Chapter IV

Invariant Cohomology

In this chapter G denotes an n-dimensional Lie group with Lie algebra E.

§1. Group actions

4.1. Invariant cohomology. Consider a right action, $T: M \times G \to M$, of G on a manifold M. Recall that $\Phi \in A(M)$ is invariant if $T_a^*\Phi = \Phi$, $a \in G$, (cf. sec. 3.12) and that the invariant forms constitute a graded subalgebra, $A_I(M)$. $A_I(M)$ is stable under δ and the corresponding cohomology algebra is denoted by

$$H_I(M) = \sum_p H_I^p(M).$$

In particular, if $M = G$ and T is the group multiplication, $A_I(M)$ is denoted by $A_R(G)$, and the cohomology algebra is denoted by $H_R(G)$.

The inclusion map $i: A_I(M) \to A(M)$ induces a homomorphism

$$i_\#: H_I(M) \to H(M)$$

of graded algebras.

If $\hat{T}: N \times G \to N$ is a second right action and $\varphi: M \to N$ is equivariant, then φ^* restricts to a homomorphism

$$\varphi_I^*: A_I(M) \leftarrow A_I(N),$$

and so induces a homomorphism

$$\varphi_I^\#: H_I(M) \leftarrow H_I(N).$$

Assume that $\psi: M \to N$ is a second equivariant map, and that

$$H: \mathbb{R} \times M \to N$$

is a homotopy connecting φ and ψ and satisfying

$$H(t, x \cdot a) = H(t, x) \cdot a, \qquad t \in \mathbb{R}, \quad x \in M, \quad a \in G$$

(H is called an *equivariant homotopy*). Then the associated homotopy operator $h: A(M) \leftarrow A(N)$ (cf. sec. 0.14), satisfies

$$h \circ \hat{T}_a^* = T_a^* \circ h.$$

Hence it restricts to a linear map

$$h_I: A_I(M) \leftarrow A_I(N).$$

Now we have (cf. sec. 0.14)

$$\psi_I^* - \varphi_I^* = h_I \circ \delta + \delta \circ h_I,$$

whence $\psi_I^\# = \varphi_I^\#$.

Next, assume that $M = U \cup V$ where U and V are open sets, stable under the action of G. Then so is $U \cap V$ and, as in sec. 5.4, volume I, we can form the sequence

$$0 \longrightarrow A_I(M) \xrightarrow{\alpha} A_I(U) \oplus A_I(V) \xrightarrow{\beta} A_I(U \cap V) \longrightarrow 0,$$

where

$$\alpha(\Phi) = (\Phi|_U, \Phi|_V), \qquad \beta(\Phi, \Psi) = \Phi|_{U \cap V} - \Psi|_{U \cap V}.$$

Lemma I: If G is compact, then the above sequence is short exact.

Proof: Let (f, g) be a partition of unity for M with

$$\operatorname{carr} f \subset U \qquad \text{and} \qquad \operatorname{carr} g \subset V.$$

Let Δ be the unique left invariant n-form on G ($n = \dim G$) such that, with respect to some left orientation, $\int_G \Delta = 1$.

Define new functions $f_I, g_I \in \mathscr{S}(M)$ by

$$f_I(x) = \int_G f(x \cdot a)\, da, \qquad g_I(x) = \int_G g(x \cdot a)\, da.$$

According to Example 2, sec. 3.18, f_I and g_I are invariant. Moreover,

$$\operatorname{carr} f_I \subset (\operatorname{carr} f) \cdot G \subset U \qquad \text{and} \qquad \operatorname{carr} g_I \subset (\operatorname{carr} g) \cdot G \subset V.$$

Finally,

$$f_I(x) + g_I(x) = \int_G (f+g)(x \cdot a)\, da = \int_G da = 1.$$

It follows that (f_I, g_I) is again a partition of unity for M subordinate to the open cover U, V. Now mimic the proof of Lemma I, sec. 5.4, volume I, using f_I and g_I.

Q.E.D.

