X, sec. 6.14).

3. If G is connected, we have $(\vee E^*)_I = (\vee E^*)_{\theta=0}$, where θ is the representation of E in $\vee E^*$ given by

$$\theta(h)\,(h_1^* \vee \cdots \vee h_p^*) = -\sum_{i=1}^{p} h_1^* \vee \cdots \vee \operatorname{ad}(h)^* h_i^* \vee \cdots \vee h_p^*, \qquad h_1^*, \ldots, h_p^* \in E^*,$$

(cf. Example 2, sec. 1.9). Hence, in this case, $h_{\mathscr{P}}$ is a homomorphism from $(\vee E^*)_{\theta=0}$ into $H(B)$.

4. Suppose G is compact and connected. Then the cohomology algebra $H(P)$ is determined by the graded differential algebra $(A(B), \delta)$ and the Weil homomorphism $h_{\mathscr{P}}$. Moreover, given $A(B)$ and $h_{\mathscr{P}}$ it is possible to determine $H(P)$ explicitly. This will be done in volume III.

Theorem II: Let $\varphi\colon \mathscr{P} \to \hat{\mathscr{P}}$ be a homomorphism of principal bundles with the same group G and let $\psi\colon B \to \hat{B}$ be the induced map. Then the diagram,

$$\begin{array}{ccc} & & H(\hat{B}) \\ & \overset{h_{\hat{\mathscr{P}}}}{\nearrow} & \\ (\vee E^*)_I & & \big\downarrow \psi^* \\ & \underset{h_{\mathscr{P}}}{\searrow} & \\ & & H(B), \end{array}$$

commutes ($h_{\mathscr{P}}$ and $h_{\hat{\mathscr{P}}}$ denote the Weil homomorphisms for $\mathscr{P}$ and $\hat{\mathscr{P}}$).

Proof: In fact, let $\hat{\omega}$ be a connection form for $\hat{\mathscr{P}}$ and let $\omega = \varphi^*\hat{\omega}$ be the induced connection form for $\mathscr{P}$ (cf. sec. 6.15). Then $\Omega = \varphi^*\hat{\Omega}$. This relation implies that

$$\gamma = \varphi^*\hat{\gamma} \qquad \text{and} \qquad \gamma_B = \psi^*\hat{\gamma}_B ,$$

whence

$$h_{\mathscr{P}} = \psi^\# h_{\hat{\mathscr{P}}} .$$

Q.E.D.

Corollary: Let $h_{\mathscr{P}}^+$ denote the restriction of $h_{\mathscr{P}}$ to $(\vee^+ E^*)_I$. Then $\pi^\# \circ h_{\mathscr{P}}^+ = 0$. ($(\vee^+ E^*)_I = \sum_{j>0} (\vee^j E^*)_I$.)

Proof: Regard the action $T\colon P \times G \to P$ as a homomorphism from the product bundle $\hat{\mathscr{P}} = (P \times G, \pi_P, P, G)$ to $\mathscr{P}$, inducing $\pi\colon P \to B$ between the base manifolds. Since $\hat{\mathscr{P}}$ is trivial, we have $h_{\hat{\mathscr{P}}}^+ = 0$ (cf. Remark 2, above), whence $\pi^\# h_{\mathscr{P}}^+ = h_{\hat{\mathscr{P}}}^+ = 0$.

Q.E.D.

6.20. Change of connection. Let ω_0 and ω_1 be connection forms in $\mathscr{P}$ and set $\theta = \omega_1 - \omega_0$. Then

$$i(h)\theta = \omega_1(Z_h) - \omega_0(Z_h) = h - h = 0, \qquad h \in E,$$

$$T_a^*\theta = (\operatorname{Ad} a^{-1})\theta, \qquad a \in G,$$

and

$$\theta(h)\theta = -(\operatorname{ad} h)\theta, \qquad h \in E.$$

In particular, θ is a basic E-valued 1-form on P (cf. sec. 6.6).

Now adopt the notation established in the proof of Theorem I, sec. 6.19, and observe that the connection form ω in $P \times \mathbb{R}$ can be written

$$\omega = \omega_0 \times 1 + \theta \times f.$$

Theorem I implies that, for each $\Gamma \in (\vee^p E^*)_I$, there exists a $\Phi \in A^{2p-1}(B)$ such that $(\gamma_1)_B\Gamma - (\gamma_0)_B\Gamma = \delta\Phi$.

In this section we construct an explicit Φ. Use $i_S\colon \vee^p E^* \to \otimes^p E^*$ to identify $\vee^p E^*$ with the p-linear symmetric functions in E (cf, sec. 6.18). We shall use the notation

$$\langle \Gamma, \Psi_1^{k_1} \vee \cdots \vee \Psi_r^{k_r} \rangle = \Gamma(\underbrace{\Psi_1 \cdots \Psi_1}_{(k_1 \text{ arguments})}, \ldots, \underbrace{\Psi_r \cdots \Psi_r}_{(k_r \text{ arguments})}),$$

$\Gamma \in \vee^p E^*$, $\Psi_1, \ldots, \Psi_r \in A(P; E)$, cf. sec. 6.16.

Proposition XIV: With the notation and hypotheses above,

$$(\gamma_1)_B\Gamma - (\gamma_0)_B\Gamma = \delta\Phi,$$

where Φ is the $(2p - 1)$-form on B determined by

$$\pi^*\Phi = \sum_{i+j+k=p-1} \frac{1}{i+2j+1} \langle \Gamma, \theta \vee \frac{1}{i!}(\nabla_0\theta)^i \vee \frac{1}{j!}(\tfrac{1}{2}[\theta, \theta])^j \vee \frac{1}{k!}(\Omega_0)^k \rangle .$$

Proof: Since the homotopy connecting i_0 and i_1 is just the identity map of $B \times \mathbb{R}$, we have $i_1^* - i_0^* = k \circ \delta + \delta \circ k$, where

$$(k\Psi)(x; \xi_1, \dots, \xi_{p-1}) = \int_0^1 \Psi(x, t; d/dt, \xi_1, \dots, \xi_{p-1})\, dt,$$

$$\Psi \in A^p(B \times \mathbb{R}), \qquad \xi_i \in T_x(B),$$

(cf. sec. 0.14). It follows that

$$(\gamma_1)_B\Gamma - (\gamma_0)_B\Gamma = (i_1^* - i_0^*)\gamma_B\Gamma = \delta\Phi,$$

where

$$\Phi(x; \xi_1, \dots, \xi_{2p-1}) = \int_0^1 (\gamma_B\Gamma)(x, t; d/dt, \xi_1, \dots, \xi_{2p-1})\, dt.$$

Hence (cf. sec. 6.16)

$$(\pi^*\Phi)(z; \zeta_1, \dots, \zeta_{2p-1}) = \int_0^1 (\gamma_I\Gamma)(z, t; d/dt, \zeta_1, \dots, \zeta_{2p-1})\, dt$$

$$= \int_0^1 \frac{1}{p!} \Gamma(\Omega, \dots, \Omega)(z, t; d/dt, \zeta_1, \dots, \zeta_{2p-1})\, dt.$$

On the other hand, the Maurer–Cartan formula (Proposition XI, sec. 6.14) applied to the relation above for ω yields

$$\Omega = \Omega_0 \times 1 + (\delta\theta + [\omega_0, \theta]) \times f + \tfrac{1}{2}[\theta, \theta] \times f^2 - \theta \times \delta f.$$

Since θ is basic, we obtain from the corollary to Proposition IX, sec. 6.13, that

$$\Omega = \Omega_0 \times 1 + \nabla_0\theta \times f + \tfrac{1}{2}[\theta, \theta] \times f^2 - \theta \times \delta f.$$

This implies that

$$\Gamma(\Omega,\dots,\Omega) = -\sum_{i+j+k=p-1} \frac{p!}{i!\,j!\,k!}\frac{1}{2^j}\langle \Gamma, \theta \vee (\nabla_0\theta)^i \vee [\theta,\theta]^j \vee \Omega_0^k\rangle \times f^{i+2j}\delta f$$

$$+ \sum_{i+j+k=p} \frac{p!}{i!\,j!\,k!}\frac{1}{2^j}\langle \Gamma, (\nabla_0\theta)^i \vee [\theta,\theta]^j \vee \Omega_0^k\rangle \times f^{i+2j}.$$

It follows that

$$\pi^*\Phi = \sum_{i+j+k=p-1} \langle \Gamma, \theta \vee \frac{1}{i!}(\nabla_0\theta)^i \vee \frac{1}{j!}(\tfrac{1}{2}[\theta,\theta])^j \vee \frac{1}{k!}(\Omega_0)^k\rangle$$

$$\times \int_0^1 (f^{i+2j}\delta f)\left(t; \frac{d}{dt}\right) dt.$$

But

$$\int_0^1 (f^{i+2j}\delta f)\left(t; \frac{d}{dt}\right) dt = \int_0^1 t^{i+2j}\, dt = \frac{1}{i+2j+1}.$$

The proposition follows.

Q.E.D.

Corollary: Suppose $\mathscr{P}$ admits a connection form ω_0 whose curvature Ω_0 is zero. Let ω_1 be any connection form in $\mathscr{P}$ and set $\theta = \omega_1 - \omega_0$. Then, for $\Gamma \in (\vee^p E^*)_I$, $(\gamma_1)_B\Gamma = \delta\Phi$, where

$$\pi^*\Phi = \sum_{i+j=p-1} \frac{1}{p+j}\langle \Gamma, \theta \vee \frac{1}{i!}(\nabla_0\theta)^i \vee \frac{1}{j!}\left(\tfrac{1}{2}[\theta,\theta]\right)^j\rangle.$$

Example: Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle. Let ω be a connection form in $\mathscr{P}$ with curvature form Ω. Consider the trivial bundle $\hat{\mathscr{P}} = (P \times G, \pi_P, P, G)$ and let ω_0 denote the connection form on $\hat{\mathscr{P}}$ corresponding to the horizontal subbundle $T_P \times G$. Then

$$\omega_0(z, a; \zeta, \eta) = L_a^{-1}(\eta)$$

and the corresponding curvature form is zero as follows from Corollary II to Proposition X, sec. 6.14.

On the other hand, since $T: P \times G \to P$ is a homomorphism of principal bundles inducing $\pi: P \to B$, it follows that $\omega_1 = T^*\omega$ is a connection form in $\hat{\mathscr{P}}$ with curvature $\Omega_1 = T^*\Omega$.

A straightforward calculation shows that, in this case,

$$\theta(z, a) = ((\operatorname{Ad} a^{-1})\omega \times 1)(z, a), \qquad z \in P, \quad a \in G,$$

($\theta = \omega_1 - \omega_0$). It follows that

$$\nabla_0\theta(z, a) = ((\mathrm{Ad}\, a^{-1})(\delta\omega) \times 1)(z, a)$$

and

$$[\theta, \theta](z, a) = ((\mathrm{Ad}\, a^{-1})[\omega, \omega] \times 1)(z, a).$$

Now let $\Gamma \in (\vee^p E^*)_I$ $(p > 0)$. Then, since Γ is invariant, the corollary to Proposition XIV reads

$$\gamma_I\Gamma \times 1 = \delta\left\{\sum_{i+j=p-1} \frac{1}{p+j} \left\langle \Gamma, \omega \vee \frac{1}{i!}(\delta\omega)^i \vee \frac{1}{j!}\left(\tfrac{1}{2}[\omega, \omega]\right)^j \right\rangle \times 1\right\};$$

i.e.,

$$\pi^*\gamma_B\Gamma = \gamma_I\Gamma = \delta\left\{\sum_{i+j=p-1} \frac{1}{p+j} \left\langle \Gamma, \omega \vee \frac{1}{i!}(\delta\omega)^i \vee \frac{1}{j!}\left(\tfrac{1}{2}[\omega, \omega]\right)^j \right\rangle\right\}.$$

Substitution of the relation $\Omega = \delta\omega + \frac{1}{2}[\omega, \omega]$ yields the formula

$$\begin{aligned}\pi^*(\gamma_B\Gamma) &= \gamma_I\Gamma \\ &= \frac{(p-1)!}{(2p-1)!}\,\delta\left\{\sum_{k=0}^{p-1} (-\tfrac{1}{2})^{p-1-k}\binom{2p-1}{k}\left\langle \Gamma, \omega \vee \Omega^k \vee [\omega, \omega]^{p-1-k}\right\rangle\right\}.\end{aligned}$$

(The calculation is long but elementary except for the observation that

$$\sum_{l=0}^{r}\binom{r}{l}\frac{(-1)^l}{p+l} = \int_0^1 x^{p-1}(1-x)^r\,dx = \frac{r!}{p\cdots(p+r)}.)$$

6.21. Formal power series and the Taylor homomorphism. Consider the infinite sequences

$$\Gamma = (\Gamma_0, \Gamma_1, \ldots) \qquad \text{with} \quad \Gamma_k \in \vee^k E^*.$$

Define addition and multiplication by

$$(\Gamma + \hat{\Gamma})_k = \Gamma_k + \hat{\Gamma}_k \qquad \text{and} \qquad (\Gamma \cdot \hat{\Gamma})_k = \sum_{i+j=k} \Gamma_i \vee \hat{\Gamma}_j \qquad (k = 0, 1, \ldots).$$

The associative algebra so obtained is called the *algebra of formal power series in* E^* and is denoted by $\vee^{**} E^*$.

