Chapter V

Bundles with Structure Group

§1. Principal bundles

5.1. Definition. Let G be a Lie group. A (smooth) *principal bundle with structure group* G is a pair $(\mathscr{P}, T)$, where

(i) $\mathscr{P} = (P, \pi, B, G)$ is a smooth fibre bundle.
(ii) $T\colon P \times G \to P$ is a right action of G on P.
(iii) $\mathscr{P}$ admits a coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ such that

$$\psi_\alpha(x, ab) = \psi_\alpha(x, a) \cdot b, \qquad x \in U_\alpha, \quad a, b \in G.$$

(Note that we write $T(z, a) = z \cdot a$.)

The action T is called the *principal action* and a coordinate representation satisfying condition (iii) is called a *principal coordinate representation.*

Condition (iii) implies that

$$\pi(z \cdot a) = \pi(z), \qquad z \in P, \quad a \in G.$$

Moreover, it follows that the action T is free and that the orbit of G through a point $z \in P$ is the fibre containing z. In particular, the orbits are submanifolds of P. They will be denoted by $G_x = \pi^{-1}(x)$ $(x \in B)$, (since the action is free there is no confusion with the notation for isotropy subgroups). Note that $G_x \mapsto x$ defines a set bijection between the orbits and B.

Let $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, \hat{B}, G)$ be a second principal bundle with principal action $\hat{T}$. A smooth equivariant map $\varphi\colon P \to \hat{P}$ is called a *homomorphism of principal bundles*. Such a homomorphism is orbit preserving, and hence fibre preserving. Thus it induces a smooth map $\psi\colon B \to \hat{B}$ such that $\hat{\pi} \circ \varphi = \psi \circ \pi$ (cf. sec. 1.13, volume I).

Moreover, φ restricts to smooth maps $\varphi_x : G_x \to G_{\psi(x)}$ $(x \in B)$. The relations

$$\varphi_x(z \cdot a) = \varphi_x(z) \cdot a, \qquad z \in G_x, \quad a \in G,$$

imply that each φ_x is a diffeomorphism. It follows that φ is a diffeomorphism if and only if ψ is. In this case φ^{-1} is also a homomorphism of principal bundles and φ and φ^{-1} are called *isomorphisms of principal bundles*. If $B = \hat{B}$ and $\psi = \iota$, then φ is called a *strong isomorphism of principal bundles.*

Examples: 1. *The product bundle*: The trivial bundle,

$$(B \times G, \pi, B, G),$$

together with the right action

$$(x, a) \cdot b = (x, ab), \qquad x \in B, \quad a, b \in G$$

is a principal bundle. It is called the *trivial*, or *product bundle.*

2. *Homogeneous spaces:* Let K be a closed subgroup of G. Then the fibre bundle $(G, \pi, G/K, K)$ (cf. sec. 2.13), together with the action of K on G by right multiplication, is a principal bundle with structure group K.

3. *Frame bundles*: Let $\xi = (E, \rho, B, F)$ be a vector bundle, and, for $x \in B$, let G_x denote the set of linear isomorphisms from F to F_x. We shall construct a principal bundle, $(P, \pi, B, GL(F))$, where $P = \bigcup_x G_x$ and π is the projection which carries G_x to x.

In fact let $\{(U_\alpha, \psi_\alpha)\}$ be a coordinate representation for ξ. The isomorphisms $\psi_{\alpha,x} : F \xrightarrow{\cong} F_x$ determine set bijections

$$\varphi_{\alpha,x} : GL(F) \to G_x, \qquad x \in U_\alpha,$$

by

$$\varphi_{\alpha,x}(\varphi) = \psi_{\alpha,x} \circ \varphi, \qquad \varphi \in GL(F).$$

Thus set bijections $\varphi_\alpha : U_\alpha \times GL(F) \to \pi^{-1}(U_\alpha)$ are given by

$$\varphi_\alpha(x, \varphi) = \psi_{\alpha,x} \circ \varphi, \qquad x \in U_\alpha, \quad \varphi \in GL(F).$$

Evidently

$$(\varphi_\alpha^{-1} \circ \varphi_\beta)(x, \varphi) = (x, \psi_{\alpha,x}^{-1} \circ \psi_{\beta,x} \circ \varphi),$$
$$x \in U_\alpha \cap U_\beta, \quad \varphi \in GL(F).$$

It follows that $\varphi_\alpha^{-1} \circ \varphi_\beta$ is a diffeomorphism of $(U_\alpha \cap U_\beta) \times GL(F)$. Hence (cf. Proposition X, sec. 1.13, volume I), there is a unique smooth structure on the set P such that $(P, \pi, B, GL(F))$ becomes a smooth bundle.

Finally, define a right action of $GL(F)$ on each set G_x by setting

$$\varphi_x \cdot \varphi = \varphi_x \circ \varphi, \qquad \varphi_x \in G_x, \quad \varphi \in GL(F).$$

These actions define a right action of $GL(F)$ on the set P. Moreover,

$$\varphi_\alpha(x, \varphi) \cdot \varphi_1 = \varphi_\alpha(x, \varphi \circ \varphi_1), \qquad x \in U_\alpha, \quad \varphi, \varphi_1 \in GL(F).$$

It follows that the action of $GL(F)$ on P is smooth and that $\mathscr{P} = (P, \pi, B, GL(F))$ is a *principal* bundle.

Fix a basis $e_1, \dots, e_r$ of F. Then a bijection from G_x to the set of bases (or *frames*) of F_x is given by

$$\varphi \mapsto (\varphi e_1, \dots, \varphi e_r).$$

For this reason $\mathscr{P}$ is often called the *frame bundle* associated with ξ. Frame bundles are discussed again in article 5 of this chapter, and then extensively in article 7 of Chapter VIII.

5.2 Elementary properties. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle admitting a cross-section σ over an open set $U \subset B$. σ determines the homomorphism $\varphi: U \times G \to P$ of principal bundles, given by

$$\varphi(x, a) = \sigma(x) \cdot a, \qquad x \in U, \quad a \in G.$$

φ may be regarded as a strong isomorphism from the trivial bundle to the restriction of $\mathscr{P}$ to U. In particular, if $\mathscr{P}$ admits a cross-section, it is the trivial bundle.

If τ is a second cross-section over a second open set V, then there is a unique smooth map

$$g_{UV}: U \cap V \to G$$

such that $\varphi(x, g_{UV}(x)) = \tau(x)$. We have

$$\tau(x) = \sigma(x) \cdot g_{UV}(x), \qquad x \in U \cap V,$$

and this equation determines g_{UV}.

Lemma I: Let $\mathscr{P} = (P, \pi, B, G)$ be a smooth bundle. Let T be a smooth free right action of G on P, whose orbits coincide with the fibres of the bundle. Then $\mathscr{P}$ is a principal bundle with principal action T.

Proof: Let $\{U_\alpha\}$ be an open cover of B such that each U_α admits a cross-section $\sigma_\alpha : U_\alpha \to P$. Define $\psi_\alpha : U_\alpha \times G \xrightarrow{\cong} \pi^{-1}(U_\alpha)$ by setting

$$\psi_\alpha(x, a) = \sigma_\alpha(x) \cdot a.$$

Then $\{(U_\alpha, \psi_\alpha)\}$ is a coordinate representation satisfying condition (iii).

Q.E.D.

Next, let $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, \hat{B}, G)$ be a principal bundle, and let $\psi: B \to \hat{B}$ be a smooth map. We shall construct a principal bundle (P, π, B, G) together with a homomorphism, $\varphi: P \to \hat{P}$, of principal bundles which induces ψ.

In fact, let P be the disjoint union,

$$P = \bigcup_{x \in B} (\{x\} \times G_{\psi(x)}),$$

and define π by setting $\pi(\{x\} \times G_{\psi(x)}) = x$. Define a right action, T, of G on the set P and an equivariant set map $\varphi: P \to \hat{P}$ by

$$T((x, z), a) = (x, z \cdot a) \qquad \text{and} \qquad \varphi(x, z) = z,$$

$$z \in G_{\psi(x)}, \quad x \in B, \quad a \in G.$$

Give P a smooth structure, as follows. Choose an open cover $\{V_\nu\}$ of $\hat{B}$ such that each V_ν admits a cross-section $\sigma_\nu : V_\nu \to \hat{P}$. Set $U_\nu = \psi^{-1}(V_\nu)$ and define bijections $\chi_\nu : U_\nu \times G \to \pi^{-1}(U_\nu)$ by

$$\chi_\nu(x, a) = (x, \sigma_\nu(\psi(x)) \cdot a).$$

Then for $x \in U_\nu \cap U_\mu$,

$$(\chi_\mu^{-1} \circ \chi_\nu)(x, a) = (x, g_{\mu\nu}(\psi(x))a),$$

where $g_{\mu\nu} : V_\mu \cap V_\nu \to G$ is the smooth map satisfying

$$\sigma_\nu(y) = \sigma_\mu(y) \cdot g_{\mu\nu}(y), \qquad y \in V_\mu \cap V_\nu.$$

We can thus apply Proposition X, sec. 1.13, volume I, to obtain a unique smooth structure on P such that $\mathscr{P} = (P, \pi, B, G)$ is a smooth bundle with coordinate representation $\{(U_\nu, \chi_\nu)\}$. Since the maps χ_ν are equivariant, T is a smooth action and $(\mathscr{P}, T)$ is a principal bundle. Moreover, φ is a homomorphism of principal bundles.

$\mathscr{P}$ is called the *pull-back* of $\hat{\mathscr{P}}$ to B via ψ and it is often written $\psi^*\hat{\mathscr{P}}$.

Let $\mathscr{P}_1 = (P_1, \pi_1, B, G)$ be a second principal bundle over B which admits a homomorphism $\varphi_1: P_1 \to \hat{P}$ of principal bundles inducing $\psi: B \to \hat{B}$. Then a strong isomorphism $\varphi_2: P \xrightarrow{\cong} P_1$ is defined by

$$\varphi_2(z) = ((\varphi_1)_x^{-1} \circ \varphi_x)(z), \qquad z \in \pi^{-1}(x).$$

Note that $\varphi_1 \circ \varphi_2 = \varphi$.

§2. Associated bundles

Notation convention: In this article $\mathscr{P} = (P, \pi, B, G)$ denotes a fixed principal bundle with principal action T. Moreover,

$$S: G \times F \to F$$

will denote a fixed left action of G on a manifold F.

5.3. Associated bundles. Consider the right action, Q, of G on the product manifold $P \times F$ given by

$$Q_a(z, y) = (z, y) \cdot a = (z \cdot a, a^{-1} \cdot y), \qquad z \in P, \quad y \in F, \quad a \in G.$$

Q will be called the *joint action of* G. The set of orbits for the joint action will be denoted by $P \times_G F$ and

$$q: P \times F \to P \times_G F$$

will denote the corresponding projection; i.e., $q(z, y)$ is the orbit through (z, y).

The map q determines a map $\rho: P \times_G F \to B$ via the commutative diagram

$$\begin{array}{ccc} P \times F & \xrightarrow{q} & P \times_G F \\ \downarrow{\scriptstyle \pi_P} & & \downarrow{\scriptstyle \rho} \\ P & \xrightarrow[\pi]{} & B, \end{array} \tag{5.1}$$

where π_P is the obvious projection. Denote $\rho^{-1}(x)$ by F_x, $x \in B$.

Proposition I: There is a unique smooth structure on $P \times_G F$ such that

(1) $\xi = (P \times_G F, \rho, B, F)$ is a smooth fibre bundle.

(2) $q: P \times F \to P \times_G F$ is a smooth fibre preserving map, restricting to diffeomorphisms

$$q_z: z \times F \xrightarrow{\cong} F_{\pi(z)}, \qquad z \in P,$$

on each fibre.