Corollary: There is an exact triangle

$$\begin{array}{ccc} H_I(M) & \longrightarrow & H_I(U) \oplus H_I(V) \\ & \nwarrow \qquad \swarrow & \\ & H_I(U \cap V). & \end{array}$$

4.2. Group projection. Consider again an action $T: M \times G \to M$ (with no additional hypothesis on G) and assume that M is connected. Fix a point $z \in M$ and consider the map $A_z: G \to M$ given by $A_z(a) = z \cdot a$. It induces a homomorphism

$$A_z^\#: H(M) \to H(G).$$

If $w \in M$ is a second point then a path $x(t)$ joining z to w provides a homotopy,

$$H: (t, a) \mapsto A_{x(t)}(a),$$

joining A_z and A_w. Hence $A_z^\# = A_w^\#$.

It follows that the homomorphism,

$$p: H(M) \to H(G),$$

defined by $p = A_z^\#$ is independent of the choice of $z \in M$; p is called the *group projection.*

Since A_z is equivariant (with respect to the right action of G on G), it induces a homomorphism

$$(A_z^*)_I: A_I(M) \to A_R(G).$$

Moreover, because the homotopy above is also equivariant, the homomorphism,

$$p_I \colon H_I(M) \to H_R(G),$$

defined by $p_I = (A_z)_I^{\#}$ is independent of the choice of z; p_I is called the *invariant group projection.*

Example: Let M be a connected Lie group and let G be a subgroup. Consider the right action of G on M given by restricting the group multiplication. Then the map $A_z \colon G \to M$ is given by

$$A_z(a) = za, \qquad z \in M, \quad a \in G.$$

In particular, $A_e = j_G$ is the inclusion map of G into M. Hence

$$p = A_e^{\#} = j_G^{\#}.$$

4.3. Compact groups. Assume that G is compact and that $T\colon M \times G \to G$ is an action. We shall construct a linear map

$$\rho \colon A(M) \to A_I(M)$$

homogeneous of degree zero, and satisfying (cf. sec. 4.1 for i)

$$\rho \circ i = \iota.$$

Orient G and let Δ be the unique left invariant n-form ($n = \dim G$) on G such that (with respect to some left orientation)

$$\int_G \Delta = 1.$$

Regard $(M \times G, \pi_M, M, G)$ as a trivial, oriented bundle, and let $\pi_G \colon M \times G \to G$ denote the projection. Then a linear map, homogeneous of degree zero,

$$I_\Delta \colon A(M \times G) \to A(M),$$

is defined by

$$I_\Delta \Omega = \oint_G \Omega \wedge \pi_G^* \Delta$$

(cf. sec. 0.15).

Thus we can consider the linear map $I_\Delta \circ T^* \colon A(M) \to A(M)$; it is given by

$$(I_\Delta \circ T^*)(\Phi) = \oint_G T^*\Phi \wedge \pi_G^* \Delta.$$

Lemma II: Fix $x \in M$. Then $(I_\Delta \circ T^*)(\Phi)(x) = \int_G (T_a^*\Phi)(x)\, da$.

Proof: The retrenchment of $T^*\Phi \wedge \pi_G^*\Delta$ to $x \times G$ is a $\wedge T_x(M)^*$-valued n-form on G:

$$(T^*\Phi \wedge \pi_G^*\Delta)_x \in A^n(G; \wedge T_x(M)^*).$$

A short, straightforward computation shows that

$$(T^*\Phi \wedge \pi_G^*\Delta)_x = f \cdot \Delta,$$

where $f \in \mathscr{S}(G; \wedge T_x(M)^*)$ is given by $f(a) = (T_a^*\Phi)(x)$.

It follows that

$$[(I_\Delta \circ T^*)\Phi](x) = \int_G (T^*\Phi \wedge \pi_G^*\Delta)_x = \int_G (T_a^*\Phi)(x)\, da.$$

Q.E.D.

Proposition I: (1) If $\Phi \in A(M)$, then $(I_\Delta \circ T^*)\Phi \in A_I(M)$.
(2) If $\Phi \in A_I(M)$, then $(I_\Delta \circ T^*)\Phi = \Phi$.