Next, recall from sec. 1.9, volume I, that $\mathscr{S}_0(E)$ denotes the algebra of smooth function germs at 0. That is, an element of $\mathscr{S}_0(E)$ is an equivalence class of functions $f \in \mathscr{S}(E)$ under the following equivalence relation: $f \sim g$ if $f - g$ is zero in a neighbourhood of 0. If U is a neighbourhood of 0 in E and $g \in \mathscr{S}(U)$, then there is a unique germ, $[g]_0 \in \mathscr{S}_0(E)$, such that any $f \in [g]_0$ agrees with g sufficiently close to 0. We say g is a *representative* of $[g]_0$.

Now let $f \in \mathscr{S}(U)$ (U, a neighbourhood of 0 in E). Then the kth derivative of f is the smooth map $f^{(k)} \in \mathscr{S}(U; \vee^k E^*)$ defined inductively by

$$f^{(0)} = f$$

and

$$f^{(k)}(x; h_1, \ldots, h_k) = \lim_{t \to 0} \frac{f^{(k-1)}(x + th_1; h_2, \ldots, h_k) - f^{(k-1)}(x; h_2, \ldots, h_k)}{t}.$$

(Note that we identify $\vee^k E^*$ with $S^k(E)$ via i_S as described in sec. 6.18.)

The Leibniz formula states that

$$(fg)^{(k)} = \sum_{i+j=k} f^{(i)} \vee g^{(j)}, \qquad f, g \in \mathscr{S}(U);$$

i.e., the map,

$$f \mapsto (f(0), f'(0), f''(0), \ldots),$$

is a homomorphism of $\mathscr{S}(U)$ into $\vee^{**}E^*$. Since the derivatives of f at 0 depend only on the germ of f at 0, this homomorphism determines a homomorphism

$$\text{Tay}: \mathscr{S}_0(E) \to \vee^{**}E^*$$

called the *Taylor homomorphism.*

Next recall that G acts on E by the automorphisms $\text{Ad}\, a$. Thus an action of G on $\mathscr{S}_0(E)$ is defined by

$$a \cdot [f]_0 = [(\text{Ad}\, a^{-1})^* f]_0, \qquad f \in \mathscr{S}(E), \quad a \in G.$$

The corresponding invariant subalgebra is denoted by $\mathscr{S}_0(E)_I$. On the other hand, we have an induced action of G on $\vee^{**}E^*$. Clearly, the Taylor homomorphism is equivariant with respect to these actions and hence it restricts to a homomorphism,

$$\text{Tay}_I: \mathscr{S}_0(E)_I \to (\vee^{**}E^*)_I,$$

called the *invariant Taylor homomorphism.*

6.22. The homomorphisms $h_{\mathscr{P}}^{**}$ and $s_{\mathscr{P}}$. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle over an n-manifold B and consider the Weil homomorphism

$$h_{\mathscr{P}}: (\vee E^*)_I \to H(B).$$

Since $H^p(B) = 0$, $p > n$, $h_{\mathscr{P}}$ extends to a homomorphism

$$h_{\mathscr{P}}^{**}: (\vee^{**}E^*)_I \to H(B).$$

Clearly the image of $h_{\mathscr{P}}^{**}$ coincides with the image of $h_{\mathscr{P}}$.

On the other hand, we have the invariant Taylor homomorphism

$$(\mathrm{Tay})_I\colon \mathscr{S}_0(E)_I \to (\vee^{**}E^*)_I$$

Composing these homomorphisms we obtain a homomorphism

$$s_{\mathscr{P}}\colon \mathscr{S}_0(E)_I \to H(B).$$

Explicitly, $s_{\mathscr{P}}[f]_0 = \sum_{p=0}^{\infty} h_{\mathscr{P}}(f^{(p)}(0))$.

If $\varphi\colon P \to \hat{P}$ is a homomorphism of principal bundles inducing $\psi\colon B \to \hat{B}$, then $\psi^{\#} \circ h_{\hat{\mathscr{P}}}^{**} = h_{\mathscr{P}}^{**}$ and $\psi^{\#} \circ s_{\hat{\mathscr{P}}} = s_{\mathscr{P}}$ as follows from Theorem II, sec. 6.19, and the definitions.

Remark: The advantage of using $h_{\mathscr{P}}^{**}$ or $s_{\mathscr{P}}$ rather than $h_{\mathscr{P}}$ is the following: Let $[f]_0 \in \mathscr{S}_0(E)_I$, $\Gamma \in (\vee^{**}E^*)_I$, $\alpha \in H(B)$. These elements are invertible in their respective algebras if and only if $f(0) \neq 0$ (respectively $\Gamma_0 \neq 0$, $\alpha_0 \neq 0$, where α_0 is the component of α in $H^0(B)$). Moreover, if $f(0) \neq 0$, then

$$s_{\mathscr{P}}([f]_0^{-1}) = (s_{\mathscr{P}}([f]_0))^{-1}.$$

On the other hand, an element $\Gamma \in (\vee E^*)_I$ is only invertible if $\Gamma_0 \neq 0$ and $\Gamma_i = 0$, $i > 0$, while $h_{\mathscr{P}}(\Gamma)$ is invertible whenever $\Gamma_0 \neq 0$. Hence, if $\Gamma_0 \neq 0$, and $\Gamma_i \neq 0$ for some $i > 0$, then $(h_{\mathscr{P}}(\Gamma))^{-1}$ exists but it is expressible in the $h_{\mathscr{P}}(\Gamma_i)$ only via a complicated polynomial. To obtain simple expressions it is necessary to introduce $(\vee^{**}E^*)_I$.

§7. Special cases

6.23. Principal bundles with abelian structure group. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle whose structure group G is abelian. Let ω be a connection form in $\mathscr{P}$ with curvature form Ω. Then

$$i(h)\Omega = 0 \quad \text{and} \quad T_a^*\Omega = \Omega, \quad a \in G. \tag{6.1}$$

Moreover, the Maurer–Cartan equation (Proposition XI, sec. 6.14) reduces to $\delta\omega = \Omega$. In particular, it follows that $\delta\Omega = 0$.

In view of Proposition III, sec. 6.3, relations (6.1) imply that there is a (unique) E-valued 2-form Ω_B on B such that $\Omega = \pi^*\Omega_B$. Since

$$\pi^*\,\delta\Omega_B = \delta\pi^*\Omega_B = \delta\Omega = 0,$$

it follows that $\delta\Omega_B = 0$.

Next observe that, since G is abelian, $(\vee E^*)_I = \vee E^*$ and so γ_I and γ_B become homomorphisms

$$\gamma_I\colon \vee E^* \to A_B(P) \quad \text{and} \quad \gamma_B\colon \vee E^* \to A(B).$$

Evidently (cf. sec. 6.18)

$$\gamma_B(\Gamma) = \frac{1}{p!}\,\Gamma(\Omega_B ,\ldots, \Omega_B), \qquad \Gamma \in \vee^p E^*.$$

In particular,

$$\gamma_B(h^*) = \langle h^*, \Omega_B\rangle, \qquad h^* \in E^*. \tag{6.2}$$

Proposition XV: For every $h^* \in E^*$, let χ_{h^*} denote the 1-form on P given by

$$\chi_{h^*}(Z) = \langle h^*, \omega(Z)\rangle, \qquad Z \in \mathscr{X}(P).$$

Then

$$\delta\chi_{h^*} = \pi^*\gamma_B(h^*).$$

Proof: In fact,

$$\begin{aligned}\pi^*\gamma_B(h^*) &= \pi^*\langle h^*, \Omega_B\rangle = \langle h^*, \Omega\rangle \\ &= \langle h^*, \delta\omega\rangle = \delta\langle h^*, \omega\rangle = \delta\chi_{h^*}.\end{aligned}$$

Q.E.D.

Remark: In volume III it will be shown that Proposition XV generalizes to principal bundles with compact connected structure group.

Example: Assume that $G = S^1$. Let e^* be the basis vector of E^* which generates the invariant 1-form whose integral over S^1 equals 1. Then X_{e^*} is a 1-form on P satisfying

$$\oint_{S^1} X_{e^*} = 1 \qquad \text{and} \qquad \delta X_{e^*} = \pi^*\gamma_B(e^*).$$

Hence, $-\gamma_B(e^*)$ represents the Euler class, $\chi_{\mathscr{P}}$, of the circle bundle $\mathscr{P}$ (cf. sec. 8.2, volume I).

This shows that $\chi_{\mathscr{P}} = -h_{\mathscr{P}}(e^*)$ and that $\chi_{\mathscr{P}}$ is represented by the 2-form

$$\Phi = -\langle e^*, \Omega_B \rangle.$$

6.24. The cohomology of $\mathbb{C}P^n$. Recall from sec. 5.20 the Hopf fibration $\mathscr{P} = (S^{2n+1}, \pi, \mathbb{C}P^n, S^1)$. The principal action of S^1 is the restriction to S^{2n+1} of the representation R of S^1 in $\mathbb{C}^{n+1}$ given by

$$R(e^{i\theta}) \cdot z = e^{i\theta} z, \qquad z \in \mathbb{C}^{n+1}.$$

Next we define a connection in $\mathscr{P}$. Identify the Lie algebra of the principal S^1-bundle $\mathscr{P}$ with $\mathbb{R}$ so that the invariant 1-form generated by 1^* has integral 1. Let Z (respectively, $\hat{Z}$) denote the fundamental fields generated by 1 on S^{2n+1} (respectively, $\mathbb{C}^{n+1}$). Then

$$Z(z) = \hat{Z}(z), \qquad z \in S^{2n+1},$$

and

$$\hat{Z}(z) = (z, 2\pi i z), \qquad z \in \mathbb{C}^{n+1}.$$

Define a 1-form θ on $\mathbb{C}^{n+1}$ by

$$\theta(z; \zeta) = -\frac{1}{2\pi}\operatorname{Im}\langle z, \zeta\rangle,$$

where $\langle\ ,\ \rangle$ denotes the Hermitian inner product. Then θ is S^1-invariant and

$$\theta(z; \hat{Z}(z)) = \langle z, z\rangle, \qquad z \in \mathbb{C}^{n+1}.$$

Thus, if ω denotes the restriction of θ to S^{2n+1},

$$\omega(Z) = 1,$$

and so ω is a connection form in $\mathscr{P}$. Since S^1 is abelian, the corresponding curvature form is given by $\Omega = \delta\omega$, and (cf. sec. 6.23) we have

$$\delta\omega = \pi^*\Omega_B .$$

Proposition XVI: Let $\chi_{\mathscr{P}}$ denote the Euler class of the S^1-bundle $\mathscr{P}$. Then (1) the classes $1, \chi_{\mathscr{P}}, \dots, (\chi_{\mathscr{P}})^n$ form a basis for $H(\mathbb{C}P^n)$.

(2) $(\chi_{\mathscr{P}})^n$ is an orientation class for $\mathbb{C}P^n$.