(3) $(P \times F, q, P \times_G F, G)$ is a smooth principal bundle with principal action Q.

(4) π_P is a homomorphism of principal bundles.

Definition: ξ is called the *fibre bundle with fibre F and structure group G associated with $\mathscr{P}$*; q is called the *principal map.*

Proof of Proposition I: If a smooth structure satisfies 3, it makes $P \times_G F$ into a quotient manifold of $P \times F$ under q. Hence, by the corollary to Proposition V, sec. 3.9, volume I, it is uniquely determined.

Proof of (1): We construct a smooth structure on $P \times_G F$ for which ξ is a smooth bundle. Let $\{U_\alpha\}$ be an open cover of B and consider cross-sections $\sigma_\alpha\colon U_\alpha \to P$. These are related by

$$\sigma_\beta(x) = \sigma_\alpha(x) \cdot g_{\alpha\beta}(x), \qquad x \in U_\alpha \cap U_\beta,$$

where $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ are smooth maps. Define set maps,

$$\varphi_\alpha\colon U_\alpha \times F \to \rho^{-1}(U_\alpha),$$

by setting

$$\varphi_\alpha(x, y) = q(\sigma_\alpha(x), y), \qquad x \in U_\alpha, \quad y \in F.$$

Then $\rho(\varphi_\alpha(x, y)) = x$ and so φ_α restricts to set maps

$$\varphi_{\alpha,x}\colon F \to \rho^{-1}(x), \qquad x \in U_\alpha .$$

Moreover, to each orbit in $\rho^{-1}(x)$ there corresponds a unique $y \in F$ such that the orbit passes through $(\sigma_\alpha(x), y)$. Hence $\varphi_{\alpha,x}$ is bijective, and so φ_α is bijective.

Further, the relations $q(z \cdot a, y) = q(z, a \cdot y)$ imply that

$$\varphi_\alpha^{-1} \circ \varphi_\beta(x, y) = (x, g_{\alpha\beta}(x) \cdot y), \qquad x \in U_\alpha \cap U_\beta, \quad y \in F.$$

Thus Proposition X, sec. 1.13, volume I, yields a smooth structure on $P \times_G F$ for which ξ is a smooth bundle with coordinate representation $\{(U_\alpha, \varphi_\alpha)\}$.

Proof of (3): To show that $(P \times F, q, P \times_G F, G)$, is a smooth principal bundle with principal action Q consider the commutative diagrams,

$$\begin{array}{ccc} U_\alpha \times G \times F & \xrightarrow[\cong]{\psi_\alpha \times \iota} & \pi^{-1}(U_\alpha) \times F \\ \downarrow{\scriptstyle \iota \times S} & & \downarrow{\scriptstyle q} \\ U_\alpha \times F & \xrightarrow[\varphi_\alpha]{\cong} & \rho^{-1}(U_\alpha), \end{array} \tag{5.2}$$

where $\psi_\alpha(x, a) = \sigma_\alpha(x) \cdot a$. Set $V_\alpha = \rho^{-1}(U_\alpha)$; then

$$q^{-1}(V_\alpha) = (\pi \circ \pi_P)^{-1}(U_\alpha) = \pi^{-1}(U_\alpha) \times F.$$

Thus diffeomorphisms $\chi_\alpha\colon V_\alpha \times G \xrightarrow{\cong} q^{-1}(V_\alpha)$ are given by

$$\chi_\alpha(\varphi_\alpha(x, y), a) = (\psi_\alpha(x, a), a^{-1} \cdot y).$$

They satisfy the relations

$$(q \circ \chi_\alpha)(w, a) = w, \qquad \text{and} \qquad \chi_\alpha(w, ab) = Q(\chi_\alpha(w, a), b), \qquad w \in V_\alpha, \quad a, b \in G$$

(cf. diagram 5.2). (3) follows.

Proof of (2): The commutative diagram (5.1) shows that q is fibre preserving, while the commutative diagrams (5.2) imply that the maps

$$q_z\colon F \xrightarrow{\cong} F_{\pi(z)}$$

are diffeomorphisms.

Proof of (4): This is obvious.

Q.E.D.

5.4. Equivariant maps. Assume $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, \hat{B}, G)$ is a second principal bundle and that $\hat{S}$ is a left action of G on a manifold $\hat{F}$. Suppose further that

$$\varphi\colon P \to \hat{P} \qquad \text{and} \qquad \alpha\colon F \to \hat{F}$$

are smooth equivariant maps.

Then the map $\varphi \times \alpha\colon P \times F \to \hat{P} \times \hat{F}$ is equivariant with respect to the joint actions of G; i.e., it is a homomorphism of principal bundles. Thus it induces a smooth map,

$$\varphi \times_G \alpha : P \times_G F \to \hat{P} \times_G \hat{F},$$

which makes the diagram,

$$\begin{array}{ccc} P \times F & \xrightarrow{\varphi \times \alpha} & \hat{P} \times \hat{F} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \hat{q}} \\ P \times_G F & \xrightarrow[\varphi \times_G \alpha]{} & \hat{P} \times_G \hat{F} \end{array} ,$$

commute.

Let $\psi: B \to \hat{B}$ be the smooth map induced by φ. Then the diagram,

$$\begin{array}{ccc} P \times_G F & \xrightarrow{\varphi \times_G \alpha} & \hat{P} \times_G \hat{F} \\ \rho \downarrow & & \downarrow \hat{\rho} \\ B & \xrightarrow[\psi]{} & \hat{B} \end{array},$$

commutes; i.e., $\varphi \times_G \alpha$ is a fibre preserving map between the associated bundles. The commutative diagrams

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & \hat{F} \\ q_z \downarrow \cong & & \cong \downarrow \hat{q}_{\varphi(z)} \\ F_x & \xrightarrow[(\varphi \times_G \alpha)_x]{} & \hat{F}_{\psi(x)} \end{array}, \qquad x = \pi(z), \qquad z \in P,$$

show that, if α is a diffeomorphism, then so is each $(\varphi \times_G \alpha)_x$.

The case that $\mathscr{P} = \hat{\mathscr{P}}$ and $\varphi = \iota$, is of particular importance; in this case we obtain a fibre preserving map,

$$(\iota \times_G \alpha) : P \times_G F \to P \times_G \hat{F},$$

which induces the identity map in B.

5.5. Examples: 1. $F = \{point\}$. Then $P \times_G F = B$ and the principal bundle $(P \times F, q, P \times_G F, G)$ coincides with $\mathscr{P}$.

2. Assume the action of G on F is trivial. Then $\xi = (B \times F, \rho, B, F)$ is trivial. Also, if the principal bundle $\mathscr{P}$ is trivial, then so is ξ.

3. Suppose $y \in F$ is fixed under the action of G: $a \cdot y = y$, $a \in G$. Then the inclusion $j : \{y\} \to F$ is equivariant. It induces (sec. 5.4) a smooth commutative diagram

$$\begin{array}{ccc} P \times_G \{y\} & \xrightarrow{\sigma} & P \times_G F \\ \cong \downarrow & & \downarrow \rho \\ B & \xrightarrow[\iota]{} & B \end{array};$$

thus σ is a cross-section in ξ.

4. *λ-extension:* Let $\lambda: G \to K$ be a homomorphism of Lie groups. Then G acts from the left on K by

$$a \cdot y = \lambda(a)y, \qquad a \in G, \quad y \in K.$$

Thus we obtain a bundle $\mathscr{P}_\lambda = (P \times_G K, \rho, B, K)$.

On the other hand, the multiplication map of K determines a right action

$$(P \times K) \times K \to P \times K.$$

This map factors over q to yield a free right action

$$T_\lambda : (P \times_G K) \times K \to P \times_G K.$$

The orbits of T_λ are precisely the fibres of $P \times_G K$. Thus it follows from Lemma I, sec. 5.2, that $(\mathscr{P}_\lambda, T_\lambda)$ is a principal K-bundle. It is called the *λ-extension of $\mathscr{P}$*.

Next, define a smooth map φ_λ: $P \to P \times_G K$ by setting $\varphi_\lambda(z) = q(z, e)$. The diagram,

$$\begin{array}{ccc} P \times G & \xrightarrow{\iota \times \lambda} & P \times K \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle q} \\ P & \xrightarrow{\varphi_\lambda} & P \times_G K \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \rho} & \\ & B & \end{array} \qquad (5.3)$$

commutes (cf. diagram 5.1, sec. 5.3). This shows that φ_λ is a fibre preserving map from P to $P \times_G K$, inducing the identity in B.

In particular, consider the case that $G = K$ and $\lambda = \iota$; thus G acts on itself by left multiplication. In this case φ_λ is a strong isomorphism of principal bundles, and the diagram shows that $(P \times G, q, P \times_G G, G)$ is the trivial principal bundle.

5. *Reduction of structure group*: Again, let λ: $G \to K$ be a homomorphism of Lie groups. Assume that $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, B, K)$ is a principal bundle. A *reduction of the structure group of $\hat{\mathscr{P}}$ from K to G via λ* is a principal bundle $\mathscr{P} = (P, \pi, B, G)$ and a smooth fibre preserving map φ: $P \to \hat{P}$, inducing the identity in the base, and satisfying

$$\varphi(z \cdot a) = \varphi(z) \cdot \lambda(a), \qquad a \in G.$$

Such a reduction induces an obvious isomorphism of principal bundles from the λ-extension of $\mathscr{P}$ to $\hat{\mathscr{P}}$ (cf. Example 4). Conversely, if $\mathscr{P} = (P, \pi, B, G)$ is any principal bundle with λ-extension $\mathscr{P}_\lambda = (P \times_G K, \rho, B, K)$, then the homomorphism φ_λ of Example 4 is a reduction of the structure group of $\mathscr{P}_\lambda$ from K to G.

5.6. Associated vector bundles. Assume now that F is a finite-dimensional (real or complex) vector space and S is a representation of G in F. In this case $P \times_G F$ is a *vector bundle*.

In fact, for each $x \in B$, $z \in \pi^{-1}(x)$, the diffeomorphisms $q_z\colon F \xrightarrow{\cong} F_x$ are connected by

$$q_{z \cdot a} = q_z \circ S(a), \qquad a \in G.$$

Since each map $S(a)$ is a linear isomorphism, there is a unique linear structure in F_x for which the maps q_z are linear isomorphisms. The zero vector of F_x is given by $0_x = q(z, 0)$, $z \in \pi^{-1}(x)$.

Each $\varphi_{\alpha,x}$ of the coordinate representation $\{(U_\alpha, \varphi_\alpha)\}$ for ξ defined in sec. 5.3 is a linear isomorphism. Hence ξ is a vector bundle with vector bundle coordinate representation $\{(U_\alpha, \varphi_\alpha)\}$. Since q restricts to isomorphisms in the fibres, the trivial bundle $(P \times F, \pi_P, P, F)$ is the pull-back of ξ to P via π (cf. sec. 2.5, volume I).

To the trivial representation S corresponds the trivial vector bundle.

Next, consider a representation of G in a second vector space H and let $\alpha\colon F \to H$ be an equivariant linear map. Then the induced map (cf. sec. 5.4),

$$\iota \times_G \alpha : P \times_G F \to P \times_G H,$$

is linear in each fibre, and so it is a (strong) bundle map.