Proof: (1) Fix $a \in G$, $x \in M$, and write

$$\alpha_a = (\wedge dT_a)_x^* : \wedge T_x(M)^* \leftarrow \wedge T_{x \cdot a}(M)^*.$$

Set $\Psi = (I_\Delta \circ T^*)\Phi$. Then using Lemma II and the linearity of α_a we find

$$\begin{aligned}(T_a^*\Psi)(x) &= \alpha_a(\Psi(x \cdot a)) = \alpha_a\left(\int_G (T_b^*\Phi)(x \cdot a)\, db\right)\\ &= \int_G \alpha_a(T_b^*\Phi(x \cdot a))\, db\\ &= \int_G (T_{ab}^*\Phi)(x)\, db.\end{aligned}$$

Thus formula (1.2), sec. 1.15, yields

$$(T_a^*\Psi)(x) = \int_G (T_b^*\Phi)(x)\, db = \Psi(x),$$

and so Ψ is invariant.

(2) If Φ is invariant, Lemma II yields

$$[(I_\Delta \circ T^*)\Phi](x) = \Phi(x) \int_G da = \Phi(x).$$

Q.E.D.

Proposition I shows that $I_\Delta \circ T^*$ may be regarded as a linear map

$$\rho\colon A(M) \to A_I(M)$$

satisfying $\rho \circ i = \iota$.

Theorem I: Let $M \times G \to M$ be a right action of a compact Lie group. Then

$$i_\#\colon H_I(M) \to H(M)$$

is injective. If G is connected, then $i_\#$ is an isomorphism.

Proof: Recall from sec. 0.15 that $\oint_G \circ \delta = \delta \circ \oint_G$. Hence $\rho \circ \delta = \delta \circ \rho$, where $\rho\colon A(M) \to A_I(M)$ is the operator constructed above. Thus it induces $\rho_\# : H(M) \to H_I(M)$. Since $\rho \circ i = \iota$, clearly

$$\rho_\# \circ i_\# = \iota$$

and so $i_\#$ is injective.

Next, assume that G is connected. In Theorem II, sec. 4.4 below, we shall construct an operator,

$$h_M\colon A(M \times G) \to A(M),$$

homogeneous of degree -1, such that

$$I_\Delta - j_e^* = \delta h_M + h_M \delta$$

($j_e\colon M \to M \times G$ is inclusion opposite e).

Since $T \circ j_e = \iota$, precomposing both sides of this relation with T^* yields

$$i \circ \rho - \iota = I_\Delta \circ T^* - \iota = \delta h_M T^* + h_M T^* \delta.$$

It follows that $i_\# \circ \rho_\# = \iota_{H(M)}$ and hence $i_\#$ is an isomorphism.

Q.E.D.

Remark: Theorem I applies equally well to left actions.

4.4. The operator I_Φ. Let M, N be manifolds with N connected, oriented, and of dimension n. Each $\Phi \in A_c^n(N)$ which satisfies

$$\int_N \Phi = 1$$

determines the operator $I_\Phi: A(M \times N) \to A(M)$, given by

$$I_\Phi(\Omega) = \oint_N \Omega \wedge \pi_N^*\Phi.$$

I_Φ is linear and homogeneous of degree zero. It follows from Propositions IX and X, sec. 7.13, volume I (or sec. 0.15), that

$$I_\Phi \circ \delta = \delta \circ I_\Phi \qquad \text{and} \qquad I_\Phi \circ \pi_M^* = \iota.$$

Now fix $b \in N$ and let $j_b: M \to M \times N$ denote inclusion opposite b.