Proof: (1) Recall the Gysin sequence

$$\begin{array}{ccccc} \downarrow & & & & \\ H^p(S^{2n+1}) & \xrightarrow{\oint_{S^1}^*} & H^{p-1}(\mathbb{C}P^n) & \xrightarrow{D} & H^{p+1}(\mathbb{C}P^n) \\ & & & & \Big\downarrow \pi^* \\ & & & & H^{p+1}(S^{2n+1}) \longrightarrow , \end{array}$$

from sec. 8.2, volume I, where D is given by

$$D\alpha = \alpha \cdot \chi_{\mathscr{P}}, \qquad \alpha \in H(\mathbb{C}P^n).$$

Observe that, if $\alpha \in H^+(\mathbb{C}P^n)$, then $\alpha \in \sum_{j=1}^{2n} H^j(\mathbb{C}P^n)$. It follows that $\pi^\#\alpha \in \sum_{j=1}^{2n} H^j(S^{2n+1})$ and so $\pi^\#\alpha = 0$. Hence the Gysin sequence yields the exact sequences,

$$0 \longrightarrow H^1(\mathbb{C}P^n) \longrightarrow 0$$

and

$$0 \longrightarrow H^p(\mathbb{C}P^n) \xrightarrow{D} H^{p+2}(\mathbb{C}P^n) \longrightarrow 0 \qquad (0 \leqslant p \leqslant 2n-2).$$

This shows that the elements $1, \chi_{\mathscr{P}}, \dots, \chi_{\mathscr{P}}^n$ form a basis for $H(\mathbb{C}P^n)$.

(2) We must show that for a suitable orientation of $\mathbb{C}P^n$,

$$\int_{\mathbb{C}P^n} \Omega_B^n = 1.$$

Orient the bundle $\mathscr{P}$ by ω and give $\mathbb{C}P^n$ the orientation such that the induced local product orientation in S^{2n+1} (cf. sec. 7.6, volume I or sec. 0.15) is the standard orientation. Then

$$\oint_{S^1} \omega = 1,$$

and so the Fubini theorem together with Stokes' theorem (cf. sec. 4.17, volume I, and sec. 7.14, volume I) imply that

$$\int_{\mathbb{C}P^n} \Omega_B^n = \int_{S^{2n+1}} (\pi^*\Omega_B)^n \wedge \omega$$
$$= \int_{S^{2n+1}} \omega \wedge (\delta\omega)^n = \int_B (\delta\theta)^{n+1},$$

where B is the unit ball in $\mathbb{R}^{2n+2}$.

Next we show that

$$(\delta\theta)^{n+1} = \frac{(n+1)!}{\pi^{n+1}}\Delta,$$

where Δ denotes the normed positive determinant function in $\mathbb{R}^{2n+2}$. In fact, fix an orthonormal basis e_ν ($\nu = 1, \ldots, n+1$) in $\mathbb{C}^{n+1}$ and let X_ν, Y_ν ($\nu = 1, \ldots, n+1$) be the constant vector fields corresponding to the vectors e_ν, ie_ν. Then, if a vector $z \in \mathbb{C}^{n+1}$ is written

$$z = \sum_\nu \xi^\nu e_\nu + \sum_\nu \eta^\nu(ie_\nu), \qquad \xi^\nu, \eta^\nu \in \mathbb{R},$$

we have

$$\langle\theta, X_\nu\rangle(z) = -\frac{1}{2\pi}\eta^\nu \quad \text{and} \quad \langle\theta, Y_\nu\rangle(z) = \frac{1}{2\pi}\xi^\nu.$$

These relations yield

$$\delta\theta(X_\nu, X_\mu) = 0, \qquad \delta\theta(Y_\nu, Y_\mu) = 0, \qquad \text{and} \qquad \delta\theta(X_\nu, Y_\mu) = \frac{1}{\pi}\delta_{\nu\mu}.$$

It follows that

$$(\delta\theta)^{n+1}(X_1, Y_1, \ldots, X_{n+1}, Y_{n+1}) = (n+1)!/\pi^{n+1},$$

whence $(\delta\theta)^{n+1} = [(n+1)!/\pi^{n+1}]\Delta$.

Finally, recall from Example 2, sec. 4.15, volume I, that

$$\int_{S^{2n+1}} i(T)\Delta = 2\pi^{n+1}/n!,$$

where T is the vector field in $\mathbb{C}^{n+1}$ given by $T(z) = (z, z)$. Moreover,

$$\delta\, i(T)\Delta = \theta(T)\Delta = 2(n+1)\Delta.$$

These relations yield

$$\int_{\mathbb{C}P^n} \Omega_B^n = \int_B (\delta\theta)^{n+1} = \frac{(n+1)!}{\pi^{n+1}}\int_B \Delta = \frac{(n+1)!}{2(n+1)\pi^{n+1}}\int_{S^{2n+1}} i(T)\Delta = 1.$$

Q.E.D.

Corollary I: The Euler class of the Hopf fibration (S^3, π, S^2, S^1) is an orientation class of S^2.

Corollary II: The inclusion maps, $i: \mathbb{C}P^k \to \mathbb{C}P^n$ $(k \leqslant n)$, induce linear isomorphisms

$$i^\#: H^p(\mathbb{C}P^k) \xleftarrow{\cong} H^p(\mathbb{C}P^n) \qquad (0 \leqslant p \leqslant 2k).$$

6.25. Reduction of structure group. Let $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, B, K)$ be a second principal bundle over the same base. Assume that $\sigma: K \to G$ is a homomorphism and that $\varphi: \hat{P} \to P$ is a smooth fibre preserving map inducing the identity in B and satisfying

$$\varphi(z \cdot a) = \varphi(z) \cdot \sigma(a), \qquad z \in P, \quad a \in K;$$

thus, $(\hat{P}, \varphi)$ is a reduction of structure group from G to K via σ (cf. Example 5, sec. 5.5).

Denote the Lie algebra of K by F. The derivative $\sigma': F \to E$ induces a homomorphism

$$(\sigma')^\vee: \vee F^* \leftarrow \vee E^*.$$

Since

$$\operatorname{Ad} \sigma(a) \circ \sigma' = \sigma' \circ \operatorname{Ad} a, \qquad a \in K,$$

$(\sigma')^\vee$ restricts to a homomorphism

$$\sigma_I: (\vee F^*)_I \leftarrow (\vee E^*)_I .$$

Theorem III: With the notation and hypotheses above, the diagram,

$$\begin{array}{ccc} (\vee F^*)_I & \xleftarrow{\sigma_I} & (\vee E^*)_I \\ & {\scriptstyle h_{\hat{\mathscr{P}}}} \searrow \quad \swarrow {\scriptstyle h_{\mathscr{P}}} & \\ & H(B) & \end{array} ,$$

commutes.

Corollary: Let $\lambda: G \to H$ be a homomorphism from G into a Lie group H with Lie algebra L. Let $\mathscr{P}_\lambda$ be the λ-extension of $\mathscr{P}$ (cf. Example 4, sec. 5.5). Then

$$h_{\mathscr{P}} \circ \lambda_I = h_{\mathscr{P}_\lambda}.$$

The proof of Theorem III is preceded by three lemmas.

Lemma IV: There are principal coordinate representations $\{(U_\alpha, \hat{\psi}_\alpha)\}$ and $\{(U_\alpha, \psi_\alpha)\}$ for $\hat{\mathscr{P}}$ and for $\mathscr{P}$ such that the diagrams,

$$\begin{array}{ccc} U_\alpha \times K & \xrightarrow{\iota \times \sigma} & U_\alpha \times G \\ \hat{\psi}_\alpha \downarrow \cong & & \cong \downarrow \psi_\alpha \\ \hat{\pi}^{-1}U_\alpha & \xrightarrow[\varphi]{} & \pi^{-1}U_\alpha \end{array},$$

commute.

Proof: Let $\{(U_\alpha, \hat{\psi}_\alpha)\}$ be any principal coordinate representation for $\hat{\mathscr{P}}$. Consider the cross-sections $U_\alpha \to P$ defined by

$$x \mapsto \varphi(\hat{\psi}_\alpha(x, e))$$

and define maps ψ_α: $U_\alpha \times G \to P$ by

$$\psi_\alpha(x, b) = \varphi(\hat{\psi}_\alpha(x, e)) \cdot b, \qquad x \in U_\alpha, \quad b \in G.$$

Then $\{(U_\alpha, \psi_\alpha)\}$ is a principal coordinate representation for $\mathscr{P}$. Moreover,

$$\begin{aligned} \varphi\hat{\psi}_\alpha(x, a) &= \varphi(\hat{\psi}_\alpha(x, e) \cdot a) \\ &= \varphi(\hat{\psi}_\alpha(x, e)) \cdot \sigma(a) = \psi_\alpha(x, \sigma(a)), \qquad a \in K, \quad x \in B, \end{aligned}$$

as desired.

Q.E.D.

Lemma V: There are principal connections $\hat{V}$ for $\hat{\mathscr{P}}$ and V for $\mathscr{P}$ such that.

$$d\varphi \circ \hat{V} = V \circ d\varphi.$$

In particular, if W is a vector space the operators $\hat{H}^*$, $\hat{\nabla}$ in $A(\hat{P}; W)$ and H^*, ∇ in $A(P; W)$ satisfy

$$\hat{H}^* \circ \varphi^* = \varphi^* \circ H^* \qquad \text{and} \qquad \hat{\nabla} \circ \varphi^* = \varphi^* \circ \nabla.$$

Proof: If the principal bundles are trivial, $\hat{P} = B \times K$, $P = B \times G$ and if φ is given by $\varphi = \iota \times \sigma$, then the connections

$$\hat{V}(\xi, \eta) = (0, \eta), \qquad \xi \in T_x(B), \quad \eta \in T_a(K),$$

and

$$V(\xi, \zeta) = (0, \zeta), \qquad \xi \in T_x(B), \quad \zeta \in T_b(G),$$

satisfy the conditions above.

In the general case let $\{U_\alpha\}$ be the covering of B used in Lemma IV. Then, in view of that lemma, there are principal connections $\hat{V}_\alpha$ and V_α in the restrictions of $\hat{\mathscr{P}}$ and $\mathscr{P}$ to U_α which satisfy

$$d\varphi \circ \hat{V}_\alpha = V_\alpha \circ d\varphi.$$

Choose a partition of unity $\{p_\alpha\}$ in B subordinate to the open covering $\{U_\alpha\}$ and set

$$\hat{V} = \sum_\alpha (\hat{\pi}^* p_\alpha)\, \hat{V}_\alpha, \qquad V = \sum_\alpha (\pi^* p_\alpha)\, V_\alpha.$$

Q.E.D.

Lemma VI: Let V, $\hat{V}$ be principal connections in $\mathscr{P}$ and $\hat{\mathscr{P}}$ satisfying the condition of Lemma V. Then the corresponding connection forms, ω and $\hat{\omega}$, and curvature forms, Ω and $\hat{\Omega}$, are related by the equations

(1) $(\sigma')_* \hat{\omega} = \varphi^*\omega$

and

(2) $(\sigma')_* \hat{\Omega} = \varphi^*\Omega.$

Proof: (1) It follows from Lemma V that both sides of (1) give zero when applied to horizontal vectors. Thus it is sufficient to check that

$$(\sigma'_* \hat{\omega})(\hat{Z}_h) = (\varphi^*\omega)(\hat{Z}_h), \qquad h \in F.$$

The equations $\varphi(z \cdot a) = \varphi(z) \cdot \sigma(a)$ $(z \in \hat{P},\ a \in K)$ imply that

$$\hat{Z}_h \underset{\varphi}{\sim} Z_{\sigma'(h)}.$$

Hence (for $h \in F$)

$$(\sigma'_* \hat{\omega})(\hat{Z}_h) = \sigma'(h) = \omega(Z_{\sigma'(h)}) = (\varphi^*\omega)(\hat{Z}_h).$$

(2) In fact,

$$\begin{aligned}(\sigma')_* \hat{\Omega} &= (\sigma')_* \hat{\nabla}\hat{\omega} = \hat{\nabla}(\sigma')_* \hat{\omega} \\ &= \hat{\nabla}\varphi^*\omega = \varphi^*\nabla\omega = \varphi^*\Omega.\end{aligned}$$

Q.E.D.