Denote the vector bundles corresponding to F and H by ξ and η and consider the induced representations of G in the spaces

$$F \oplus H, \qquad F \otimes H, \qquad L(F; H), \qquad F^*, \qquad \wedge F.$$

The associated vector bundles corresponding to these representations are given, respectively, by

$$\xi \oplus \eta, \qquad \xi \otimes \eta, \qquad L(\xi; \eta), \qquad \xi^*, \qquad \wedge \xi.$$

The various canonical maps between these vector spaces, such as

evaluation:	$L(F; H) \otimes F$	$\to$	H,
composition:	$L_F \otimes L_F$	$\to$	L_F,
projection:	$F \oplus H$	$\to$	F,
trace:	L_F	$\to$	$\mathbb{R}$,

commute with the representations of G. Thus they induce maps between the corresponding vector bundles. For the four examples above we have (cf. sec. 2.10, volume I).

evaluation:	$L(\xi;\eta)\otimes\xi$	$\to$	η,
composition:	$L_\xi\otimes L_\xi$	$\to$	L_ξ,
projection:	$\xi\oplus\eta$	$\to$	ξ,
trace:	L_ξ	$\to$	$\mathscr{S}(B)$.

§3. Bundles and homogeneous spaces

In this article K denotes a closed subgroup of a Lie group G. Their Lie algebras are denoted by F and E ($F \subset E$). The corresponding principal bundle (cf. Example 2, sec. 5.1) is denoted by $\mathscr{P}_K = (G, \pi_K, G/K, K)$ and we write $\bar{e} = \pi_K(e)$ (e, the unit element of G). The left action of G on G/K is denoted by T.

5.7. Bundles with fibre a homogeneous space. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle with principal action R. The left action of G on G/K determines an associated bundle

$$\xi = (P \times_G (G/K), \rho, B, G/K)$$

(cf. sec. 5.3). To simplify notation we shall write

$$P \times_G (G/K) = P/K.$$

Consider the commutative diagram,

$$\begin{array}{ccc} P \times G/K & \xrightarrow{\;q\;} & P/K \\ {\scriptstyle \pi_P}\big\downarrow & & \big\downarrow{\scriptstyle \rho} \\ P & \xrightarrow[\pi]{\qquad} & B, \end{array}$$

and define $p: P \to P/K$ by $p(z) = q(z, \bar{e})$.

Proposition II: With the notation above, $(P, p, P/K, K)$ is a principal bundle whose principal action is the restriction of R to K.

Proof: It is sufficient to show that each $w \in P/K$ has a neighbourhood W such that $(p^{-1}(W), p, W, K)$ is a principal bundle.

Let $\{(U_\alpha, \varphi_\alpha)\}$ be the coordinate representation for ξ defined in sec. 5.3. Set $W = \rho^{-1}(U_\alpha)$, where α is chosen so that $w \in W$. Then

$$p^{-1}(W) = \pi^{-1}(U_\alpha).$$

Finally, let $j\colon G \to G \times G/K$ be the inclusion opposite $\bar{e}$. From diagram (5.2) of sec. 5.3 we obtain the commutative diagram

$$\begin{array}{ccccc} U_\alpha \times G & \xrightarrow{\iota\times j} & U_\alpha\times G\times G/K & \xrightarrow[\cong]{\psi_\alpha\times\iota} & \pi^{-1}U_\alpha \times G/K \\ {\scriptstyle \iota\times\pi_K}\big\downarrow && \big\downarrow{\scriptstyle \iota\times T} && \big\downarrow{\scriptstyle q}\\ U_\alpha\times G/K & \xrightarrow[\iota]{} & U_\alpha\times G/K & \xrightarrow[\varphi_\alpha]{\cong} & \rho^{-1}(U_\alpha). \end{array}$$

It restricts to the commutative diagram,

$$\begin{array}{ccc} U_\alpha \times G & \xrightarrow[\cong]{\psi} & p^{-1}(W)\times\{\bar{e}\} \\ {\scriptstyle \iota\times\pi_K}\big\downarrow && \big\downarrow{\scriptstyle p}\\ U_\alpha\times G/K & \xrightarrow[\varphi_\alpha]{\cong} & W, \end{array}$$

where $\psi = (\psi_\alpha \times \iota)\circ(\iota\times j)$.

Now $\mathscr{P}_K$ is a principal K-bundle, and ψ is equivariant with respect to the given actions of K. It follows that $(p^{-1}(W), p, W, K)$ is a principal K-bundle.

Q.E.D.

Next, fix $z \in P$ and write $\pi(z) = x$. Then the fibre inclusion, $j_{G/K}: G/K \to P/K$, for the bundle ξ is given by

$$j_{G/K} = q_z : G/K \xrightarrow{\cong} \rho^{-1}(x).$$

Let $j_G\colon G \to P$ and $j_K\colon K \to P$ denote the fibre inclusions given by

$$j_G(b) = z\cdot b \qquad \text{and} \qquad j_K(a) = z\cdot a, \qquad b\in G, \quad a \in K,$$

and let $i\colon K \to G$ be the inclusion map. Then the diagram,

$$\begin{array}{ccccc} K & = & K && \\ {\scriptstyle i}\big\downarrow && \big\downarrow{\scriptstyle j_K} && \\ G & \xrightarrow{j_G} & P & \xrightarrow{\pi} & B \\ {\scriptstyle \pi_K}\big\downarrow && \big\downarrow{\scriptstyle p} && \| \\ G/K & \xrightarrow[j_{G/K}]{} & P/K & \xrightarrow[\rho]{} & B, \end{array} \tag{5.4}$$

commutes. Moreover j_G is a homomorphism of principal K-bundles.

5.8. Subgroup of a subgroup. Assume now that G is a closed subgroup of a Lie group H, and apply the results of sec. 5.7 to the principal bundle $\mathscr{P} = (H, \pi, H/G, G)$. We obtain an associated bundle,

$$\xi = (H \times_G G/K, \rho, H/G, G/K),$$

and a principal bundle

$$\hat{\mathscr{P}} = (H, p, H \times_G G/K, K).$$

The left action of H on H/K restricts to a smooth map,

$$H \times G/K \to H/K,$$

which factors to yield a diffeomorphism

$$H \times_G G/K \xrightarrow{\cong} H/K$$

(equivariant with respect to the left actions of H, cf. sec. 5.9). We identify these manifolds via this diffeomorphism and write

$$\xi = (H/K, \rho, H/G, G/K), \qquad \hat{\mathscr{P}} = (H, p, H/K, K).$$

Then $\hat{\mathscr{P}}$ is the standard principal bundle, while ρ is given by

$$\rho(aK) = aG, \qquad a \in H.$$

Moreover, diagram (5.4) reads

$$\begin{array}{ccccc} K & = & K & & \\ \downarrow & & \downarrow & & \\ G & \longrightarrow & H & \xrightarrow{\pi} & H/G \\ {\scriptstyle \pi_K}\downarrow & & \downarrow{\scriptstyle p} & & \| \\ G/K & \longrightarrow & H/K & \xrightarrow[\rho]{} & H/G. \end{array} \tag{5.5}$$

Now suppose that K is normal in G. Then a smooth free right action of the factor group G/K on H/K is given by

$$\bar{x} \cdot \bar{a} = \overline{x \cdot a}, \qquad x \in H, \quad a \in G.$$

The orbits of G/K under this action coincide with the fibres in the bundle $\xi = (H/K, \rho, H/G, G/K)$. It follows from Lemma I, sec. 5.2, that ξ is a principal G/K-bundle.

5.9. Bundles with base a homogeneous space. Let K act from the left on a manifold N. There is a unique left action,

$$\Lambda : G \times (G \times_K N) \to G \times_K N,$$

of G that makes the diagram,

$$\begin{array}{ccc} G \times G \times N & \xrightarrow{\mu \times \iota} & G \times N \\ {\scriptstyle \iota \times q}\downarrow & & \downarrow{\scriptstyle q} \\ G \times (G \times_K N) & \xrightarrow[\Lambda]{} & G \times_K N, \end{array}$$

commute. Clearly Λ, together with T, is an action of G on the bundle $\xi = (G \times_K N, \rho, G/K, N)$ associated with $\mathscr{P}_K$; i.e., G acts on the total and base spaces and the projection is equivariant:

$$\rho \circ \Lambda = T \circ (\iota \times \rho).$$

Let $N_{\bar{e}} = \rho^{-1}(\bar{e})$. Since $a \cdot \bar{e} = \bar{e}$ $(a \in K)$, it follows that Λ restricts to a left action

$$K \times N_{\bar{e}} \to N_{\bar{e}}.$$

The projection q restricts to a K-equivariant diffeomorphism,

$$q_e : N \xrightarrow{\cong} N_{\bar{e}}$$

(cf. Proposition I, (2), sec. 5.3).

Conversely, assume that $\eta = (M, \rho_M, G/K, Q)$ is a smooth bundle over G/K and that Λ (with T) is a left action of G on η. Then we can construct the bundle,

$$\xi = (G \times_K Q_{\bar{e}}, \rho, G/K, Q_{\bar{e}}),$$

via the induced action of K on $Q_{\bar{e}}$.

Λ restricts to a smooth map $G \times Q_{\bar{e}} \to M$. This factors over q to yield an equivariant fibre preserving diffeomorphism,

$$\psi : G \times_K Q_{\bar{e}} \xrightarrow{\cong} M,$$

which induces the identity map in G/K.

5.10. Vector bundles. In this section we apply the results of sec. 5.9 to vector bundles. Each representation of K in a vector space N yields a vector bundle over G/K associated with $\mathscr{P}_K$ (cf. sec. 5.6) in which G acts by bundle maps. Conversely, if G acts by bundle maps in a vector bundle η over G/K so as to induce the standard action in G/K, then the action restricts to a representation of K in the fibre over $\tilde{e}$.

If these two constructions are applied consecutively, starting off with a representation of K (respectively, a vector bundle over G/K acted on by G), we obtain a representation (respectively, a vector bundle acted on by G) which is equivariantly (respectively, equivariantly and strongly) isomorphic to the original.

Examples: **1.** If the representation of K in N is trivial, then

$$G \times_K N = G/K \times N$$

and ξ is trivial.

2. Assume that the representation of K in N extends to a representation of G in N. Define a diffeomorphism φ of $G \times N$ by setting

$$\varphi(b, y) = (b, b^{-1} \cdot y), \qquad b \in G, \quad y \in N.$$

Then (letting Q denote the joint action of K in $G \times N$)

$$\varphi \circ (\rho_a \times \iota) = Q_a \circ \varphi, \qquad a \in K$$

(ρ_a denotes the right translation of G by a). It follows that φ induces a diffeomorphism

$$\psi\colon G/K \times N \xrightarrow{\cong} G \times_K N.$$

Evidently ψ is a strong vector bundle isomorphism. Moreover,

$$\psi(b \cdot z, b \cdot y) = b \cdot \psi(z, y), \qquad b \in G, \quad z \in G/K, \quad y \in N,$$

(where G acts on $G \times_K N$ as defined in sec. 5.9).

5.11. Tangent bundle of a homogeneous space. Recall that the Lie algebras of K and G are denoted by F and E. The adjoint representation of G restricts to a representation, $\mathrm{Ad}_{G,K}$, of K in E. Since the Lie algebra F is stable under the maps $\mathrm{Ad}_{G,K}(a)$, $a \in K$, we obtain a representation, $\mathrm{Ad}^{\perp}$, of K in E/F. The sequence

$$0 \to F \to E \to E/F \to 0$$

is short exact and K-equivariant with respect to the representations Ad, $\mathrm{Ad}_{G,K}$, and $\mathrm{Ad}^{\perp}$ of K.

Now form the vector bundles

$$\xi = (G \times_K (E/F), \rho_\xi, G/K, E/F) \qquad \text{and} \qquad \eta = (G \times_K F, \rho_\eta, G/K, F).$$

G acts on both ξ and η. On the other hand, the left action, T, of G on G/K induces a left action, dT, of G on the tangent bundle $\tau_{G/K}$ (cf. Example 7, sec. 3.2).