Theorem II: There exists a linear map $h_M : A(M \times N) \to A(M)$, homogeneous of degree -1, and such that

$$I_\Phi - j_b^* = \delta h_M + h_M \delta.$$

Proof: Let $(U, u, \mathbb{R}^n)$ be a chart on N such that $u^{-1}(0) = b$. Choose $\Psi \in A_c^n(U)$ so that

$$\int_U \Psi = \int_N \Psi = 1.$$

Then (cf. Theorem II, sec. 5.13, volume I) there is an $(n - 1)$-form $X \in A_c^{n-1}(N)$ satisfying

$$\Phi - \Psi = \delta X.$$

Fix one such X and define an operator, $k_M: A(M \times N) \to A(M)$, by

$$k_M(\Omega) = (-1)^p \oint_N \Omega \wedge \pi_N^* X, \qquad \Omega \in A^p(M \times N).$$

Then $\delta k_M + k_M \delta = I_\Phi - I_\Psi$.

Next let $\lambda: M \times U \to M \times N$ denote the inclusion. Since $\Psi \in A_c^n(U)$ it determines an operator,

$$I_\Psi: A(M \times U) \to A(M),$$

which we denote by $\tilde{I}_\Psi$ to avoid confusion. Evidently $\tilde{I}_\Psi \circ \lambda^* = I_\Psi$; hence

$$I_\Phi - \tilde{I}_\Psi \circ \lambda^* = \delta k_M + k_M \delta. \tag{4.1}$$

Finally, fix a homotopy $H: U \times \mathbb{R} \to U$ which connects the identity map with the constant map $U \to b$. Then $\iota \times H$ is a homotopy connecting the identity map of $M \times U$ with $\tilde{j}_b \circ \tilde{\pi}_M$ ($\tilde{\pi}_M: M \times U \to M$ is the projection and $\tilde{j}_b : M \to M \times U$ is the inclusion opposite b).

The induced homotopy operator $\tilde{h}_M$ satisfies

$$\tilde{\pi}_M^* \circ \tilde{j}_b^* - \iota = \delta \tilde{h}_M + \tilde{h}_M \delta.$$

It follows as above that $\tilde{I}_\Psi \circ \tilde{\pi}_M^* = \iota$; hence

$$\tilde{j}_b^* - \tilde{I}_\Psi = \delta \tilde{I}_\Psi \tilde{h}_M + \tilde{I}_\Psi \tilde{h}_M \delta.$$

Precompose both sides with λ^* to obtain

$$j_b^* - \tilde{I}_\Psi \circ \lambda^* = \delta \, \tilde{I}_\Psi \tilde{h}_M \lambda^* + \tilde{I}_\Psi \tilde{h}_M \lambda^* \, \delta. \tag{4.2}$$

Thus, setting $h_M = k_M - \tilde{I}_\Psi \tilde{h}_M \lambda^*$, we find, on subtracting (4.2) from (4.1), that

$$I_\Phi - j_b^* = \delta h_M + h_M \delta.$$

Q.E.D.

§2. Left invariant forms on a Lie group

4.5. Left invariant differential forms. Consider the *left* action of G on itself by left translations. The differential forms on G that are invariant under this action will be called *left invariant*, and the graded subalgebra of left invariant differential forms will be denoted by $A_L(G)$. Thus $\Phi \in A_L(G)$ if and only if

$$a \cdot \Phi = \lambda_a^* \Phi = \Phi, \qquad a \in G,$$

or equivalently (when $\deg \Phi = p$) if

$$\Phi(a; L_a h_1, \ldots, L_a h_p) = \Phi(e; h_1, \ldots, h_p), \qquad h_1, \ldots, h_p \in E, \quad a \in G.$$

Now let X_h be a left invariant vector field (cf. sec. 1.2). Since, for each $a \in G$,

$$(\lambda_a)_* X_h = X_h,$$

Proposition III, sec. 4.4, volume I, (or sec. 0.13) implies that the algebra $A_L(G)$ is stable under the operators $i(X_h)$ and $\theta(X_h)$. It is clearly stable under δ. The corresponding cohomology algebra $H(A_L(G), \delta)$ will be denoted by $H_L(G)$.

Proposition II: The correspondence, $\Phi \mapsto \Phi(e)$, defines an isomorphism,

$$\tau_L: A_L(G) \xrightarrow{\cong} \wedge E^*,$$

of graded algebras. In particular, the left invariant functions are constant.