6.26. Proof of Theorem III: Choose V, ω, Ω and $\hat{V}$, $\hat{\omega}$, $\hat{\Omega}$ as in the lemmas above. Let

$$\beta\colon \otimes E^* \to A(P), \qquad \hat{\beta}\colon \otimes F^* \to A(\hat{P})$$

be the corresponding homomorphisms as defined in sec. 6.17. Then for $\Gamma \in \otimes^p E^*$

$$\begin{aligned}\varphi^*(\beta\Gamma) &= \varphi^*(\Gamma(\Omega, \dots, \Omega)) = \Gamma(\varphi^*\Omega, \dots, \varphi^*\Omega)\\ &= \Gamma((\sigma')_*\hat{\Omega}, \dots, (\sigma')_*\hat{\Omega}) = (\otimes^p (\sigma')^* \Gamma)(\hat{\Omega}, \dots, \hat{\Omega}).\end{aligned}$$

It follows that $\varphi^* \circ \beta = \hat{\beta} \circ \otimes (\sigma')^*$.

Thus the homomorphisms γ, γ_I, $\hat{\gamma}$ and $\hat{\gamma}_I$ (cf. sec. 6.17) are connected by the relations

$$\varphi^* \circ \gamma = \hat{\gamma} \circ (\sigma')^\vee \qquad \text{and} \qquad \varphi^* \circ \gamma_I = \hat{\gamma}_I \circ \sigma_I .$$

Since $\hat{\pi}^* = \varphi^* \circ \pi^*$, we have $\gamma_B = \hat{\gamma}_B \circ \sigma_I$ and the theorem follows.

Q.E.D.

6.27. Example. Given a principal bundle, $\mathscr{P} = (P, \pi, B, G)$, let K be a closed subgroup of G and consider the principal bundle $\mathscr{P}_1 = (P, p, P/K, K)$ (cf. sec. 5.7) and its λ-extension

$$\hat{\mathscr{P}} = (P \times_K G, \hat{\pi}, P/K, G)$$

(cf. Example 4, sec. 5.5), where $\lambda\colon K \to G$ is the inclusion. Then we have the commutative diagram,

$$\begin{array}{ccccc} & & P \times G & & \\ & \overset{\varphi_1}{\nearrow} & \downarrow q & \overset{T}{\searrow} & \\ P & \xrightarrow{\varphi} & P \times_K G & \xrightarrow{\psi} & P \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \hat{\pi}} & & \downarrow{\scriptstyle \pi} \\ P/K & \xrightarrow[\iota]{} & P/K & \xrightarrow[\rho]{} & B, \end{array}$$

where φ_1 is inclusion opposite e and T is the principal action. Thus φ is a reduction of structure group with respect to the inclusion map, $\lambda\colon K \to G$, and ψ is a homomorphism of principal bundles.

Let E and F be the Lie algebras of G and K and let

$$\lambda_I\colon (\vee F^*)_I \leftarrow (\vee E^*)_I$$

be the homomorphism induced by λ. Then, Theorem III, sec. 6.25, and Theorem II, sec. 6.19, yield the commutative diagrams

$$\begin{array}{ccc} (\vee F^*)_I & \xleftarrow{\lambda_I} & (\vee E^*)_I \\ & {\scriptstyle h_{\mathscr{P}_1}}\searrow \quad \swarrow{\scriptstyle h_{\mathscr{P}}} & \\ & H(P/K) & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & (\vee E^*)_I & \\ & {\scriptstyle h_{\mathscr{P}}}\swarrow \quad \searrow{\scriptstyle h_{\mathscr{P}}} & \\ H(P/K) & \xleftarrow[\rho^*]{} & H(B). \end{array}$$

Combining these we obtain the commutative diagram

$$\begin{array}{ccc} (\vee F^*)_I & \xleftarrow{\lambda_I} & (\vee E^*)_I \\ {\scriptstyle h_{\mathscr{P}_1}}\downarrow & & \downarrow{\scriptstyle h_{\mathscr{P}}} \\ H(P/K) & \xleftarrow[\rho^*]{} & H(B) \end{array} \quad .$$

Remark: Assume in addition that G is compact and connected and that K is a maximal torus. Then the map ρ^* is injective, as will be shown in volume III. Moreover (cf. sec. 6.23) since in this case F is abelian, the diagram above becomes

$$\begin{array}{ccc} \vee F^* & \xleftarrow{\lambda_I} & (\vee E^*)_I \\ {\scriptstyle h_{\mathscr{P}_1}}\downarrow & & \downarrow{\scriptstyle h_{\mathscr{P}}} \\ H(P/K) & \xleftarrow[\rho^*]{} & H(B) \end{array} \quad .$$

Because of the simple structure of $\vee F^*$, the following becomes an important technique: first establish properties of $h_{\mathscr{P}_1}$; then use the injectivity of $\rho^\#$ to draw conclusions about $h_{\mathscr{P}}$. This technique forms the basis of the fundamental papers [1], [2] and [3] by Borel and Hirzebruch.

6.28. Connections invariant under a group action. Suppose that $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, B, K)$ is a principal bundle with structure group a Lie group K. Denote the corresponding principal action of K on $\hat{P}$ by $\hat{T}: \hat{P} \times K \to \hat{P}$. Assume that

$$\begin{array}{ccc} G \times \hat{P} & \xrightarrow{S} & \hat{P} \\ {\scriptstyle \iota \times \hat{\pi}}\downarrow & & \downarrow{\scriptstyle \hat{\pi}} \\ G \times B & \xrightarrow[\tilde{S}]{} & B \end{array}$$

is a smooth commutative diagram in which S and $\tilde{S}$ are left actions of G. Then $(S, \tilde{S})$ is called an *action of G on the principal bundle $\hat{\mathscr{P}}$* if the maps S_g and $\hat{T}_a$ commute for each $g \in G$, $a \in K$. Assume $(S, \tilde{S})$ is such an action.

A principal connection V in $\hat{\mathscr{P}}$ will be called *G-invariant* if

$$dS_g \circ V = V \circ dS_g\,, \qquad g \in G.$$

This holds if and only if the connection form satisfies

$$S_g^*\omega = \omega, \qquad g \in G.$$

If V is G-invariant then $H = \iota - V$ also commutes with the operators dS_g . It follows that

$$H^* \circ S_g^* = S_g^* \circ H^*, \qquad g \in G.$$

Hence the covariant derivative ∇ satisfies

$$\nabla \circ S_g^* = S_g^* \circ \nabla, \qquad g \in G.$$

In particular, the curvature form Ω is G-invariant:

$$S_g^*\Omega = \Omega, \qquad g \in G.$$

This, in turn, implies that the homomorphism $\gamma_B\colon (\vee F^*)_I \to A(B)$ (F, the Lie algebra of K) satisfies

$$\tilde{S}_g^* \circ \gamma_B = \gamma_B\,.$$

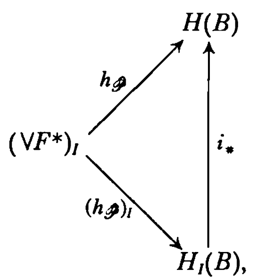
Thus γ_B can be considered as a homomorphism,

$$(\gamma_B)_I : (\vee F^*)_I \to A_I(B),$$

where $A_I(B)$ denotes the subalgebra of $A(B)$ invariant under the action of G. Since $\delta \circ \gamma_B = 0$, $(\gamma_B)_I$ induces a homomorphism

$$(h_{\mathscr{P}})_I : (\vee F^*)_I \to H_I(B).$$

The diagram,



commutes, where $i\colon A_I(B) \to A(B)$ is the inclusion.

Proposition XVII: If G is compact and acts on the principal bundle, $\mathscr{P}$, then $\mathscr{P}$ admits a G-invariant principal connection.

Proof: Let V be any principal connection. Regard V as a cross-section in the vector bundle $L_{\tau_{\hat{P}}}$ over $\hat{P}$ (whose fibre at z is the space of linear transformations of $T_z(\hat{P})$). Using the actions (cf. sec. 3.2),

$$S\colon G \times \hat{P} \to \hat{P}, \qquad dS\colon G \times T_{\hat{P}} \to T_{\hat{P}},$$

we can integrate V over G (cf. sec. 3.18) to obtain a G-invariant cross-section V^I. We show that V^I is a (G-invariant) principal connection.

For $z \in \hat{P}$, V_z^I is the endomorphism of $T_z(\hat{P})$ given by

$$V_z^I = \int_G (dS_g)_{g^{-1}z} \circ V_{g^{-1}z} \circ (dS_{g^{-1}})_z \, dg.$$

Since the vertical spaces $V_z(\hat{P})$ are dS_g-stable (because $\hat{\pi}$ is equivariant), and because each V_z is a projection of $T_z(\hat{P})$ onto $V_z(\hat{P})$, it follows from this relation that V_z^I is also a projection of $T_z(\hat{P})$ onto $V_z(\hat{P})$.

Finally, since (for $a \in K$) $d\hat{T}_a\colon T_z(\hat{P}) \to T_{z\cdot a}(\hat{P})$ is linear, we have

$$\begin{aligned} d\hat{T}_a \circ V_z^I &= \int_G (d\hat{T}_a \circ dS_g \circ V_{g^{-1}\cdot z} \circ dS_{g^{-1}})\, dg \\ &= \int_G (dS_g \circ V_{g^{-1}\cdot z\cdot a} \circ dS_{g^{-1}} \circ d\hat{T}_a)\, dg \\ &= V_{z\cdot a}^I \circ d\hat{T}_a . \end{aligned}$$

Hence V^I is a principal connection.

Q.E.D.

§8. Homogeneous spaces

In this article, K denotes a closed subgroup of G with Lie algebra F and $\mathscr{P}_K = (G, \pi, G/K, K)$ is the principal bundle defined in Example 2, sec. 5.1.

6.29. The cohomology of G/K. The principal action of K on G is denoted by μ_K :

$$\mu_K(g, a) = ga, \qquad g \in G, \quad a \in K.$$

On the other hand, the maps

$$(g_1, g_2) \mapsto g_1 g_2 \qquad \text{and} \qquad (g_1, \pi g_2) \mapsto \pi(g_1 g_2)$$

define left actions of G on G and G/K, with respect to which π is equivariant. Thus π^* restricts to a homomorphism (cf. sec. 4.18)

$$\pi_I^*: A_I(G/K) \to A_L(G).$$

Moreover, since $(G, \pi, G/K, K)$ is a principal bundle, Proposition III, sec. 6.3, shows that

$$\pi^*: A(G/K) \xrightarrow{\cong} A_B(G)$$

is an isomorphism. (Recall that $A_B(G)$ consists of those forms which are horizontal, and invariant under the right action of K on G). Thus π^* restricts to an isomorphism,

$$\pi_I^*: A_I(G/K) \xrightarrow{\cong} A_L(G) \cap A_B(G).$$

Since the action, μ_K, of K on G is right multiplication, the corresponding fundamental vector fields are the left invariant vector fields, X_k $(k \in F)$, on G. Thus the horizontal and invariant subalgebras of $A(G)$ are given by

$$\bigcap_{k \in F} \ker i(X_k) \qquad \text{and} \qquad \bigcap_{a \in K} \ker(\rho_a^* - \iota),$$

respectively. We denote them by

$$A(G)_{i_F = 0} \qquad \text{and} \qquad A(G)_{K=I}.$$

The basic subalgebra, $A_B(G)$, is their intersection.

Now recall the isomorphism $\tau_L\colon A_L(G) \xrightarrow{\cong} \wedge E^*$ of sec. 4.5. It satisfies

$$\tau_L \circ i(X_h) = i_E(h) \circ \tau_L \quad \text{and} \quad \tau_L \circ \rho_g^* = \mathrm{Ad}^\wedge(g) \circ \tau_L, \qquad h \in E, \quad g \in G,$$

(cf. sec. 4.6 and sec. 4.8). Hence it restricts to isomorphisms

$$A_L(G) \cap A(G)_{i_F=0} \xrightarrow{\cong} (\wedge E^*)_{i_F=0} \quad \text{and} \quad A_L(G) \cap A(G)_{K=I} \xrightarrow{\cong} (\wedge E^*)_{K=I}.$$

(Here $(\wedge E^*)_{K=I}$ denotes the subalgebra invariant under the operators $\mathrm{Ad}^\wedge(a)$, $a \in K$ and $(\wedge E^*)_{i_F=0} = \bigcap_{k\in F} \ker i_E(k)$.) Thus

$$\tau_L\colon A_L(G) \cap A_B(G) \xrightarrow{\cong} (\wedge E^*)_{i_F=0,K=I},$$

where $(\wedge E^*)_{i_F=0,K=I}$ denotes the intersection of $(\wedge E^*)_{i_F=0}$ and $(\wedge E^*)_{K=I}$.