Proposition III: With the hypotheses and notation above

(1) ξ is strongly and equivariantly isomorphic to $\tau_{G/K}$.
(2) The vector bundle $\xi \oplus \eta$ is trivial.

Proof: (1) According to sec. 2.11, $(d\pi_K)_e$ induces a linear isomorphism

$$E/F \xrightarrow{\cong} T_{\bar{e}}(G/K).$$

Since $\pi_K \circ \lambda_a = T_a \circ \pi_K$ and $\pi_K \circ \rho_a = \pi_K$ $(a \in K)$, we have

$$(d\pi_K)_e \circ \mathrm{Ad}_{G,K}(a) = dT_a \circ (d\pi_K)_e, \qquad a \in K.$$

Thus this isomorphism is equivariant with respect to $\mathrm{Ad}^\perp$ and dT. Now apply sec. 5.10.

(2) Since the sequence $F \to E \to E/F$ is K-equivariant, it determines a sequence of strong bundle maps

$$\eta \xrightarrow{i} G \times_K E \xrightarrow{p} \xi.$$

For each $z \in G/K$, the restriction,

$$0 \to F_z \to E_z \to (E/F)_z \to 0,$$

is short exact.

Hence, there is a strong bundle map $\sigma: \xi \to G \times_K E$ such that $p \circ \sigma = \iota$ (cf. Lemma III, sec. 2.23, volume I). Thus a strong bundle isomorphism,

$$\varphi: \xi \oplus \eta \xrightarrow{\cong} G \times_K E,$$

is defined by

$$\varphi(u, v) = \sigma(u) + i(v), \qquad u \in (E/F)_z, \quad v \in F_z, \quad z \in G/K.$$

On the other hand, the representation $\mathrm{Ad}_{G,K}$ of K in E is the restriction of a representation of G. Hence, by Example 2 of sec. 5.10, $G \times_K E$ is a trivial bundle over G/K. Thus $\xi \oplus \eta$ is trivial.

Q.E.D.

5.12. Tori. Suppose now that G is compact and connected, and that K is a torus in G. Then the adjoint representation of K is trivial and hence, so is the bundle

$$\eta = (G \times_K F, \rho_\eta, G/K, F).$$

Thus, by Proposition III, sec. 5.11, the Whitney sum of $\tau_{G/K}$ with a trivial bundle is trivial. This implies (as will be shown in sec. 7.19) that the Whitney sum of $\tau_{G/K}$ with the trivial bundle *of rank one* is trivial,

$$\tau_{G/K} \oplus \epsilon^1 \cong \epsilon^{r+1}, \qquad r = \dim G/K. \tag{5.6}$$

Now we distinguish two cases:

Case I: K is a *maximal* torus (cf. sec. 2.15). Then the Euler–Poincaré characteristic of G/K is positive (cf. sec. 4.21). Hence Theorem II, sec. 10.1, volume I, implies that every vector field on G/K has at least one zero. In particular, the tangent bundle of G/K is nontrivial.

Case II: K is *not* maximal. Then K is properly contained in a maximal torus, T. Since T is compact and connected, the factor group T/K is again a torus (cf. Proposition XIII, sec. 1.12).

Thus according to sec. 5.8 we can form the principal T/K-bundle

$$\mathscr{P} = (G/K, \pi, G/T, T/K).$$

Write $T_{G/K} = H_{G/K} \oplus V_{G/K}$, where $V_{G/K}$ is the vertical subbundle and $H_{G/K}$ is a horizontal bundle (cf. sec. 0.15).

Since $\mathscr{P}$ is a principal bundle, the vertical subbundle is trivial (as will be shown in sec. 6.1),

$$V_{G/K} = \epsilon^m, \qquad m = \dim T/K.$$

By hypothesis, K is properly contained in T and so we have $m \geqslant 1$. On the other hand, $H_{G/K}$ is the pull-back of $\tau_{G/T}$ under π.

It follows that $\tau_{G/K}$ is the pull-back of $\tau_{G/T} \oplus \epsilon^m$. In view of relation (5.6), with K replaced by T, the bundle $\tau_{G/T} \oplus \epsilon^1$ is trivial. Hence so is $\tau_{G/K}$.

Thus if K is a nonmaximal torus, then the homogeneous space G/K has trivial tangent bundle.

§4. The Grassmannians

5.13. The Grassmann manifolds. Let Γ be one of the fields $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ and consider the vector space $\Gamma^n = \Gamma \oplus \cdots \oplus \Gamma$. Introduce a positive definite inner product $\langle\ ,\ \rangle$ in Γ^n which is Euclidean, Hermitian, or quaternionic according as $\Gamma = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. In the case $\Gamma = \mathbb{R}$ also choose an orientation in Γ^n.

A *k-plane* in Γ^n is a Γ-subspace of Γ-dimension k. The set of all k-planes in Γ^n is denoted by $G_\Gamma(n; k)$. An *oriented k-plane* in $\mathbb{R}^n$ is a k-plane F together with an orientation of F. The set of oriented k-planes in $\mathbb{R}^n$ will be denoted by $\tilde{G}(n; k)$ if $k < n$. Finally, we define $\tilde{G}(n; n)$ to be the set consisting of a single element, namely the oriented vector space $\mathbb{R}^n$.

This article deals with each of the four cases listed below. In each case, Γ, $I(n)$, $G(n; k)$ is to be interpreted as described below.

Case	Γ	$I(n)$	$G(n; k)$
I	$\mathbb{R}$	$O(n)$	$G_{\mathbb{R}}(n; k)$
II	$\mathbb{R}$	$SO(n)$	$\tilde{G}(n; k)$
III	$\mathbb{C}$	$U(n)$	$G_{\mathbb{C}}(n; k)$
IV	$\mathbb{H}$	$Q(n)$	$G_{\mathbb{H}}(n; k)$

Observe that in each case the Lie algebra of $I(n)$ consists of the Γ-linear transformations of Γ^n that are skew with respect to the inner product $\langle\ ,\ \rangle$. The Lie algebra of $I(n)$ is denoted by $E(n)$.

The set $G(n; k)$ is made into a manifold in the following way: First define a transitive left action of the Lie group $I(n)$ on $G(n; k)$ by setting

$$(\varphi, F) \mapsto \varphi(F), \qquad \varphi \in I(n), \qquad F \in G(n; k).$$

This yields a surjection, $\alpha: I(n) \to G(n; k)$, given by

$$\alpha(\varphi) = \varphi(\Gamma^k), \qquad \varphi \in I(n)$$

(where Γ^k is regarded as the subspace of Γ^n consisting of those vectors whose last $n - k$ components are zero).

Denote $(\Gamma^k)^\perp$ by Γ^{n-k}: $\Gamma^n = \Gamma^k \oplus \Gamma^{n-k}$. This decomposition determines an inclusion, $I(k) \times I(n-k) \to I(n)$, and clearly

$$\alpha^{-1}(\Gamma^k) = I(k) \times I(n-k).$$

Hence α induces a commutative diagram,

$$\begin{array}{ccc} & I(n) & \\ {}^{\pi}\swarrow & & \searrow^{\alpha} \\ I(n)/(I(k) \times I(n-k)) & \xrightarrow[\beta]{\cong} & G(n;k), \end{array}$$

and β is an equivariant bijection. Give $G(n; k)$ the unique manifold structure such that β is a diffeomorphism. The manifold so obtained is called the *Grassmannian of k-planes in* Γ^n. Since β is equivariant the action of $I(n)$ on $G(n; k)$ defined above is smooth.

Observe that the canonical isomorphism

$$I(k) \times I(n-k) \xrightarrow{\cong} I(n-k) \times I(k)$$

induces, via β, a diffeomorphism

$$\Omega\colon G(n;k) \xrightarrow{\cong} G(n; n-k).$$

If $F \in G(n; k)$, then $\Omega(F)$ is the orthogonal complement of F in Γ^n.

5.14. Examples: 1. *The Grassmannian of k-planes in* $\mathbb{R}^n$: Assume that $0 < k < n$. Then an involution, ω, of $\tilde{G}(n; k)$ is defined as follows: If F is an oriented k-plane, then $\omega(F)$ is the same k-plane with the opposite orientation. On the other hand, a projection,

$$p\colon \tilde{G}(n;k) \to G_{\mathbb{R}}(n;k),$$

is defined by forgetting the orientations of the elements of $\tilde{G}(n; k)$. Evidently, p is a double covering and ω is the involution that interchanges the two points in each $p^{-1}(F)$.

To see that p and ω are smooth note that $SO(n)$ acts transitively on $G_{\mathbb{R}}(n; k)$, and that the isotropy subgroup at $\mathbb{R}^k$ is the group

$$K = SO(n) \cap (O(k) \times O(n-k)).$$

This group consists of two components,

$$K_0 = \{(\varphi, \psi) \mid \det \varphi = 1, \det \psi = 1\} = SO(k) \times SO(n-k),$$

and

$$K_1 = \{(\varphi, \psi) \mid \det \varphi = -1, \quad \det \psi = -1\}.$$

The commutative diagram,

$$\begin{array}{ccc} SO(n)/(SO(k) \times SO(n-k)) & \xrightarrow{\cong} & \tilde{G}(n; k) \\ \pi \big\downarrow & & \big\downarrow p \\ SO(n)/K & \xrightarrow{\cong} & G_{\mathbb{R}}(n; k), \end{array}$$

shows that p is smooth, a local diffeomorphism and a double covering. Hence ω is also smooth.

The dimension of $G_{\mathbb{R}}(n; k)$ is given by

$$\dim G_{\mathbb{R}}(n; k) = \binom{n}{2} - \binom{k}{2} - \binom{n-k}{2} = k(n-k).$$

2. *Real projective space*: Assume that $n \geqslant 2$ and consider the manifold $\tilde{G}(n; 1)$. Its points are the oriented lines in $\mathbb{R}^n$ through the origin. Identifying each such line with its positive unit vector, we obtain an $SO(n)$-equivariant bijection between $\tilde{G}(n; 1)$ and S^{n-1}. Since $SO(n)$ acts smoothly on S^{n-1}, the commutative diagram (cf. Example 2, sec. 3.6)

$$\begin{array}{ccccc} & & SO(n)/SO(n-1) & & \\ & \cong\swarrow & & \searrow\cong & \\ \tilde{G}(n; 1) & & \xrightarrow{\quad\cong\quad} & & S^{n-1} \end{array}$$

shows that this identification is a diffeomorphism.

Moreover, the involution, ω, in $\tilde{G}(n; 1)$ defined in Example 1 corresponds under this diffeomorphism to the antipodal involution of S^{n-1}. Thus we obtain a diffeomorphism

$$G_{\mathbb{R}}(n; 1) \xrightarrow{\cong} \mathbb{R}P^{n-1}$$

(cf. Example 2, sec. 1.4, volume I). Hence $G_{\mathbb{R}}(n; 1)$ is diffeomorphic to the real projective space of dimension $n - 1$.

3. *Complex and quaternionic projective space*: Let $n \geqslant 2$. The manifolds $G_{\mathbb{C}}(n; 1)$ (respectively, $G_{\mathbb{H}}(n; 1)$) of complex (respectively, quaternionic) lines in $\mathbb{C}^n$ (respectively, $\mathbb{H}^n$) through the origin are called *complex* (*quaternionic*) *projective space* and are denoted by $\mathbb{C}P^{n-1}$ and $\mathbb{H}P^{n-1}$ respectively.