Proof: According to Proposition I, sec. 1.2, a strong bundle isomorphism,

$$G \times E \xrightarrow{\cong} T_G,$$

is given by $(a, h) \mapsto X_h(a)$. It induces a strong bundle isomorphism,

$$\varphi: G \times \wedge E^* \xrightarrow{\cong} \wedge T_G^*,$$

and so we obtain an isomorphism,

$$\varphi_*: \mathscr{S}(G; \wedge E^*) \xrightarrow{\cong} A(G).$$

A simple computation (using the left invariance of the vector fields X_h) shows that the diagrams

$$\begin{array}{ccc} \mathscr{S}(G; \wedge E^*) & \xrightarrow[\cong]{\varphi_*} & A(G) \\ \lambda_a^* \downarrow & & \downarrow \lambda_a^* \\ \mathscr{S}(G; \wedge E^*) & \xrightarrow[\cong]{\varphi_*} & A(G), \end{array} \qquad a \in G,$$

commute (cf. sec. 0.13). It follows that the left invariant forms correspond under φ_* to the constant functions $G \to \wedge E^*$. The proposition follows.

Q.E.D.

Corollary: $A_L(G)$ is the exterior algebra over the vector space $A_L^1(G)$.

4.6. The differential algebra $\wedge E^*$. Since $A_L(G)$ is stable under the operators $i(X_h)$, $\theta(X_h)$ $(h \in E)$, and δ, there are uniquely determined operators $i_E(h)$, $\theta_E(h)$, and δ_E in $\wedge E^*$, such that

$$\tau_L \circ i(X_h) = i_E(h) \circ \tau_L, \qquad \tau_L \circ \theta(X_h) = \theta_E(h) \circ \tau_L, \qquad h \in E,$$

and

$$\tau_L \circ \delta = \delta_E \circ \tau_L.$$

$i_E(h)$ and δ_E are antiderivations in $\wedge E^*$, homogeneous of degrees -1 and $+1$, respectively, while $\theta_E(h)$ is a derivation, homogeneous of degree zero.

From Proposition I, sec. 4.2, and Proposition II, sec. 4.3, both of volume I (or sec. 0.13), we obtain the relations

$$i_E([h, k]) = \theta_E(h)\, i_E(k) - i_E(k)\, \theta_E(h), \qquad \theta_E([h, k]) = \theta_E(h)\, \theta_E(k) - \theta_E(k)\, \theta_E(h),$$

$$\theta_E(h) = i_E(h)\, \delta_E + \delta_E i_E(h)$$

and

$$\delta_E^2 = 0, \qquad h, k \in E.$$

The second formula shows that θ_E is a representation of E in the vector space $\wedge E^*$. Since δ_E is an antiderivation in $\wedge E^*$ whose square is zero, $(\wedge E^*, \delta_E)$ is a graded differential algebra. The corresponding cohomology algebra is called the *cohomology algebra of the Lie algebra E* and will be denoted by $H(E)$.

It follows from the definitions that

$$\tau_L: A_L(G) \xrightarrow{\cong} \wedge E^*$$

is an isomorphism of differential algebras. Thus it induces an isomorphism

$$(\tau_L)_*: H_L(G) \xrightarrow{\cong} H(E).$$

Now we shall determine the operators $i_E(h)$, $\theta_E(h)$, and δ_E explicitly.

Proposition III: The operators $i_E(h)$, $\theta_E(h)$, and δ_E are given by

(1) $[i_E(h)\Phi](h_1, ..., h_{p-1}) = \Phi(h, h_1, ..., h_{p-1})$.

(2) $[\theta_E(h)\Phi](h_1, ..., h_p) = -\sum_{i=1}^{p} \Phi(h_1, ..., [h, h_i], ..., h_p)$.