Composing the isomorphisms τ_L and π_I^*, we obtain the commutative diagram

$$\begin{array}{ccccc}
A(G) & \longleftarrow & A_L(G) & \xrightarrow[\cong]{\tau_L} & \wedge E^* \\
\uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi_I^*} & & \uparrow{\scriptstyle k} \\
A(G/K) & \longleftarrow & A_I(G/K) & \xrightarrow[\tau_L \circ \pi_I^*]{\cong} & (\wedge E^*)_{i_F=0,K=I}.
\end{array}$$

The right-hand square coincides with the diagram of Proposition XI, sec. 4.18 and k is the inclusion.

Next, assume that K is connected. Then (cf. Proposition VI, sec. 3.13) the subalgebra, $A(G)_{K=I}$, is given by

$$A(G)_{K=I} = A(G)_{\theta_F=0} = \bigcap_{k\in F} \ker \theta(X_k).$$

Set (cf. sec. 4.6)

$$(\wedge E^*)_{\theta_F=0} = \bigcap_{k\in F} \ker \theta_E(k) \quad \text{and} \quad (\wedge E^*)_{i_F=0,\theta_F=0} = (\wedge E^*)_{i_F=0} \cap (\wedge E^*)_{\theta_F=0}.$$

Then we can rewrite the diagram above in the form

$$\begin{array}{ccccc}
A(G) & \longleftarrow & A_L(G) & \xrightarrow[\cong]{\tau_L} & \wedge E^* \\
\uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi_I^*} & & \uparrow{\scriptstyle k} \\
A(G/K) & \longleftarrow & A_I(G/K) & \xrightarrow[\tau_L \circ \pi_I^*]{\cong} & (\wedge E^*)_{i_F=0,\theta_F=0}.
\end{array}$$

Theorem IV: Let K be a closed connected subgroup of a compact connected Lie group G. Then, in the commutative diagram,

$$\begin{array}{ccccc} H(G) & \xleftarrow{\cong} & H_L(G) & \xrightarrow[\cong]{(\tau_L)_*} & H(E) \\ \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi_I^*} & & \uparrow{\scriptstyle k_*} \\ H(G/K) & \xleftarrow{\cong} & H_I(G/K) & \xrightarrow[(\tau_L \circ \pi_I^*)_*]{\cong} & H((\wedge E^*)_{i_F=0,\theta_F=0}) \end{array}$$

all the horizontal maps are algebra isomorphisms.

Proof: This is a restatement of Theorem V, sec. 4.19.

Q.E.D.

6.30. Connections in $(G, \pi, G/K, K)$. Recall that π: $G \to G/K$ is equivariant with respect to the left actions of G. We shall find the G-invariant principal connections for the principal bundle $(G, \pi, G/K, K)$ (cf. sec. 6.28).

Let V be a G-invariant principal connection. Since the vertical space at e is given by

$$V_e(G) = \ker(d\pi)_e = F$$

(cf. sec. 2.11), it follows that the restriction V_e of V to E is a projection

$$V_e: E \to F.$$

Moreover, since V is a G-invariant principal connection,

$$\operatorname{Ad} a \circ V_e = L_a \circ R_a^{-1} \circ V_e = V_e \circ \operatorname{Ad} a, \qquad a \in K.$$

In particular, $\ker V_e$ is stable under the operators $\operatorname{Ad} a$ ($a \in K$). Note that $\ker V_e$ is the horizontal subspace at e.

Proposition XVIII: The map α: $V \mapsto \ker V_e$ is a bijection from the set of G-invariant principal connections to the set of K-stable subspaces of E complementing F.

Proof: If W, V are two such connections with $\ker V_e = \ker W_e$ then, since

$$\operatorname{Im} V_e = F = \operatorname{Im} W_e\,,$$

we have $V_e = W_e$. Now the G-invariance implies that $V = W$ and so α is injective.

On the other hand, assume $F_1 \subset E$ is a subspace stable under $\text{Ad}\, a$ $(a \in K)$ and complementary to F:

$$E = F_1 \oplus F.$$

Let $V_e\colon E \to F$ be the projection with kernel F_1 and define a G-invariant strong bundle map, V, in T_G by

$$V_g = L_g \circ V_e \circ L_g^{-1}, \qquad g \in G.$$

V_g is a projection onto $L_g(F)$. But since π is equivariant, $L_g(= d\lambda_g)$ maps F isomorphically to the vertical space at g; i.e., V_g is a projection onto the vertical subspace. Moreover since F_1 is stable under $\text{Ad}\, a$ $(a \in K)$, it follows that

$$\text{Ad}\, a \circ V_e = V_e \circ \text{Ad}\, a, \qquad a \in K.$$

Since $R_a \circ L_g = L_g \circ R_a$ (cf. sec. 1.1) this yields

$$R_a \circ V_g = V_{g \cdot a} \circ R_a\,, \qquad g \in G, \quad a \in K.$$

Thus V is a G-invariant principal connection. By definition, $\ker V_e = F_1$, and so α is surjective.

Q.E.D.

Corollary I: $(G, \pi, G/K, K)$ admits a G-invariant connection if and only if there is a K-stable subspace $F_1 \subset E$ such that $E = F_1 \oplus F$.

Corollary II: If K is connected, the G-invariant principal connections are in one-to-one correspondence with the subspaces $F_1 \subset E$ such that

$$(\text{ad}\, h) F_1 \subset F_1 \quad (h \in F) \qquad \text{and} \qquad E = F_1 \oplus F.$$

Corollary III: If K is compact, the bundle, $\mathscr{P}_K$, admits a G-invariant principal connection.

Proof: Apply Proposition XVII, sec. 1.17.

Q.E.D.

6.31. Curvature and the Weil homomorphism. Assume that E admits a decomposition $E = F_1 \oplus F$, where F_1 is stable under the operators $\text{Ad}\, a$, $a \in K$. Let $p\colon E \to F$ and $p_1\colon E \to F_1$ be the projections.

Then p and p_1 are precisely the vertical and horizontal projections in $T_e(G)$ corresponding to the induced G-invariant principal connection V.

It follows that the connection form ω is the unique left invariant 1-form in $A^1(G; F)$ which satisfies

$$\omega(e; h) = p(h), \qquad h \in E.$$

Next we compute the curvature Ω of V. Observe that if X_h, X_k are left invariant vector fields on G, then

$$\delta\omega(X_h, X_k) = -\omega([X_h, X_k]) = -\omega(e; [h, k])$$

(because the functions $\omega(X_h)$, $\omega(X_k)$ are left invariant, and so constant). Similarly,

$$\tfrac{1}{2}[\omega, \omega](X_h, X_k) = [\omega(e; h), \omega(e; k)].$$

It follows from Proposition XI, sec. 6.14 that Ω is the unique left invariant E-valued 2-form such that

$$\Omega(e; h, k) = [p(h), p(k)] - p([h, k]), \qquad h, k \in E.$$

Thus if $h, k \in F_1$, then

$$\Omega(e; h, k) = -p([h, k]).$$

Finally, consider the invariant Weil homomorphism

$$(h_{\mathscr{P}_K})_I : (\vee F^*)_I \to H_I(G/K).$$

If $\Gamma \in (\vee^k F^*)_I$, then $(h_{\mathscr{P}_K})_I(\Gamma)$ is represented by the unique left invariant differential form $\Phi \in A_I^{2k}(G/K)$ which satisfies (cf. sec. 6.18)

$$\pi^*\Phi(e; h_1, \ldots, h_{2k}) = \frac{(-1)^k}{2^k\, k!} \sum_{\sigma \in S^{2k}} \epsilon_\sigma \Gamma(p([h_{\sigma(1)}, h_{\sigma(2)}]), \ldots, p([h_{\sigma(2k-1)}, h_{\sigma(2k)}])),$$

for $h_i \in F_1$. Clearly this differential form also represents $h_{\mathscr{P}_K}(\Gamma)$ in $H(G/K)$.

6.32. Symmetric spaces. Suppose that φ is an automorphism of G such that

$$\varphi^2 = \iota \qquad \text{and} \qquad \varphi \neq \iota.$$

The elements $a \in G$ satisfying $\varphi(a) = a$ form a closed subgroup; let K be its one-component. Then the Lie algebra, F, of K is the subspace of vectors $h \in E$ satisfying $\varphi'(h) = h$.

The homogeneous space G/K is called a *symmetric space with connected fibre*. If G is compact, we say G/K has *compact type*.

Since φ is an involution, so is φ'. Hence, setting

$$E^+ = \ker(\varphi' - \iota) \qquad \text{and} \qquad E^- = \ker(\varphi' + \iota),$$

we have $E = E^+ \oplus E^-$ and $E^+ = F$. Now we show that

$$[E^+, E^-] \subset E^- \qquad \text{and} \qquad [E^-, E^-] \subset E^+$$

(where, for subspaces $A, B \subset E$, $[A, B]$ is the space spanned by vectors of the form $[h, k]$, $h \in A$, $k \in B$).

In fact, for $h \in E^+$, $k \in E^-$,

$$\varphi'([h, k]) = [\varphi'(h), \varphi'(k)] = -[h, k],$$

whence $[h, k] \in E^-$. The second relation is proved in the same way. It follows that E^- is stable under the operators ad h ($h \in F$) and so, by Corollary II to Proposition XVIII, sec. 6.30, it determines a G-invariant principal connection on $(G, \pi, G/K, K)$. It is called the *symmetric space connection.*

Examples: 1. The Grassmann manifolds (cf. sec. 5.13)

$$SO(n)/(SO(k) \times SO(n-k)), \qquad U(n)/(U(k) \times U(n-k))$$

and

$$Q(n)/(Q(k) \times Q(n-k))$$

are symmetric spaces of compact type. In fact, consider the first case. Choose a decomposition $W = W_1 \oplus W_1^\perp$ of a Euclidean space, W, with $\dim W = n$, $\dim W_1 = k$. Define a rotation $\tau: W \to W$ by

$$\tau(w) = \begin{cases} w, & w \in W_1 \\ -w, & w \in W_1^\perp. \end{cases}$$

Then define an involution $\varphi: SO(n) \to SO(n)$ by

$$\varphi(\sigma) = \tau \circ \sigma \circ \tau^{-1}.$$

Evidently $\varphi(\sigma) = \sigma$ if and only if σ stabilizes W_1 and $W_1^\perp$. Thus the one-component of the subgroup left fixed by φ is $SO(k) \times SO(n-k)$.

The other two cases are established in the same way.

2. Endow $\mathbb{R}^n$ with a Euclidean metric. Define an involution, φ, of $GL^+(n; \mathbb{R})$ by setting

$$\varphi(\sigma) = (\sigma^*)^{-1}, \qquad \sigma \in GL^+(n; \mathbb{R}),$$

where σ^* denotes the dual of σ with respect to the inner product. The subgroup left fixed by φ is $SO(n)$.

Since φ' is given by $\varphi'(\alpha) = -\alpha^*$, $\alpha \in L(n)$, we have

$$L(n)^+ = \mathrm{Sk}(n) \qquad \text{and} \qquad L(n)^- = S(n)$$

($S(n)$ is the space of symmetric transformations of $\mathbb{R}^n$).

In this case the invariant connection leads to a homomorphism

$$(\vee\, \mathrm{Sk}(n)^*)_I \xrightarrow{(h_{\mathscr{P}})_I} H_I(GL^+(n; \mathbb{R})/SO(n)).$$

This homomorphism is in general *nontrivial*, as will be shown in volume III.

On the other hand, according to Example 1 of sec. 4.11, $GL^+(n; \mathbb{R})/SO(n)$ is diffeomorphic to the vector space $S(n)$. Thus its cohomology is trivial, as is the Weil homomorphism $h_{\mathscr{P}}$.

Problems

G is a Lie group with Lie algebra E.

1. Trivial bundles. Let $\mathscr{P} = (B \times G, \pi, B, G)$ be a trivial principal bundle. With each connection form, ω, associate the E-valued 1-form θ on B defined by

$$\omega(x, e; \xi, h) = h + \theta(x; \xi), \qquad x \in B, \quad \xi \in T_x(B), \quad h \in E.$$

(i) Show that this correspondence defines a bijection between principal connections in $\mathscr{P}$ and elements of $A^1(B; E)$.