4. *Complex and quaternionic projective lines*: We shall construct diffeomorphisms

$$\mathbb{C}P^1 \xrightarrow{\cong} S^2 \qquad \text{and} \qquad \mathbb{H}P^1 \xrightarrow{\cong} S^4.$$

Define a map $\mathbb{C} \to \mathbb{C}P^1$ by sending $z \in \mathbb{C}$ to the one-dimensional complex subspace of $\mathbb{C}^2$ generated by the pair $(1, z)$. This is a smooth embedding. Since $\dim \mathbb{C} = 2 = \dim \mathbb{C}P^1$, it is a diffeomorphism onto an open subset of $\mathbb{C}P^1$. The only point which is not in the image is the one-dimensional subspace of $\mathbb{C}^2$ generated by $(0, 1)$. Since $\mathbb{C}P^1$ is compact, it is the one-point compactification of $\mathbb{C}$; i.e., $\mathbb{C}P^1$ is diffeomorphic to S^2.

Similarly, $\mathbb{H}P^1$ is the one-point compactification of $\mathbb{H}$ and hence it is diffeomorphic to S^4.

5.15. Canonical vector bundles over $G(n; k)$. Recall that in secs. 2.1 and 2.22, volume I, we defined real and complex vector bundles. Quaternionic vector bundles are defined in a similar way, and the definition of all three may be given simultaneously as follows: A *Γ-vector bundle* is a smooth bundle $\xi = (M, \pi, B, F)$, in which F and F_x $(x \in B)$ are Γ-vector spaces, and which admits a coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ such that each map,

$$\psi_{\alpha,x}: F \xrightarrow{\cong} F_x,$$

is a Γ-linear isomorphism.

We shall construct canonical Γ-vector bundles over $G(n; k)$. It will be important to distinguish between a k-plane, F, as a subspace of Γ^n, and as a point in $G(n; k)$.

Consider the disjoint union

$$M = \bigcup_{F \in G(n;k)} F.$$

Thus a point of M is a pair (F, v) with $v \in F$. Let $\rho: M \to G(n; k)$ be the projection given by

$$\rho(F, v) = F.$$

Observe that a left action of $I(n)$ on the *set* M is given by

$$\varphi(F, v) = (\varphi(F), \varphi(v)), \qquad \varphi \in I(n), \quad (F, v) \in M.$$

We shall make $\xi_k = (M, \rho, G(n; k), \Gamma^k)$ into a Γ-vector bundle so that this action becomes a smooth action.

Consider the representation of $I(k) \times I(n-k)$ in Γ^k given by

$$(\varphi, \psi)(u) = \varphi(u), \qquad \varphi \in I(k), \quad \psi \in I(n-k), \quad u \in \Gamma^k.$$

It determines a Γ-vector bundle (cf. sec. 5.10),

$$\hat{\xi}_k = (I(n) \times_{I(k)\times I(n-k)} \Gamma^k, \hat{\rho}, I(n)/(I(k) \times I(n-k)), \Gamma^k),$$

which admits a canonical left action of $I(n)$. Now define a surjective set map,

$$\Phi\colon I(n) \times \Gamma^k \to M,$$

by setting

$$\Phi(\varphi, v) = (\varphi(\Gamma^k), \varphi(v)), \qquad \varphi \in I(n), \quad v \in \Gamma^k.$$

Factoring through the joint action, we obtain the commutative diagram,

$$\begin{array}{ccc} I(n) \times_{I(k)\times I(n-k)} \Gamma^k & \xrightarrow{\Psi} & M \\ {\scriptstyle\hat{\rho}}\downarrow & & \downarrow{\scriptstyle\rho} \\ I(n)/(I(k) \times I(n-k)) & \xrightarrow[\beta]{\cong} & G(n; k), \end{array}$$

where β is the equivariant diffeomorphism of sec. 5.13 and Ψ is an $I(n)$-equivariant bijection restricting to linear isomorphisms on the fibres.

Give M the manifold structure for which Ψ is a diffeomorphism. Then ξ_k becomes a vector bundle acted on by $I(n)$ and

$$\Psi\colon \hat{\xi}_k \xrightarrow{\cong} \xi_k$$

is an equivariant isomorphism.

Similarly, we obtain a vector bundle $\xi_k^\perp = (M^\perp, \rho, G(n; n-k), \Gamma^{n-k})$ by setting

$$M^\perp = \bigcup_{F\in G(n;k)} F^\perp.$$

It admits an action of $I(n)$ and is equivariantly isomorphic to the bundle

$$(I(n) \times_{I(k)\times I(n-k)} \Gamma^{n-k}, \hat{\rho}, I(n)/(I(k) \times I(n-k)), \Gamma^{n-k}).$$

(Replace Γ^k by $\Gamma^{n-k} = (\Gamma^k)^\perp$ in the discussion above.) ξ_k and $\xi_k^\perp$ are called the *canonical k-plane and $(n-k)$-plane bundles over* $G(n; k)$. The direct decomposition,

$$\Gamma^k \oplus \Gamma^{n-k} \xrightarrow{\cong} \Gamma^n,$$

determines a strong bundle isomorphism

$$\xi_k \oplus \xi_k^{\perp} \xrightarrow{\cong} G(n; k) \times \Gamma^n.$$

Finally, the actions of $I(n)$ on ξ_k and $\xi_k^{\perp}$ defined above, together with the standard actions of $I(n)$ on $G(n; k)$ and Γ^n define actions on the bundles $\xi_k \oplus \xi_k^{\perp}$ and $G(n; k) \times \Gamma^n$. Moreover, the isomorphism defined above is equivariant.

5.16. The tangent bundle of $G(n; k)$. Given two Γ-vector bundles ξ and η over the same base B, we can form the (real) vector bundle $L_\Gamma(\xi; \eta)$ whose fibre at $x \in B$ consists of the Γ-linear maps between the fibres of ξ and η at x.

Proposition IV: The tangent bundle of $G(n; k)$ satisfies

$$\tau_{G(n;k)} \cong L_\Gamma(\xi_k; \xi_k^{\perp}).$$

Proof: Identify $G(n; k)$ with $I(n)/(I(k) \times I(n-k))$. According to sec. 5.11 its tangent bundle is obtained from the representation $\mathrm{Ad}^{\perp}$ of $I(k) \times I(n-k)$ in $E(n)/(E(k) \oplus E(n-k))$.

On the other hand, $L_\Gamma(\xi_k; \xi_k^{\perp})$ is obtained from the representation of $I(k) \times I(n-k)$ in $L_\Gamma(\Gamma^k; \Gamma^{n-k})$ given by

$$(\sigma, \tau)(\varphi) = \tau \circ \varphi \circ \sigma^{-1}, \qquad \sigma \in I(k), \quad \tau \in I(n-k), \quad \varphi \in L_\Gamma(\Gamma^k; \Gamma^{n-k}).$$

Thus we must construct an $(I(k) \times I(n-k))$-linear isomorphism

$$L_\Gamma(\Gamma^k; \Gamma^{n-k}) \cong E(n)/(E(k) \oplus E(n-k)).$$

Recall that $E(n)$ is the real vector space of Γ-linear skew transformations of Γ^n, and that $\Gamma^{n-k} = (\Gamma^k)^{\perp}$. The Lie algebras $E(k)$ of $I(k)$ and $E(n-k)$ of $I(n-k)$ (considered as subalgebras of $E(n)$) are given by

$$E(k) = \{\alpha \in E(n) \mid \alpha(\Gamma^{n-k}) = 0\} \qquad \text{and} \qquad E(n-k) = \{\alpha \in E(n) \mid \alpha(\Gamma^k) = 0\}.$$

Define a subspace $L \subset E(n)$ by setting

$$L = \{\alpha \in E(n) \mid \alpha(\Gamma^k) \subset \Gamma^{n-k} \text{ and } \alpha(\Gamma^{n-k}) \subset \Gamma^k\}.$$

Then

$$E(n) = E(k) \oplus E(n-k) \oplus L.$$

Moreover, since the adjoint representation of $I(n)$ is given by

$$(\mathrm{Ad}\,\sigma)\alpha = \sigma \circ \alpha \circ \sigma^{-1}, \qquad \sigma \in I(n), \quad \alpha \in E(n),$$

it follows that L is stable under $I(k) \times I(n-k)$. In particular, there is an isomorphism of $(I(k) \times I(n-k))$-spaces

$$L \cong E(n)/(E(k) \oplus E(n-k)).$$

Finally, define a linear isomorphism,

$$\Phi: L_\Gamma(\Gamma^k; \Gamma^{n-k}) \xrightarrow{\cong} L,$$

by setting

$$\Phi(\alpha)(x \oplus y) = \alpha(x) - \tilde{\alpha}(y), \qquad \alpha \in L_\Gamma(\Gamma^k; \Gamma^{n-k}), \quad x \in \Gamma^k, \quad y \in \Gamma^{n-k},$$

where $\tilde{\alpha}$ denotes the adjoint of α. Since $I(k)$ and $I(n-k)$ consist of isometries, it follows easily that Φ is $(I(k) \times I(n-k))$-equivariant. Hence

$$L_\Gamma(\Gamma^k; \Gamma^{n-k}) \cong L \cong E(n)/(E(k) \oplus E(n-k)),$$

which completes the proof.

Q.E.D.

Corollary: There are isomorphisms

$$\tau_{G_{\mathbb{R}}(n;k)} \cong \xi_k \otimes_{\mathbb{R}} \xi_k^\perp, \qquad \tau_{\tilde{G}(n;k)} \cong \xi_k \otimes_{\mathbb{R}} \xi_k^\perp, \qquad \text{and} \qquad \tau_{G_{\mathbb{C}}(n;k)} \cong \xi_k^* \otimes_{\mathbb{C}} \xi_k^\perp$$

(where ξ_k is interpreted as a vector bundle over the appropriate manifold, and ξ_k^* is the complex dual of ξ_k).

§5. The Stiefel manifolds

We continue the notational conventions of article 4.

5.17. Stiefel manifolds. An *orthonormal k-frame* in Γ^n is a sequence of k vectors, $(u_1, ..., u_k)$, such that

$$\langle u_i, u_j \rangle = \delta_{ij} .$$

An n-frame in the oriented space, $\mathbb{R}^n$, is called *positive*, if it represents the orientation of $\mathbb{R}^n$.

We extend the conventions of this article by letting $V(n; k)$ denote any one of the sets $\hat{V}_{\mathbb{R}}(n; k)$, $V_{\mathbb{R}}(n; k)$, $V_{\mathbb{C}}(n; k)$, and $V_{\mathbb{H}}(n; k)$ defined by:

Case I	$\hat{V}_{\mathbb{R}}(n; k)$	Orthonormal k-frames in $\mathbb{R}^n$.
Case II	$V_{\mathbb{R}}(n; k)$	Orthonormal k-frames in $\mathbb{R}^n$ if $k < n$; positive orthonormal n-frames in $\mathbb{R}^n$ if $k = n$.
Case III	$V_{\mathbb{C}}(n; k)$	Orthonormal k-frames in $\mathbb{C}^n$.
Case IV	$V_{\mathbb{H}}(n; k)$	Orthonormal k-frames in $\mathbb{H}^n$.