(ii) Fix a principal connection, V, in $\mathscr{P}$ with corresponding 1-form $\theta \in A^1(B; E)$. Show that the linear map H_z at $z = (x, y)$ is given by

$$H_z(\xi, \eta) = (\xi, -R_y\theta(x; \xi)), \qquad \xi \in T_x(B), \quad \eta \in T_y(G).$$

(iii) Consider the E-valued 2-form Φ on B given by $\Phi = \delta\theta + \frac{1}{2}[\theta, \theta]$. Show that

$$(\pi^*\Phi)(x, y) = (\operatorname{Ad} y(\Omega(x, y)), \qquad x \in B, \quad y \in G,$$

where Ω is the curvature of V.

(iv) Let $z(t) = (x(t), y(t))$ $(0 \leqslant t \leqslant 1)$ be a smooth path in $B \times G$. Show that $\dot{z}(t)$ is horizontal if and only if

$$\dot{y}(t) = -R_{y(t)}\theta(x(t); \dot{x}(t)).$$

2. Local formulae for principal connections. Let $\{(U_\alpha, \psi_\alpha)\}$ be a principal coordinate representation for a principal bundle $\mathscr{P} = (P, \pi, B, G)$. Fix a principal connection in $\mathscr{P}$.

(i) As in problem 1, use the connection form to define local 1-forms $\theta_\alpha \in A^1(U_\alpha; E)$.

(ii) Find the relation between the restrictions $\theta_\alpha |_{U_\alpha \cap U_\beta}$ and $\theta_\beta |_{U_\alpha \cap U_\beta}$.

(iii) Set $\Phi_\alpha = \delta\theta_\alpha + \frac{1}{2}[\theta_\alpha, \theta_\alpha]$. Find the relation between $\Phi_\alpha |_{U_\alpha \cap U_\beta}$ and $\Phi_\beta |_{U_\alpha \cap U_\beta}$.

3. Horizontal lifts. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle with a fixed principal connection V. A *horizontal lift* of a path $x(t)$ $(0 \leqslant t \leqslant 1)$

in B is a smooth path $z(t)$ $(0 \leqslant t \leqslant 1)$ in P such that $\pi z(t) = x(t)$ and each tangent vector $\dot{z}(t)$ is horizontal.

(i) Let $x(t)$, $0 \leqslant t \leqslant 1$, be a smooth path in B. Given $z_0 \in G_{x(0)}$, show that there is a unique horizontal lift $z(t)$ of $x(t)$ such that $z(0) = z_0$. (Hint: cf. problem 1, (iv)), and problem 21, Chap. I).

(ii) Let $\psi: \mathbb{R}^2 \to B$ be a smooth map. Fix $z_0 \in G_{\psi(0,0)}$. Let $z(\tau)$ $(\tau \in \mathbb{R})$ be the horizontal lift of $\psi(\tau, 0)$ that satisfies $z(0) = z_0$. For fixed τ, let $z(\tau, t)$ be the horizontal lift of $\psi(\tau, t)$ that satisfies $\psi(\tau, 0) = z(\tau)$. Show that the map $\varphi: \mathbb{R}^2 \times G \to P$ given by

$$\varphi(\tau, t, a) = z(\tau, t) \cdot a$$

is a homomorphism of principal bundles.

(iii) Let τ and t denote the first and second coordinate functions in $\mathbb{R}^2$, with gradients $\delta\tau$ and δt. Let $\hat{V}$ be the principal connection in $\mathbb{R}^2 \times G$ induced via φ from V. Let $\theta \in A^1(\mathbb{R}^2; E)$ be the corresponding 1-form (cf. problem 1). Show that $\theta = f \cdot \delta\tau$, where $f \in \mathscr{S}(\mathbb{R}^2; E)$ satisfies $f(\tau, 0) = 0$. Conclude that, if Ω is the curvature of V, then

$$(\varphi^*\Omega)(\tau, t, e) = \frac{\partial f}{\partial t}(\tau, t)\, \delta t \wedge \delta\tau.$$

Conclude that $\Omega = 0$ implies that $\theta = 0$.

4. **Homotopic paths.** Let $\mathscr{P}$, V be as in problem 3. Let α and β be smooth paths in B such that

$$\alpha(0) = \beta(0) = x_0 \qquad \text{and} \qquad \alpha(1) = \beta(1) = x_1 .$$

Assume that Φ is a homotopy connecting α and β such that

$$\Phi(0, t) = x_0 \qquad \text{and} \qquad \Phi(1, t) = x_1 , \qquad t \in \mathbb{R}.$$

(i) Assume that the curvature of V is zero. Prove that if $\hat{\alpha}$ and $\hat{\beta}$ are horizontal lifts of α and β, both starting at the same point, then

$$\hat{\alpha}(1) = \hat{\beta}(1).$$

(ii) Establish the converse.

5. **Holonomy groups I.** Let $(\mathscr{P}, V)$ be as in problem 3. Assume that B is connected. Fix base points $x \in B$ and $z \in G_x$. Identify G with G_x via $a \mapsto z \cdot a$.

A *loop* in B, based at x, is a smooth map $\gamma: t \mapsto \gamma(t)$ $(0 \leqslant t \leqslant 1)$ such that $\gamma(0) = \gamma(1) = x$. Two loops based at x are called *homotopic* if they are homotopic in the sense of problem 4. A loop is called *conctractible* if it is homotopic to the constant loop. If every loop is contractible, B is called *simply connected.*

(i) Let γ be a loop based at x. Show that there is a unique element $a(\gamma) \in G$ such that for every horizontal lift, $\hat{\gamma}$, of γ

$$\hat{\gamma}(0) \cdot a(\gamma) = \hat{\gamma}(1).$$

(ii) If γ_τ is a homotopy of loops based at x, show that $\tau \mapsto a(\gamma_\tau)$ is a smooth path in G. (*Hint*: Use problem 3, (ii).)

(iii) Let $\mathscr{L}$ denote the set of loops, based at x, and let $\mathscr{L}_0$ denote the subset of contractible loops. Show that $\Gamma = \{a(\gamma) \mid \gamma \in \mathscr{L}\}$ is a subgroup of G and that $\Gamma_0 = \{a(\gamma) \mid \gamma \in \mathscr{L}_0\}$ is a normal subgroup of Γ. Show that Γ/Γ_0 is finite or countable.

(iv) Show that Γ_0 is a connected Lie subgroup of G (use problem 6, Chap. II). Conclude that Γ is a Lie subgroup of G with Γ_0 as 1-component.

Γ is called the *holonomy group* of the connection with respect to z.

6. **Holonomy groups II.** Adopt the hypotheses and notation of problem 5.

(i) Reduce the structure group of $\mathscr{P}$ to Γ; i.e., construct a principal bundle, $\mathscr{P}_1 = (P_1, \pi_1, B, \Gamma)$, and a Γ-equivariant inclusion map $\varphi: P_1 \to P$ such that $\pi \circ \varphi = \pi_1$.

Interpret P_1 as a maximal connected integral submanifold of an involutive distribution on P.

(ii) Let ω be the connection form of V. Prove that $\varphi^*\omega$ takes values in the Lie algebra of Γ_0. Conclude that $\varphi^*\omega$ is a connection form in P_1.

(iii) *Ambrose-Singer*: Assume that $\Gamma = G$ and let Ω be the curvature of V. Show that the vectors $\Omega(z; h, k)$, $z \in P$, $h, k \in T_z(P)$, span the Lie algebra E. (*Hint*: Use problem 3).

(iv) Suppose that two principal connections have the same curvature. Show that their holonomy groups have the same 1-component.

7. **Zero curvature.** Let $(\mathscr{P}, V)$ be as in problem 3, with B connected.

(i) Show that the following conditions are equivalent: (a) the curvature Ω is zero; (b) the holonomy group, Γ, is discrete; (c) the horizontal subbundle is an involutive distribution on P.

(ii) Assume $\Omega = 0$. Let M be a maximal connected integral manifold for the horizontal subbundle, and construct a principal covering projection (M, π, B, Γ) (cf. Problem 18, Chap. I).

(iii) Assume $\Omega = 0$ and B is simply connected. Construct an isomorphism $P = B \times G$ of principal bundles, which carries V to the standard connection in $B \times G$.

8. **Principal bundles with abelian structure group.** Assume that $\mathscr{P} = (P, \pi, B, G)$ is a principal bundle with abelian structure group. Let ω be a connection form and let Ω_B be the corresponding curvature form in B. Let $\varphi: D \to B$ be a smooth map of the two-dimensional disk D into B. Denote by γ the image of ∂D (∂D is the boundary of D) under φ and let $x_0 \in \partial D$ be a fixed point.

(i) Show that, for some fixed $a \in G$,

$$\hat{\gamma}(1) = \hat{\gamma}(0) \cdot a,$$

where $\hat{\gamma}$ is any horizontal lift of γ.

(ii) Show that $a = \exp_G(-\int_D \varphi^* \Omega_B)$.

9. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle with principal connection V.

(i) Show that every horizontal vector field X on P can be written as a finite sum $\sum_i f_i \cdot X_i$, where the X_i are horizontal and invariant and $f_i \in \mathscr{S}(P)$.

(ii) Assume that G is connected. Show that a differential form Φ is in $A_B(P)$ if and only if $\theta(X)\Phi = 0$ for every vertical vector field X.

10. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle and let $G \times F \to F$ be a left action of G on a manifold F. Let $\xi = (M, \rho, B, F)$ be the associated bundle ($M = P \times_G F$).

(i) Let $\mathbf{H}_P$ be the horizontal subbundle associated with a principal connection in $\mathscr{P}$. Show that the vector spaces

$$\mathbf{H}_{q(z,y)}(M) = (dq)_{(z,y)}\mathbf{H}_z(P), \qquad z \in P, \quad y \in F$$

($q: P \times F \to M$ is the principal map) are the fibres of a subbundle $\mathbf{H}_M$ of τ_M. Show that $\tau_M = \mathbf{H}_M \oplus \mathbf{V}_M$. $\mathbf{H}_M$ is called the *associated horizontal subbundle for* M.

(ii) With the aid of $\mathbf{H}_M$, define the notion of horizontal lifts in the bundle ξ. Establish an analogue of problem 3, (i) for ξ.

11. Let (P, π, B, G) be a principal bundle with a principal connection and corresponding homomorphism $\gamma_B\colon (\vee E^*)_I \to A(B)$. Suppose $\Gamma \in (\vee^p E^*)_I$ is an element such that $\gamma_B(\Gamma) = 0$. Show that Γ determines a closed $(2p - 1)$-form on P; hence obtain an element of $H_I^{2p-1}(P)$.

12. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle and let K be a closed subgroup of G. Consider the bundles $\mathscr{P}_K = (G, \pi_K, G/K, K)$ and $\mathscr{P}_1 = (P, p, P/K, K)$.

(i) Show that a G-invariant principal connection in $\mathscr{P}_K$ and a principal connection in $\mathscr{P}$ together determine the principal connection in $\mathscr{P}_1$ given by

$$\omega_1(z; \zeta) = \omega_K(e; \omega(z; \zeta)),$$

where ω_1, ω_K, ω are the connection forms.

(ii) Describe the horizontal subbundle, the horizontal projection, and the curvature.

13. Define connections in the principal bundles of article 5, Chap. V. Obtain the corresponding curvatures.

14. Let G be a compact connected Lie group with maximal torus T. Show that the principal bundle $G \to G/T$ admits a unique G-invariant connection and determine its curvature.

15. Bundles with compact support. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle. Let $O \subset B$ be an open set so that $B - O$ is compact and let $\sigma\colon O \to P$ be a cross-section over O. Then the pair $(\mathscr{P}, \sigma)$ is called a *principal bundle with compact support*. If $U \subset O$ is any open set such that $B - U$ is compact, then U is called an *open complement* for $(\mathscr{P}, \sigma)$.

A *homomorphism between principal bundles* $(\mathscr{P}, \sigma)$ *and* $(\hat{\mathscr{P}}, \hat{\sigma})$ *with compact support* is a homomorphism, $\varphi\colon \mathscr{P} \to \hat{\mathscr{P}}$, of principal bundles such that

(a) The induced map $\psi\colon B \to \hat{B}$ is proper.