A transitive left action of $I(n)$ on $V(n; k)$ is given by

$$\varphi \cdot (u_1, ..., u_k) = (\varphi(u_1), ..., \varphi(u_k)),$$
$$\varphi \in I(n), \quad (u_1, ..., u_k) \in V(n; k).$$

In particular, write $\Gamma^n = \Gamma^k \oplus \Gamma^{n-k}$ and let $(e_1, ..., e_k)$ be a fixed orthonormal basis of Γ^k. Then the subgroup of $I(n)$ which fixes the k-frame $(e_1, ..., e_k)$ is exactly $I(n - k)$ (cf. sec. 5.13). Thus the action of $I(n)$ on $V(n; k)$ determines an equivariant bijection

$$I(n)/I(n - k) \xrightarrow{\cong} V(n; k).$$

Assign $V(n; k)$ the unique manifold structure such that this bijection is a diffeomorphism. (Then the action above is smooth.) The manifold $V(n; k)$ is called the *Stiefel manifold of orthonormal k-frames in n-space.*

5.18. The universal frame bundle over $G(n; k)$. A canonical principal bundle,

$$\mathscr{P}(n; k) = (V(n; k), \pi_k, G(n; k), I(k)),$$

is defined as follows:

If $(u_1, \ldots, u_k) \in V(n; k)$, let $\pi_k(u_1, \ldots, u_k)$ be the (oriented) k-plane with $u_1, \ldots, u_k$ as (positive) basis. Then $\pi_k: V(n; k) \to G(n; k)$ is a well defined map. Moreover, we have the smooth commutative diagram,

$$\begin{array}{ccc} I(n)/I(n-k) & \xrightarrow{\cong} & V(n; k) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_k} \\ I(n)/(I(k) \times I(n-k)) & \xrightarrow{\cong} & G(n; k), \end{array}$$

where the horizontal diffeomorphisms are defined in sec. 5.17 and sec. 5.13 respectively, and $p(\sigma \cdot I(n-k)) = \sigma \cdot (I(k) \times I(n-k))$, $\sigma \in I(n)$.

We can apply sec. 5.8 to obtain a smooth principal bundle

$$(I(n)/I(n-k), p, I(n)/(I(k) \times I(n-k)), I(k)).$$

Thus the diagram above shows that π_k is the projection of a smooth principal bundle, $\mathscr{P}(n; k) = (V(n; k), \pi_k, G(n; k), I(k))$. Note that, if $F \in G(n; k)$ then $\pi_k^{-1}(F)$ consists of the (positive) orthonormal k-frames in F. For this reason $\mathscr{P}(n; k)$ is called the *universal frame bundle over* $G(n; k)$.

The inclusion maps,

$$\Gamma^n \to \Gamma^{n+1} \to \Gamma^{n+2} \to \cdots,$$

determine smooth commutative diagrams,

$$\begin{array}{ccccc} V(n; k) & \longrightarrow & V(n+1; k) & \longrightarrow & \cdots \\ {\scriptstyle \pi_k}\downarrow & & \downarrow{\scriptstyle \pi_k} & & \\ G(n; k) & \longrightarrow & G(n+1; k) & \longrightarrow & \cdots, \end{array}$$

which are, in fact, homomorphisms of principal $I(k)$-bundles.

The vector bundle, η_k, associated with $\mathscr{P}(n; k)$ via the action of $I(k)$ in Γ^k, is canonically isomorphic to the bundle $\xi_k = (M, \rho, G(n; k), \Gamma^k)$ of sec. 5.15. Indeed, fix a (positive) orthonormal basis $(e_1, \ldots, e_k)$ of Γ^k.

Define a map,

$$q\colon V(n; k) \times \Gamma^k \to M,$$

as follows:

$$q((u_1, ..., u_k), \sum_i \lambda^i e_i) = \sum_i \lambda^i u_i, \qquad \lambda^i \in \Gamma, \quad (u_1, ..., u_k) \in V(n; k).$$

It is easy to check that q induces an $I(n)$-equivariant, strong isomorphism

$$V(n; k) \times_{I(k)} \Gamma^k \xrightarrow{\cong} M.$$

5.19. The manifolds $I(n; k)$. Let $I(n; k)$ denote the set of isometric inclusions $\Gamma^k \to \Gamma^n$ (except in case II when $k = n$; then $I(n; n)$ will denote the set of orientation preserving isometries of $\mathbb{R}^n$). Note that $I(n)$ and $I(k)$ act, respectively, from the left and right on $I(n; k)$ via

$$\varphi \cdot \psi = \varphi \circ \psi$$

and

$$\psi \cdot \sigma = \psi \circ \sigma, \qquad \varphi \in I(n), \quad \psi \in I(n; k), \quad \sigma \in I(k).$$

Now fix a (positive) orthonormal basis, $e_1, ..., e_k$, of Γ^k. Then an $I(n)$-equivariant bijection,

$$I(n; k) \to V(n; k),$$

is given by $\varphi \mapsto (\varphi e_1, ..., \varphi e_k)$. We use this bijection to make $I(n; k)$ into a smooth manifold, and to identify it with $V(n; k)$.

In particular, we may write

$$\mathscr{P}(n; k) = (I(n; k), \pi_k, G(n; k), I(k)).$$

Then $\pi_k(\varphi) = \varphi(\Gamma^k)$, $\varphi \in I(n; k)$. Moreover the principal action of $I(k)$ is the right action given above.

Finally, observe that the isomorphism $I(n; k) \times_{I(k)} \Gamma^k \xrightarrow{\cong} \xi_k$ of sec. 5.18 is induced by the map, $q : I(n; k) \times \Gamma^k \to M$, given by

$$q(\varphi, v) = \varphi(v), \qquad \varphi \in I(n; k), \quad v \in \Gamma^k.$$

Proposition V: Let $\mathscr{P} = (P, \pi, B, I(k))$ be a principal bundle. Then, for some $n \geq k$, there is a homomorphism of principal bundles

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & I(n; k) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_k} \\ B & \xrightarrow[\psi]{} & G(n; k). \end{array}$$

Definition: ψ is called a *classifying map* for the principal bundle $\mathscr{P}$.

According to the proposition, $\mathscr{P}$ is the pull-back of $\mathscr{P}(n; k)$ to B via ψ.
Before proving the proposition we establish

Lemma II: Let $\xi = (P \times_{I(k)} \Gamma^k, \rho_\xi, B, \Gamma^k)$ be the vector bundle associated with $\mathscr{P}$ via the action of $I(k)$ on Γ^k. Then, for some $n \geq k$, there is a strong bundle map $\sigma: \xi \to B \times \Gamma^n$ restricting to Γ-linear injections on the fibres.

Proof: This lemma is proved in sec. 2.23, volume I, in case I and case II. The same argument holds in cases III and IV, using Hermitian and quaternionic inner products.

Q.E.D.

Proof of the proposition: Let σ be the bundle map constructed in Lemma II and let

$$q: P \times \Gamma^k \to P \times_{I(k)} \Gamma^k$$

be the principal map(cf. sec. 5.3). Then a smooth map $\varphi: P \to I(n; k)$ is defined by the relation

$$(\pi(z), \varphi(z)u) = \sigma(q(z, u)), \qquad z \in P, \quad u \in \Gamma^k.$$

Clearly,

$$\varphi(z \cdot \tau)u = \varphi(z)(\tau(u)) = (\varphi(z) \circ \tau)u, \qquad z \in P, \quad \tau \in I(k), \quad u \in \Gamma^k.$$

Hence $\varphi(z \cdot \tau) = \varphi(z) \circ \tau$ and so φ is equivariant; i.e., φ is a homomorphism of principal bundles.

Q.E.D.

5.20. Examples: 1. *Hopf fiberings*: A point of $V(n; 1)$ is just a unit vector in Γ^n. Thus, if $n > 1$,

$$V_{\mathbb{R}}(n; 1) = S^{n-1}, \qquad V_{\mathbb{C}}(n; 1) = S^{2n-1}, \qquad V_{\mathbb{H}}(n; 1) = S^{4n-1}.$$

Moreover, the left action (cf. sec. 5.17) of $O(n)$, $U(n)$, and $Q(n)$ on these spheres is the standard one.

Next observe that in cases I, III, and IV, $I(1)$ can be identified with the unit sphere of Γ ($\Gamma = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$) as follows: For each unit vector $a \in \Gamma$, define $\mu_a \in I(1)$ by

$$\mu_a(z) = za^{-1}, \qquad z \in \Gamma.$$

Then $a \mapsto \mu_a$ is an isomorphism of Lie groups (cf. Example 2, sec. 2.6 and Example 3, sec. 2.7). Thus the universal 1-frame bundles become

$$(S^{n-1}, \pi, \mathbb{R}P^{n-1}, S^0), \qquad (S^{2n-1}, \pi, \mathbb{C}P^{n-1}, S^1), \qquad \text{and} \qquad (S^{4n-1}, \pi, \mathbb{H}P^{n-1}, S^3).$$

Notice that the first bundle is simply the double covering of Example 2, sec. 5.14. Moreover $\mathbb{C}P^1 = S^2$ and $\mathbb{H}P^1 = S^4$ (cf. Example 4, sec 5.14). Thus the bundles $\mathscr{P}_{\mathbb{C}}(2; 1)$ and $\mathscr{P}_{\mathbb{H}}(2; 1)$ can be written

$$(S^3, \pi, S^2, S^1) \qquad \text{and} \qquad (S^7, \pi, S^4, S^3).$$

Consider the right action of S^0 (respectively, S^1, S^3) on $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$, $\mathbb{H}^n$) given by

$$(z_1, ..., z_n) \cdot z = (z^{-1}z_1, ..., z^{-1}z_n), \qquad z_i \in \Gamma.$$

This action restricts to an action of S^0 (respectively, S^1, S^3) on S^{n-1} (respectively, S^{2n-1}, S^{4n-1}). We shall show that these actions are the principal actions of S^0, S^1, and S^3 on the 1-frame bundles.

In fact, let $\sigma \in I(n; 1)$ and write $\sigma(1) = (z_1, ..., z_n)$. Then $\sigma(1) \in V_\Gamma(n; 1)$ and the principal action of $I(1)$ is given by (cf. sec. 5.19)

$$\sigma(1) \cdot z = (\sigma \circ \mu_z)(1) = \sigma(z^{-1}) = (z^{-1}z_1, ..., z^{-1}z_n).$$

2. *The Stiefel manifold* $V_{\mathbb{R}}(n; 2)$: Let $\Gamma = \mathbb{R}$ and consider the Stiefel manifold $V_{\mathbb{R}}(n; 2)$. Its points are the isometries $\alpha: \mathbb{R}^2 \to \mathbb{R}^n$. An embedding $\varphi: V_{\mathbb{R}}(n; 2) \to \mathbb{R}^n \oplus \mathbb{R}^n$ is defined as follows: Choose an orthonormal basis e_1, e_2 in $\mathbb{R}^2$ and set $\varphi(\alpha) = (\alpha(e_1), \alpha(e_2))$. The image of φ consists precisely of the pairs (x, y) satisfying

$$|x| = 1, \qquad |y| = 1, \qquad \langle x, y\rangle = 0.$$

On the other hand, consider the bundle $(M, \pi, S^{n-1}, S^{n-2})$, of unit tangent vectors of S^{n-1}. Then the map,

$$\psi: z \mapsto (\pi(z), z), \qquad z \in M,$$

defines an embedding of M into $\mathbb{R}^n \oplus \mathbb{R}^n$ and the images of ψ and φ coincide. Composing φ with the inverse of ψ yields a diffeomorphism of $V_{\mathbb{R}}(n; 2)$ onto M.

§6. The cohomology of the Stiefel manifolds and the classical groups

The notation conventions of articles 4 and 5 are continued in this article. We shall frequently make the identifications

$$V_{\mathbb{C}}(n; k) = U(n)/U(n-k) \qquad \text{and} \qquad V_{\mathbb{R}}(n; k) = SO(n)/SO(n-k).$$

The tensor product of graded algebras is always the anticommutative tensor product.

5.21. Complex and quaternionic Stiefel manifolds. Theorem I: The cohomology algebras of the manifolds $V_{\mathbb{C}}(n; k)$ and $V_{\mathbb{H}}(n; k)$ are exterior algebras over oddly graded subspaces (i.e., subspaces whose homogeneous elements all have odd degree). The Poincaré polynomials are given by

$$f_{V_{\mathbb{C}}(n;\, k)} = \prod_{i=1}^{k} (1 + t^{2(n-k+i)-1})$$

and

$$f_{V_{\mathbb{H}}(n;\, k)} = \prod_{i=1}^{k} (1 + t^{4(n-k+i)-1})$$

Corollary: The Poincaré polynomials for $U(n)$ and $Q(n)$ are respectively given by (since $V_{\mathbb{C}}(n; n) = U(n)$ and $V_{\mathbb{H}}(n; n) = Q(n)$)

$$f_{U(n)} = \prod_{i=1}^{n} (1 + t^{2i-1}) \qquad \text{and} \qquad f_{Q(n)} = \prod_{i=1}^{n} (1 + t^{4i-1}).$$

Proof: We consider the complex case; the argument in the quaternionic case is identical. The proof is by induction on k (for fixed n).

Since $V_{\mathbb{C}}(n, 1) = S^{2n-1}$, the theorem is clear for $k = 1$.

Suppose it holds for some k. From sec. 5.8, we obtain a bundle

$$\xi = (U(n)/U(n-k-1), \rho, U(n)/U(n-k), U(n-k)/U(n-k-1)).$$

Since $U(n-k)/U(n-k-1) = S^{2(n-k)-1}$, ξ is a sphere bundle.

Moreover, since $U(n-k)$ acts on the sphere by orientation preserving diffeomorphisms, the bundle is orientable. Thus its Euler class,

$$\chi_\xi \in H^{2(n-k)}(U(n)/U(n-k)),$$

is defined.

On the other hand, by our induction hypothesis, the theorem holds for k, and so the formula in the theorem shows that

$$H^{2(n-k)}(U(n)/U(n-k)) = 0.$$

Thus $\chi_\xi = 0$. Now it follows from Corollary II to Proposition IV of sec. 8.4, volume I, that

$$H(U(n)/U(n-k-1)) \cong H(U(n)/U(n-k)) \otimes H(S^{2(n-k)-1})$$

(as graded algebras). This closes the induction.

Q.E.D.

5.22. The Stiefel manifolds $V_{\mathbb{R}}(n; 2)$. Proposition VI: The cohomology algebra of $V_{\mathbb{R}}(n; 2)$ (for $n \geqslant 3$) is given by

$$H(V_{\mathbb{R}}(2m; 2)) \cong H(S^{2m-1}) \otimes H(S^{2m-2}) \quad \text{and} \quad H(V_{\mathbb{R}}(2m+1; 2)) \cong H(S^{4m-1}).$$

Proof: Recall from Example 2, sec. 5.20, that the sphere bundle associated with the tangent bundle of S^{n-1} is given by

$$\xi = (V_{\mathbb{R}}(n; 2), \pi, S^{n-1}, S^{n-2}).$$

Moreover (cf. Example 1, sec. 9.10, volume I)

$$\chi_\xi = \begin{cases} 0, & n-1 \text{ odd} \\ 2\omega_{n-1}, & n-1 \text{ even,} \end{cases}$$

where ω_{n-1} denotes the orientation class of S^{n-1}.

Case A: $n = 2m$, $m > 1$. Then since $\chi_\xi = 0$ there is a class $\omega \in H^{2m-2}(V_{\mathbb{R}}(2m; 2))$ such that $\oint_S^\# \omega = 1$ (cf. sec. 8.4, volume I). Moreover, the map,

$$\alpha \otimes 1 + \beta \otimes \omega_{2m-2} \to \pi^\#\alpha + \pi^\#\beta \cdot \omega, \qquad \alpha, \beta \in H(S^{2m-1}),$$

defines a linear isomorphism

$$H(S^{2m-1}) \otimes H(S^{2m-2}) \xrightarrow{\cong} H(V_{\mathbb{R}}(2m; 2)).$$

In particular, $H^{4m-4}(V_{\mathbb{R}}(2m; 2)) = 0$, and so $\omega^2 = 0$. It follows that this isomorphism is an isomorphism of graded algebras.

Case B: $n = 2m + 1$, $m \geqslant 1$. The Gysin sequence for ξ reads

$$\begin{array}{ccccc} \downarrow & & & & \\ H^i(S^{2m}) & \xrightarrow{\pi^*} & H^i(V_{\mathbb{R}}(2m+1;2)) & \xrightarrow{\oint_S^*} & H^{i-2m+1}(S^{2m}) \\ & & & & \big\downarrow D \\ & & & & H^{i+1}(S^{2m}) \rightarrow , \end{array}$$

(cf. sec. 8.2, volume I). This shows that, for $i \neq 0, 2m-1, 2m, 4m-1$,

$$H^i(V_{\mathbb{R}}(2m+1;2)) = 0.$$

Since $D(1) = \chi_\xi = 2\omega_{2m}$, D restricts to an isomorphism

$$H^0(S^{2m}) \xrightarrow{\cong} H^{2m}(S^{2m}).$$

Now the exactness of the Gysin sequence yields

$$H^{2m}(V_{\mathbb{R}}(2m+1;2)) = 0 = H^{2m-1}(V_{\mathbb{R}}(2m+1;2)).$$

Q.E.D.

5.23. Bundles with fibre $V_{\mathbb{R}}(2m+1;2)$. Let $\eta = (E, \pi, B, F)$ be an oriented bundle with $F = V_{\mathbb{R}}(2m+1;2)$. In view of sec. 5.22,

$$H(F) \cong H(S^{4m-1}).$$

Now the proofs of the results for sphere bundles established in article 1, Chap. VIII, volume I, depend only on the *cohomology* and compactness of the fibre; in particular, the identical results hold for η.

This implies that there is a class $\chi_\eta \in H^{4m}(B)$, depending only on η, and determined by the following condition: Let $\Phi \in A^{4m}(B)$ represent χ_η. Then, for some $\Omega \in A^{4m-1}(E)$,

$$\pi^*\Phi = \delta\Omega, \qquad \text{and} \qquad \oint_F \Omega = -1.$$

Moreover there is a long exact sequence,

$$\cdots \longrightarrow H^i(B) \xrightarrow{\pi^*} H^i(E) \xrightarrow{\oint_F^*} H^{i-4m+1}(B) \xrightarrow{D} H^{i+1}(B) \longrightarrow \cdots,$$

where $D\alpha = \alpha \cdot \chi_\eta$. If $\chi_\eta = 0$, then there is an isomorphism of graded algebras,

$$H(E) \cong H(B) \otimes H(S^{4m-1}).$$

5.24. The real Stiefel manifolds $V_{\mathbb{R}}(n; k)$. Theorem II: The cohomology algebra of $V_{\mathbb{R}}(n; k)$ $(k < n)$ is an exterior algebra over a graded vector space. The Poincaré polynomials are the polynomials given below.

$n = 2m, \quad k = 2l + 1, \quad l \geqslant 0$	$(1 + t^{2m-1}) \prod_{i=1}^{l} (1 + t^{4m-4i-1})$
$n = 2m + 1, \quad k = 2l, \quad l \geqslant 1$	$\prod_{i=1}^{l} (1 + t^{4m-4i+3})$
$n = 2m, \quad k = 2l, \quad m > l \geqslant 1$	$(1 + t^{2m-2l})(1 + t^{2m-1}) \prod_{i=1}^{l-1} (1 + t^{4m-4i-1})$
$n = 2m + 1, k = 2l + 1, m > l \geqslant 0$	$(1 + t^{2m-2l}) \prod_{i=1}^{l-1} (1 + t^{4m-4i+3})$

Theorem III: The Poincaré polynomials for the groups $SO(n)$ are given by

$$f_{SO(2m)} = (1 + t^{2m-1}) \prod_{i=1}^{m-1} (1 + t^{4i-1})$$

and

$$f_{SO(2m+1)} = \prod_{i=1}^{m} (1 + t^{4i-1}).$$

Proof of Theorem II: Since $V_{\mathbb{R}}(n, 1) = S^{n-1}$, the theorem is correct for $k = 1$. If $2 = k < n$ the theorem is contained in Proposition VI, sec. 5.22. Now we use induction on k. Assume the theorem holds for $V_{\mathbb{R}}(n; i)$, $i < k$, and consider two cases separately.

Case A: $n - k$ is odd. Write $n - k = 2q - 1$. Consider the bundle

$$(SO(n)/SO(n-k), \rho, SO(n)/SO(n-k+2), V_{\mathbb{R}}(2q+1; 2)).$$

By induction the theorem holds for $SO(n)/SO(n - k + 2)$. It follows that

$$H^{4q}(SO(n)/SO(n-k+2)) = H^{2n-2k+2}(SO(n)/SO(n-k+2)) = 0.$$

Hence it follows from sec. 5.23 that

$$H(SO(n)/SO(n-k)) \cong H(SO(n)/SO(n-k+2)) \otimes H(S^{2n-2k+1})$$

and the induction is closed.

Case B: $n - k = 2q$ and $q > 0$. Since (always) $k \geqslant 1$, we have $2q < n$. Now consider the sphere bundle

$$(SO(n)/SO(n-k), \rho, SO(n)/SO(n-k+1), S^{n-k}).$$

Since $n - k$ is even, we have a linear isomorphism,

$$H(SO(n)/SO(n-k)) \xrightarrow{\cong} H(SO(n)/SO(n-k+1)) \otimes H(S^{n-k}),$$

of graded vector spaces (cf. Corollary II to Proposition IV, sec. 8.4, volume I).

It follows from our induction hypothesis that

$$H^{2(n-k)}(SO(n)/SO(n-k+1)) = 0.$$

This, as in the proof of Proposition VI, implies that the isomorphism is an isomorphism of graded algebras.

Q.E.D.

Proof of Theorem III: Let $(v_1, \ldots, v_{n-1})$ be an orthonormal $(n-1)$-frame in $\mathbb{R}^n$. Then there is a unique vector, $v_n \in \mathbb{R}^n$, such that $(v_1, \ldots, v_n)$ is a positive orthonormal n-frame. This provides a diffeomorphism, $V_{\mathbb{R}}(n; n-1) \xrightarrow{\cong} SO(n)$. Now apply Theorem II.

Q.E.D.

Problems

1. **Free actions.** (i) Let G be a Lie group that acts freely and properly on a manifold M (cf. problem 5, Chap. III). Show that $(M, \pi, M/G, G)$ is a principal bundle (cf. problem 6, Chap. III).

(ii) Apply this when G is discrete and the action is discontinuous (problem 21, Chap. III). Show that the universal covering projection for any connected manifold is the projection of a principal bundle (problem 18, Chap. I).

2. (i) Show that the closed proper subgroups of S^1 are finite, and are in $1-1$ correspondence with the groups $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, $p = 1, 2, \ldots$.

(ii) Construct principal bundles $(S^1, \pi, S^1, \mathbb{Z}_p)$, where $\mathbb{Z}_p$ acts by multiplication. Let $(S^1 \times_{\mathbb{Z}_p} S^1, \rho, S^1, S^1)$ be the associated bundle (same action of $\mathbb{Z}_p$). Identify it as a principal S^1-bundle, and show that it is the trivial bundle.

(iii) Construct a principal bundle $(\mathbb{R}^2, \pi, S^1 \times_{\mathbb{Z}_p} S^1, \mathbb{Z} \times \mathbb{Z})$, where $\mathbb{Z} \times \mathbb{Z}$ acts on $\mathbb{R}^2$ by

$$(x, y) \cdot (m, n) = (x + pm + n, y + n), \qquad x, y \in \mathbb{R}, \quad m, n \in \mathbb{Z}.$$

(iv) Let $\mathbb{Z}_2$ act on S^1 via $e^{i\theta} \mapsto e^{-i\theta}$. Show that $S^1 \times_{\mathbb{Z}_2} S^1$ is not diffeomorphic to $S^1 \times S^1$.