(b) For some open complement V of $(\hat{\mathscr{P}}, \hat{\sigma})$

$$\varphi(\sigma(x)) = \hat{\sigma}(\psi(x)), \qquad x \in \psi^{-1}(V).$$

A *compact principal connection* in $(\mathscr{P}, \sigma)$ is a principal connection, V, in $\mathscr{P}$ such that for some open complement U of $(\mathscr{P}, \sigma)$

$$V_{\sigma(x)} \circ (d\sigma)_x = 0, \qquad x \in U.$$

(i) Let $(\mathscr{P}, \sigma)$ be a principal bundle with compact support. Show that a trivializing map α: $O \times G \xrightarrow{\cong} \pi^{-1}(O)$ is given by $(x, a) \mapsto \sigma(x) \cdot a$. Restate the definitions in terms of α.

(ii) Show that $(\mathscr{P}, \sigma)$ admits a compact principal connection. Show that the curvature of such a connection has support in $\pi^{-1}(K)$ for some compact subset K of B. Conclude that the induced homomorphism γ_B: $(\vee E^*)_I \to A(B)$ can be regarded as a homomorphism into $A_c(B)$.

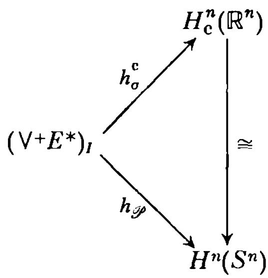
(iii) Show that γ_B induces a homomorphism, h_σ^c: $(\vee^+ E^*)_I \to H_c(B)$, the *Weil homomorphism with compact support*. Show that h_σ^c is independent of the choice of compact connection. Show that

$$\lambda_\# \circ h_\sigma^c = h_{\mathscr{P}} ,$$

where λ: $A_c(B) \to A(B)$ is the inclusion map.

(iv) Establish a naturality property for h_σ^c.

(v) Show that a compactly supported principal bundle over $\mathbb{R}^n$ determines a principal bundle over S^n and that the diagram



commutes. Hence construct an example where $h_\sigma^c \neq 0$ but $h_{\mathscr{P}} = 0$. Conclude that h_σ^c is *not* independent of σ.

16. Odd characteristic homomorphism. Let B be a manifold and let G be a Lie group with Lie algebra E. Let f: $B \to G$ be a smooth map such that for some compact subset $A \subset B, f(x) = e, x \notin A$.

(i) Set $\mathscr{P} = (B \times \mathbb{R} \times G, \pi, B \times \mathbb{R}, G)$ and $O = B \times \mathbb{R} - A \times I$, where I denotes the closed unit interval. Define σ: $O \to B \times \mathbb{R} \times G$ by

$$\sigma(x, t) = \begin{cases} (x, t, f(x)), & t \geqslant \frac{1}{2} \\ (x, t, e), & t \leqslant \frac{1}{2}. \end{cases}$$

Show that $(\mathscr{P}, \sigma)$ is a compactly supported principal bundle.

(ii) Let p: $\mathbb{R} \to [0, 1]$ be smooth and satisfy

$$p(t) = 0, \quad t \leqslant 0, \qquad \text{and} \qquad p(t) = 1, \quad t \geqslant 1.$$

Define $\theta_f \in A^1(B \times \mathbb{R}; E)$ by

$$\theta_f(x, t; \xi, \eta) = -p(t) L^{-1}_{f(x)}(df)_x \xi, \qquad \xi \in T_x(B), \quad \eta \in T_t(\mathbb{R}).$$

Show that the corresponding principal connection V_f in P is compact (cf. problem 1 and problem 15). Compute its curvature.

(iii) Define a map $\rho_E: (\vee^p E^*)_I \to (\wedge^{2p-1} E^*)_I$ by

$$(\rho_E \Gamma)(h_1, \dots, h_{2p-1}) = \frac{(-1)^{p-1}(p-1)!}{2^{p-1}(2p-1)!} \times \sum_\sigma \epsilon_\sigma \Gamma(h_{\sigma(1)}, [h_{\sigma(2)}, h_{\sigma(3)}], \dots, [h_{\sigma(2p-2)}, h_{\sigma(2p-1)}]).$$

Regard $\rho_E \Gamma$ as an element of $A_I^{2p-1}(G)$. Establish the relation

$$f^*(\rho_E \Gamma) = \oint_{\mathbb{R}} \gamma_{B\times\mathbb{R}}(\Gamma),$$

where $\gamma_{B\times\mathbb{R}}: (\vee^+ E^*)_I \to A_c(B \times \mathbb{R})$ is constructed via the connection V_f.

(iv) Obtain a map, $\bar{\rho}_E: (\vee^+ E^*)_I \to H(G)$, from ρ_E. Show that

$$\oint_{\mathbb{R}}^{\#} \circ h_\sigma^c : (\vee^+ E^*)_I \to H_c^+(B)$$

is a canonical map, independent of the connection. Show that

$$\oint_{\mathbb{R}}^{\#} \circ h_\sigma^c = f^\# \circ \bar{\rho}_E .$$

Conclude that $\oint_{\mathbb{R}}^{\#} \circ h_\sigma^c$ depends only on $f^\#$.

17. Covering by two open sets. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle. Assume $B = U \cup V$ is an open covering of B such that $\mathscr{P}$ is trivial over U and V.

(i) Let (U, ψ_U) and (V, ψ_V) be a principal coordinate representation for $\mathscr{P}$. Construct a smooth map $\varphi: U \cap V \to G$ such that

$$(\psi_V^{-1} \circ \psi_U)(x, a) = (x, \varphi(x)a), \qquad x \in U \cap V, \quad a \in G.$$

(ii) Let $\bar{\rho}_E: (\vee^+ E^*)_I \to H(G)$ be the linear map defined in problem 16, (iv). Let $\partial: H(U \cap V) \to H(B)$ be the connecting homomorphism of the Mayer–Vietoris sequence for (B, U, V). Prove that

$$h_{\mathscr{P}} = \partial \circ \varphi^\# \circ \bar{\rho}_E .$$

18. If $\mathscr{P} = (P, \pi, B, G)$ is a principal bundle with finite structure group, show that $\pi^\#$ maps $H(B)$ isomorphically onto $H_I(P)$.

19. The operator $i(a)$. Let M be a manifold. Define an $\mathscr{S}(M)$-linear map

$$i : \mathrm{Sec}\, \wedge\tau_M \to \mathrm{Hom}_M(A(M); A(M))$$

such that

(a) $i(\sigma \wedge \tau) = i(\tau) \circ i(\sigma)$, $\sigma, \tau \in \mathrm{Sec}\, \wedge\tau_M$,
(b) $i(X)$ is the substitution operator, $X \in \mathscr{X}(M)$ and
(c) $i(1) = \iota$.

(i) Show that i is uniquely determined by these conditions.

(ii) Let (P, π, B, G) be a principal bundle. Obtain canonical operators $i(a)$ $(a \in \wedge E)$ in $A(P)$ such that

$$i(h_1 \wedge \cdots \wedge h_p) = i(h_p) \circ \cdots \circ i(h_1).$$

Find expressions for the commutators $i(a) \circ \theta(h) - \theta(h) \circ i(a)$ and $i(a) \circ \delta - \delta \circ i(a)$.

(iii) Show that, for $a \in (\wedge^p E)_{\theta=0}$ and $\Phi \in A(P)_{\theta=0}$,

$$i(a)\, \delta\Phi = (-1)^p \delta i(a)(\Phi).$$

Hence obtain an operator, $i(a)_\#$, in $H(A(P))_{\theta=0}$.

(iv) Assume that G is compact and connected. Define

$$\varphi: H(A(P)_{\theta=0}) \to H(A(P)_{\theta=0}) \otimes (\wedge E^*)_{\theta=0}$$

by $\varphi(\alpha) = \sum_\nu \epsilon_{\nu,\alpha} i(a_\nu)_\# \alpha \otimes a^{*\nu}$, where a_ν, $a^{*\nu}$ is a pair of homogeneous bases of $(\wedge E)_{\theta=0}$ and $(\wedge E^*)_{\theta=0}$, and

$$\epsilon_{\nu,\alpha} = \deg a_\nu(\deg a_\nu + \deg \alpha).$$

Establish a commutative diagram,

$$\begin{array}{ccc} H(A(P)_{\theta=0}) & \xrightarrow{\ \varphi\ } & H(A(P)_{\theta=0}) \otimes (\wedge E^*)_{\theta=0} \\ \cong\big\downarrow & & \big\downarrow\cong \\ H(P) & \xrightarrow[\tau^*]{} & H(P) \otimes H(G) \end{array},$$

with vertical isomorphisms induced by inclusion maps.

(v) Extend the result of (iv) to any (i.e., not necessarily principal) action of a compact connected Lie group.

20. The operator D. Let ω be a connection form with curvature Ω in a principal bundle $\mathscr{P} = (P, \pi, B, G)$. With each representation of G in a space W, associate an operator D in $A(P; W)$ by setting

$$D\Phi = \delta\Phi + \omega(\Phi).$$

Prove the relations

(i) $D = \nabla$ in $A_B(P; E)$;

(ii) $D(\Psi(\Phi)) = D\Psi(\Phi) + (-1)^p\Psi(D\Phi)$, $\Psi \in A^p(P; E)$, $\Phi \in A(P; W)$;

(iii) $D\omega = \Omega + \frac{1}{2}[\omega, \omega]$;

(iv) $D^2\Psi = \Omega(\Psi)$, $\Psi \in A(P; W)$;

(v) $D\Omega = 0$;

(vi) if $\langle , \rangle$ is a bilinear function in W, invariant under the representation, and $\langle\!\langle , \rangle\!\rangle: A(P; W) \times A(P; W) \to A(P)$ is the induced map, then

$$\delta\langle\!\langle \Phi, \Psi \rangle\!\rangle = \langle\!\langle D\Phi, \Psi \rangle\!\rangle + (-1)^p\langle\!\langle \Phi, D\Psi \rangle\!\rangle, \qquad \Phi \in A^p(P; W), \quad \Psi \in A(P; W).$$

21. Algebraic connections. An *algebraic connection* in a principal bundle, $\mathscr{P} = (P, \pi, B, G)$, is a linear map $\chi: E^* \to A^1(P)$ satisfying the conditions:

(a) $i(h)\chi(h^*) = \langle h^*, h \rangle$, $h \in E$, $h^* \in E^*$.

(b) $T_a^* \circ \chi = \chi \circ \mathrm{Ad}^\natural(a)$, $a \in G$.

(i) Let ω be a connection form in $\mathscr{P}$. Show that an algebraic connection χ is defined by

$$\chi(h^*)(z; \zeta) = \langle h^*, \omega(z; \zeta) \rangle, \qquad z \in P, \quad \zeta \in T_z(P).$$

χ is called the *associated algebraic connection*. Show that the correspondence $\omega \mapsto \chi$ defines a bijection between principal connections and algebraic connections.

(ii) Show that an algebraic connection χ extends to the homomorphism $\chi_\wedge: \wedge E^* \to A(P)$ given by

$$(\chi_\wedge \Phi)(z; \zeta_1, \ldots, \zeta_p) = \langle \Phi, \omega(z; \zeta_1) \wedge \cdots \wedge \omega(z; \zeta_p) \rangle.$$

Show that $\chi_\wedge$ satisfies

$$i(h) \circ \chi_\wedge = \chi_\wedge \circ i_E(h),$$

$$T_a^* \circ \chi_\wedge = \chi_\wedge \circ \mathrm{Ad}^\wedge(a),$$

$$\theta(h) \circ \chi_\wedge = \chi_\wedge \circ \theta_E(h),$$

$$i(a) \circ \chi_\wedge = \chi_\wedge \circ i_E(a), \qquad a \in \wedge E.$$

(iii) Show that, for each $z \in P$, the map $\Phi \mapsto (\chi_\wedge \Phi)(z)$ defines an isomorphism $\wedge E^* \xrightarrow{\cong} \wedge \mathbf{V}_z(P)^*$. Conclude that an isomorphism

$$f: A(P)_{i=0} \otimes \wedge E^* \xrightarrow{\cong} A(P)$$

is given by $f(\Psi \otimes \Phi) = \Psi \wedge (\chi_\wedge \Phi)$.