3. Let $M(n, m; k)$ denote the set of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ of rank k.

(i) Make $M(n, m; k)$ into a smooth manifold.

(ii) Show that composition defines a smooth map

$$\rho: M(n, k; k) \times M(k, m; k) \to M(n, m; k).$$

(iii) Show that ρ is the projection of a principal bundle.

4. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle. Let G act on itself by conjugation. Show that the resulting associated bundle is a bundle over B with fibre G. Construct an example in which this bundle cannot be made into a principal bundle.

5. Let (P, π, B, G) be a principal bundle. Assume that G acts on an r-manifold, Y, and let (M, ρ, B, Y) be the associated bundle. Let $\xi = (V_M, p, M, \mathbb{R}^r)$ be the vertical subbundle of the tangent bundle τ_M.

(i) Show that $(V_M, \rho \circ p, B, T_Y)$ is a smooth bundle. Identify it with the bundle $P \times_G T_Y \to B$.

(ii) Assume that $Y = G/K$, where K is a closed subgroup of G. Identify ξ with the bundle $(P \times_K E/F, p_1, P/K, E/F)$, where E and F denote the Lie algebras of G and K.

6. (i) Let $\xi = (M, \pi, B, F)$ be a Riemannian vector bundle. Construct a principal $O(n)$-bundle whose fibre at x is the set of isometries $F \to F_x$. Show that ξ is the associated vector bundle.

(ii) Make similar constructions for real vector bundles, oriented real bundles, oriented Riemannian bundles, complex bundles, Hermitian bundles, and quaternionic vector bundles.

(iii) Apply (i) and (ii) to the tangent bundle of a manifold. Show that the resulting principal bundle has trivial tangent bundle.

7. **Flag manifolds.** A *flag* in $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$, $\mathbb{H}^n$) is a sequence of subspaces,

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{R}^n,$$

such that $\dim F_j = j$ $(1 \leqslant j \leqslant n)$.

Make the flags into a compact manifold (in each case) and identify it with a homogeneous space.

8. **Grassmann manifolds.** (i) Define an action of $GL(n; \mathbb{R})$ on $G_{\mathbb{R}}(n; k)$. Determine the orbits and isotropy subgroups.

(ii) Make a similar construction in the other three cases.

9. **Projective vector bundles.** Let $\xi = (E, \pi, B, F)$ be a real vector bundle.

(i) Define a manifold M whose points are the one-dimensional subspaces of the fibres F_x.

(ii) Show that M is the total space of a fibre bundle over B with the real projective space as typical fibre. Represent this bundle as an associated bundle.

(iii) Show that the pull-back of ξ to M can be written in the form $\eta \oplus \zeta$, where ζ is a vector bundle of rank 1. Is ζ necessarily trivial?

(iv) Establish analogous results in the complex and quaternionic cases.

10. Actions on principal bundles. A Lie group K *acts on a principal bundle* $\mathscr{P} = (P, \pi, B, G)$ if it acts from the left on P and B so that the projection π and the right translations T_a ($a \in G$) are K-equivariant.

(i) Show that an action of K on $\mathscr{P}$ induces an action of $K \times G$ on P.

(ii) Show that an action of K on $\mathscr{P}$ induces an action of K on all the associated bundles.

(iii) If K acts on $\mathscr{P}$, show that it acts on associated vector bundles by bundle maps. Obtain a geometric description of its action on the corresponding associated bundles with fibre a Grassmannian.

11. Parallelizable homogeneous spaces. Recall that a manifold M is parallelizable (respectively, stably parallelizable) if the tangent bundle τ_M is trivial (respectively, if $\tau_M \oplus \epsilon^1$ is trivial).

(i) Suppose $H \subset K \subset G$ is a sequence of closed Lie groups. If G/H is stably parallelizable, show that so is K/H.

(ii) Let G be a Lie group with Lie algebra E. Then Ad: $G \to GL(E)$. Show that $GL(E)/\text{Im Ad}$ is stably parallelizable.

(iii) Let K be a closed subgroup of a Lie group G. Assume the Lie algebra F of K satisfies

$$F = I_0 \supset I_1 \supset \cdots \supset I_k = 0,$$

where $[I_\nu, I_\nu] \subset I_{\nu+1}$. Show that G/K is stably parallelizable.

(iv) Show that the real and complex Stiefel manifolds $V(n; k)$ are parallelizable if $k \geqslant 2$. (Notice that $V_{\mathbb{R}}(n; 2)$ requires special attention.) Discuss the quaternionic case.

12. Vector fields on homogeneous spaces. Let K be a closed subgroup of a Lie group G. Let $F \subset E$ be the corresponding Lie algebras.

(i) Establish an isomorphism $f \mapsto Y_f$ between the space of K-equivariant functions f: $G \to E/F$ and vector fields on G/K. Given f_1 and f_2, describe the function f_3 satisfying $Y_{f_3} = [Y_{f_1}, Y_{f_2}]$.

(ii) Show that the zero sets $f^{-1}(0)$ and $Y_f^{-1}(0)$ are related by $f^{-1}(0) = \pi^{-1}Y_f^{-1}(0)$ (π: $G \to G/K$).

(iii) Show that the isomorphism of (i) determines an isomorphism from $(E/F)_I$ to the space of G-invariant vector fields on G/K. Describe the corresponding Lie product in $(E/F)_I$.

(iv) Let S be a q-dimensional torus in $U(n)$. Construct a family of

$(n-q)^2$, $U(n)$-invariant, vector fields on $U(n)/S$, linearly independent at each point.

13. Division algebras. Let E be an n-dimensional Euclidean space and let $e \in E$ be a fixed unit vector. Assume that a bilinear map $(x, y) \to xy$ is defined, subject to the following conditions: (a) $xe = ex = x$. (b) The maps, $y \mapsto xy$ and $y \mapsto yx$, are isomorphisms if $x \neq 0$. Then E is called a *real division algebra*. Assume E is a real division algebra, and let S^{2n-1} be the unit sphere in $E \oplus E$ (with respect to the induced inner product) and let S^n be the one-point compactification of E with z_∞ as compactifying point. If $y \neq 0$ define y^{-1} by $yy^{-1} = e$.

(i) Show that the map $\pi\colon S^{2n-1} \to S^n$ given by

$$\pi(x, y) = \begin{cases} xy^{-1}, & y \neq 0 \\ z_\infty, & y = 0 \end{cases}$$

is a smooth submersion.

(ii) Construct a smooth bundle $(S^{2n-1}, \pi, S^n, S^{n-1})$. (*Hint*: cf. problem 12, Chap. VII, volume I).

(iii) Show that in the cases $E = \mathbb{C}$, $\mathbb{H}$ this is a principal bundle.

(iv) Use the Cayley numbers to show that such a multiplication exists for $n = 8$ and construct a fibre bundle (S^{15}, π, S^8, S^7). Show that this is *not* a principal bundle (cf. problem 5, Chap. III, volume I).

14. Consider the principal bundles $(SO(n), \pi, S^{n-1}, SO(n-1))$, where π is given by $\pi(\tau) = \tau(e)$ (e, a fixed unit vector in $\mathbb{R}^n$). In the case $n = 4$ and $n = 8$ show that this principal bundle admits a cross-section and conclude that it is trivial. Conclude that S^3 and S^7 are parallelizable.

15. Coordinate representations. Let $\mathscr{B} = (M, \rho, B, F)$ be a smooth bundle, and assume that G is a Lie group acting on F from the left.

(i) Suppose $\mathscr{B}$ is the associated bundle of a principal G-bundle. Show that there is a coordinate representation (U_i, ψ_i) for $\mathscr{B}$ such that

$$\psi_i^{-1} \circ \psi_j(x, y) = (x, \gamma_{ij}(x) \cdot y),$$

where $\gamma_{ij} : U_i \cap U_j \to G$ are smooth maps satisfying

$$\gamma_{ij}(x)\, \gamma_{jk}(x) = \gamma_{ik}(x), \qquad x \in U_i \cap U_j \cap U_k .$$

(ii) Conversely, assume $\mathscr{B}$ has such a coordinate representation. Construct a principal bundle for which $\mathscr{B}$ is the associated bundle (via the given action of G on F).

(iii) If the action is effective show that the first equation of (i) implies the second.

(iv) Show that the constructions in problem 6 are special cases of the construction in (ii).

16. Let H_1 and H_2 be closed subgroups of a Lie group G such that $H_1 \subset H_2$. Consider the fibre bundle

$$\mathscr{B} = (G/H_1, \pi, G/H_2, H_2/H_1).$$

Define H_0 by

$$H_0 = \bigcap_{x \in H_2} x H_1 x^{-1}.$$

(i) Show that H_0 is the largest subgroup of H_1 which is normal in H_2. Conclude that H_0 is a closed Lie subgroup of H_1.

(ii) Show that $(G/H_0, \pi_0, G/H_2, H_2/H_0)$ is a principal bundle, and that $\mathscr{B}$ is an associated bundle.

(iii) Show that $\mathscr{B} = (V_{\mathbb{R}}(n; j), \pi, V_{\mathbb{R}}(n; k), V_{\mathbb{R}}(n - k; j - k))$ is associated with the principal bundle $(SO(n), \pi, V_{\mathbb{R}}(n; k), SO(n - k))$ and use problem 14 to conclude that $\mathscr{B}$ is trivial if $j > 1$ and $n = 4$ or 8.

17. Vector fields on fibre bundles. Let $\mathscr{B} = (M, \pi, B, F)$ be a fibre bundle. A vector field, Y, on M is called *basic*, if there is a vector field, X, on B such that

$$Y \underset{\pi}{\sim} X.$$

A vector field, Z, on M is called *vertical*, if

$$(d\pi)_z Z(z) = 0, \qquad z \in M.$$

(i) Show that the Lie product of vertical vector fields is vertical.

(ii) Show that the Lie product of a vertical and a basic vector field is vertical.

(iii) Show that the Lie product of two basic vector fields is basic.

(iv) Show that the $\mathscr{S}(M)$-module $\mathscr{X}(M)$ is generated by the basic and vertical vector fields

(v) If $\mathscr{B}$ is a principal bundle, show that Y is basic if and only if $Y - (T_a)_* Y$ is vertical for each a in the structure group.

18. Differential forms on fibre bundles. Consider the homomorphism $\pi^*: A(M) \leftarrow A(B)$ ($\mathscr{B} = (M, \pi, B, F)$, a bundle).

(i) Show that π^* is injective.

(ii) Show that, if $\Phi \in \operatorname{Im} \pi^*$, then $i(Z)\Phi = 0$ and $\theta(Z)\Phi = 0$ for every vertical vector field. Show that if F is connected, then the converse is true.

(iii) Show that if $\mathscr{B}$ admits a cross-section, then the map, $\pi^\#: H(M) \leftarrow H(B)$, is injective.

19. Let E and F be the Lie algebras of $GL(n; \mathbb{R})$ and $U(n)$.

(i) Construct an isomorphism of graded differential algebras

$$(\wedge E^* \otimes \mathbb{C}, \delta_E \otimes \iota) \cong (\wedge F^* \otimes \mathbb{C}, \delta_F \otimes \iota).$$

(ii) Compute $H_L(GL(n; \mathbb{R}))$ and compare it with $H(SO(n))$.

(iii) Compute $H_L(O(p, q))$ (cf. problem 12, Chap. II).

20. Outer automorphisms. Construct an automorphism of $U(n)$ which is not an inner automorphism. Determine its action on $H(U(n))$. Do the same for $SO(2n)$.