(iv) Consider the linear map $\mathcal{X}: E^* \to A^2(P)_{i=0}$ given by

$$\mathcal{X}(h^*)(z; \zeta_1, \zeta_2) = \langle h^*, \Omega(z; \zeta_1, \zeta_2)\rangle, \qquad z \in P, \quad \zeta_1, \zeta_2 \in T_z(P),$$

where Ω is the curvature of the principal connection corresponding to the algebraic connection χ. Show that $\mathcal{X}$ extends to a homomorphism, $\mathcal{X}_\vee: \vee E^* \to A(P)_{i=0}$, and that

$$(\mathcal{X}_\vee \Psi)(z; \zeta_1, \ldots, \zeta_{2p}) = (1/2^p) \sum_\sigma \epsilon_\sigma \Psi(\Omega(z; \zeta_{\sigma(1)}, \zeta_{\sigma(2)}), \ldots, \Omega(z; \zeta_{\sigma(2p-1)}, \zeta_{\sigma(2p)})).$$

Establish the relations ($a \in G$, $h \in E$)

$$\mathcal{X}_\vee \circ \mathrm{Ad}^\vee(a)) = T_a^* \circ \mathcal{X}_\vee \qquad \text{and} \qquad \mathcal{X}_\vee \circ \theta_S(h) = \theta(h) \circ \mathcal{X}_\vee .$$

(v) Prove that

$$\mathcal{X}(h^*) = \delta \chi h^* - \chi_\wedge \delta_E h^*, \qquad h^* \in E^*,$$

$$\nabla \chi = \mathcal{X} \qquad \text{and} \qquad \nabla \mathcal{X} = 0 \quad \text{(Bianchi identity)}.$$

(vi) Show that $\mathcal{X}_\vee$ coincides with the homomorphism γ of sec. 6.17. Thus describe the Weil homomorphism in terms of $\mathcal{X}_\vee$.

22. Horizontal projection. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle with principal connection V and associated algebraic connection χ. Let $\{e^{*\nu}\}$, $\{e_\nu\}$ be a pair of dual bases of E^* and E and let $\mu(\Phi)$ denote left multiplication by Φ ($\Phi \in A(P)$).

(i) Define operators Y_k in $A(P)$ by

$$Y_0 = \iota$$
$$Y_k = \sum_\nu \mu(\chi e^{*\nu}) \circ Y_{k-1} \circ i(e_\nu), \qquad k \geqslant 1.$$

(ii) Show that the horizontal projection H^* is given by

$$H^* = \sum_k \frac{(-1)^k}{k!} Y_k .$$

(iii) Show that, for $\Phi \in A^p(P)$,

$$H^*\Phi = (\iota - Y_1)(\iota - \tfrac{1}{2}Y_1) \cdots (\iota - (1/p)\, Y_1)\Phi.$$

(iv) Show that Y_1 is an antiderivation, and that

$$Y_1\Phi(Z_1, ..., Z_p) = \sum_{j=1}^p \Phi(Z_1, ..., V_*Z_j, ..., Z_p).$$

(v) Set $\delta_Y = Y_1 \circ \delta$. Show that, for $\Phi \in A^p(P)_{i=0}$ and $\Psi \in A(P)_{i=0}$,

$$i(h)\,\delta_Y\Phi = \theta(h)\Phi, \qquad \delta_Y\Phi = \sum_\nu (\chi e^{*\nu}) \wedge \theta(e_\nu)\Phi$$

and

$$\delta_Y(\Phi \wedge \Psi) = \delta_Y\Phi \wedge \Psi + (-1)^p\, \Phi \wedge \delta_Y\Psi.$$

23. The homomorphism g. Continue the hypotheses and notation of problem 22.

(i) Make $A(P; \wedge E^*)$ into a bigraded algebra by regarding it as the *skew* tensor product of the algebras $A(P)$ and $\wedge E^*$. Interpret the elements of $A^p(P; \wedge^q E^*)$ as functions

$$\underbrace{\mathscr{X}(P) \times \cdots \times \mathscr{X}(P)}_{(p\text{ factors})} \times \underbrace{E \times \cdots \times E}_{(q\text{ factors})} \to \mathbb{R}.$$

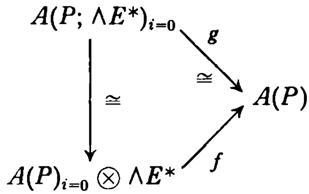
(ii) Define linear maps, $g\colon A^p(P; \wedge^q E^*) \to A^{p+q}(P)$, by setting

$$g\Phi(z; \zeta_1, ..., \zeta_{p+q}) = \frac{1}{p!\,q!} \sum_\sigma \epsilon_\sigma \Phi(z; \zeta_{\sigma(1)}, ..., \zeta_{\sigma(p)}, \omega(z; \zeta_{\sigma(p+1)}), ..., \omega(z; \zeta_{\sigma(p+q)})).$$

Show that the resulting linear map $g\colon A(P; \wedge E^*) \to A(P)$ restricts to an isomorphism $A(P; \wedge E^*)_{i=0} \xrightarrow{\cong} A(P)$.

(iii) Show that g restricts to an isomorphism $A_B(P; \wedge E^*) \xrightarrow{\cong} A_I(P)$.

(iv) Show that the diagram



commutes, where f is the isomorphism of problem 21, (iii).

(v) Let $\Phi \in A^m(P)$. Show that $g^{-1}\Phi = \sum_p \Psi_p$, where

$$\Psi_p \in A^p(P; \wedge^{m-p}E^*)_{i=0}$$

is given by

$$\Psi_p(Z_1, \dots, Z_p, h_1, \dots, h_{m-p}) = \Phi(H_*Z_1, \dots, H_*Z_p, Z_{h_1}, \dots, Z_{h_{m-p}}).$$

24. The decomposition of δ. Let V be a principal connection in a principal bundle $\mathscr{P} = (P, \pi, B, G)$.

(i) Show that antiderivations $D_\wedge$, $D_{\mathbb{X}}$, D_∇ are defined in $A(P)$ by the following equations ($\Phi \in A(P)$, $X_\nu \in \mathscr{X}(P)$):

$$D_\wedge\Phi(X_0, \dots, X_p)$$
$$= \sum_i (-1)^i (V_*X_i)(\Phi(X_0, \dots, \hat{X}_i, \dots, X_p))$$
$$+ \sum_{i<j} (-1)^{i+j}\Phi([X_i, X_j] - [H_*X_i, H_*X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p).$$

$$D_{\mathbb{X}}\Phi(X_0, \dots, X_p)$$
$$= \sum_{i<j} (-1)^{i+j}\Phi(V_*[H_*X_i, H_*X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p).$$

$$D_\nabla\Phi(X_0, \dots, X_p)$$
$$= \sum_i (-1)^i (H_*X_i)(\Phi(X_0, \dots, \hat{X}_i, \dots, X_p))$$
$$+ \sum_{i<j} (-1)^{i+j}\Phi(H_*[H_*X_i, H_*X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p).$$

(ii) Show that $\delta = D_\wedge + D_{\mathbb{X}} + D_\nabla$.

(iii) Let $\mathcal{X}$ be the associated algebraic connection. Show that, under the isomorphism $f: A(P)_{i=0} \otimes \wedge E^* \xrightarrow{\cong} A(P)$, (cf. problem 21, (iii)), $D_{\mathbb{X}}$, and D_∇ correspond to the operators,

$$\sum_\nu \mu(\mathcal{X}e^{*\nu})\omega \otimes i_E(e_\nu) \qquad \text{and} \qquad \nabla \otimes \iota,$$

where $\{e^{*\nu}\}$, $\{e_\nu\}$ is a pair of dual bases in E^* and E, ∇ is the covariant exterior derivative, and ω is the degree involution in $A(P)$.

(iv) Show that the covariant exterior derivative, ∇, satisfies

$$\nabla^2\Phi = \sum_\nu \mathbb{X}(e^{*\nu}) \wedge (\theta(e_\nu)\, H^*\Phi - \nabla i(e_\nu)\Phi), \qquad \Phi \in A(P).$$

(v) Establish the relations

$$D_\wedge^2 = 0, \qquad D_{\mathbb{X}}^2 = 0,$$
$$D_\wedge \circ D_{\mathbb{X}} + D_{\mathbb{X}} \circ D_\wedge = -D_\nabla^2,$$
$$D_\wedge \circ D_\nabla + D_\nabla \circ D_\wedge = 0,$$
$$D_{\mathbb{X}} \circ D_\nabla + D_\nabla \circ D_{\mathbb{X}} = 0.$$

(vi) Let $\mathscr{B} = (M, \rho, B, F)$ be any smooth bundle. Show that a decomposition $\tau_M = \mathbf{H}_M \oplus \mathbf{V}_M$ determines a bigradation of $A(M)$. Write $\delta_M = \sum_p \delta_p$, where δ_p is homogeneous of bidegree $(p, 1-p)$. Find expressions for the operators δ_p. Interpret the operators δ_p, when $\mathscr{B}$ is a principal bundle and $\mathbf{H}_M$ is the horizontal bundle of a principal connection.

25. The operators D_E and D_θ. Adopt the notation of problem 24. Let δ_E denote the operator $\omega \otimes \delta_E$ in $A(P; \wedge E^*)$ and let δ_θ be the operator in $A(P; \wedge E^*)$ given by $\delta_\theta = \sum_\nu \omega\theta(e_\nu) \otimes \mu(e^{*\nu})$.

(i) Show that δ_E and δ_θ are antiderivations with respect to which the algebra $A(P; \wedge E^*)_{i=0}$ is stable.

(ii) Use the isomorphism, g, of problem 23 to identify δ_E and δ_θ with operators D_E and D_θ in $A(P)$.

(iii) Show that $D_\wedge = D_E + D_\theta$.

(iv) Obtain a relation between D_θ and δ_Y (cf. problem 22).

26. Let $\mathscr{P} = (P, \pi, B \times \mathbb{R}, G)$ be a principal bundle. Fix $t \in \mathbb{R}$, and let $\mathscr{P}_t = (P_t, \pi_t, B \times \{t\}, G)$ be the restriction of $\mathscr{P}$.

(i) Construct an isomorphism from $\mathscr{P}$ to the principal bundle $(P_0 \times \mathbb{R}, \pi_0 \times \iota, B \times \mathbb{R}, G)$ (*Hint:* Use problem 3, (i).)

(ii) Conclude that $\mathscr{P}_0 \cong \mathscr{P}_t$.

(iii) Let $\varphi, \psi\colon M \to N$ be homotopic maps, where N is the base of a principal bundle. Prove that the pull-backs of this bundle to M via φ and ψ are strongly isomorphic.

(iv) Conclude that every principal bundle over a contractible space is trivial.

27. Principal bundles over S^n. Let $\mathscr{P} = (P, \pi, S^n, G)$ be a principal bundle over S^n.

(i) Show that $\mathscr{P}$ admits a coordinate representation consisting of only two elements.

(ii) Obtain a smooth map $\varphi\colon S^{n-1} \to G$ such that $\mathscr{P}$ is trivial if and only if φ is homotopic to the constant map.

(iii) Show that every principal bundle over S^3 is trivial (*Hint:* cf. problem 35, Chap. II.)

(iv) Apply problem 17.

28. Construct a fibre bundle over S^3 which is not the associated bundle of a principal bundle.

Hint: Proceed as follows:

(i) Construct a nontrivial bundle over S^4 with fibre S^3.

(ii) Pull this bundle back to a bundle $M \to S^3 \times S^1$ via a degree 1 map $S^3 \times S^1 \to S^4$.

(iii) Show that the induced projection $M \to S^3$ is the projection of the desired bundle.

29. Compact structure group. Suppose $\mathscr{P} = (P, \pi, B, G)$ is a principal bundle with compact group G. Let $\rho\colon A(P) \to A_I(P)$ denote the projection given by

$$\rho(\Phi) = \int_G T_a^* \Phi \, da, \qquad \Phi \in A(P).$$

Show that $\delta\rho\Phi = \rho\nabla\Phi$, $\Phi \in A(P)_{i=0}$, where ∇ is the covariant exterior derivative with respect to a principal connection.