(3) $[\delta_E\Phi](h_0, h_1, ..., h_p) = \sum_{i<j} (-1)^{i+j}\Phi([h_i, h_j], h_0, ..., \hat{h}_i, ..., \hat{h}_j, ..., h_p)$,

$$\Phi \in \wedge^p E^*, \; h_i \in E.$$

Proof: (1) is immediate. To obtain (2) observe that, for $\Psi \in A_L^p(G)$,

$$\begin{aligned}[\theta_E(h)\, \tau_L\Psi](h_1, ..., h_p) &= [\theta(X_h)\Psi](e; h_1, ..., h_p) \\ &= X_h(\Psi(X_{h_1}, ..., X_{h_p}))(e) - \sum_{i=1}^{p} \Psi(e; h_1, ..., [h, h_i], ..., h_p).\end{aligned}$$

Since Ψ is left invariant, so is the function $\Psi(X_{h_1}, ..., X_{h_p})$. Thus this function is constant, and the first term in the above equation is zero. This proves (2). (3) follows in the same way.

Q.E.D.

Example: Let $h^* \in E^*$. Then

$$\langle\theta_E(h)\, h^*, k\rangle = -\langle h^*, [h, k]\rangle = -\langle h^*, (\operatorname{ad} h)k\rangle$$

and

$$\langle\delta_E h^*, h \wedge k\rangle = -\langle h^*, [h, k]\rangle, \qquad h, k \in E.$$

Hence the restriction of $\theta_E(h)$ to E^* is given by

$$\theta_E(h) = -(\operatorname{ad} h)^*, \qquad h \in E,$$

while the restriction of δ_E to E^* satisfies

$$\delta_E^*(h \wedge k) = -[h, k].$$

4.7. Homomorphisms. Let $\varphi: G \to K$ be a homomorphism of Lie groups. Let F be the Lie algebra of K and recall that the derivative of φ at e,

$$\varphi': E \to F,$$

is a homomorphism of Lie algebras (sec. 1.3).

For every $h \in E$, the vector fields $X_h \in \mathscr{X}_L(G)$, $X_{\varphi'(h)} \in \mathscr{X}_L(K)$ are φ-related. Thus Proposition III, sec. 4.4, volume I (or sec. 0.13) gives the relations

$$i(X_h) \circ \varphi^* = \varphi^* \circ i(X_{\varphi'(h)}),$$

$$\theta(X_h) \circ \varphi^* = \varphi^* \circ \theta(X_{\varphi'(h)}),$$

and

$$\delta \circ \varphi^* = \varphi^* \circ \delta.$$

On the other hand, the equation $\lambda_a^* \circ \varphi^* = \varphi^* \circ \lambda_{\varphi(a)}^*$, $a \in G$, shows that φ^* restricts to a homomorphism

$$\varphi_L^*: A_L(G) \leftarrow A_L(K).$$

It is immediate from the definitions that the diagram,

$$\begin{array}{ccc} A_L(G) & \xleftarrow{\varphi_L^*} & A_L(K) \\ \tau_L \downarrow \cong & & \cong \downarrow \tau_L \\ \wedge E^* & \xleftarrow[(\wedge\varphi')^*]{} & \wedge F^* \end{array},$$

commutes.

But this yields the relations

$$i_E(h) \circ (\wedge\varphi')^* = (\wedge\varphi')^* \circ i_F(\varphi'(h)),$$

$$\theta_E(h) \circ (\wedge\varphi')^* = (\wedge\varphi')^* \circ \theta_F(\varphi'(h)),$$

and

$$\delta_E \circ (\wedge\varphi')^* = (\wedge\varphi')^* \circ \delta_F, \qquad h \in E.$$

In particular, $(\wedge\varphi')^*$ is a homomorphism of graded differential algebras. Thus it induces a homomorphism of cohomology algebras, which we denote by

$$(\varphi')^*: H(E) \leftarrow H(F).$$

It follows from the definitions that

$$\begin{array}{ccc} H_L(G) & \xleftarrow{\varphi_L^*} & H_L(K) \\ (\tau_L)_* \downarrow \cong & & \cong \downarrow (\tau_L)_* \\ H(E) & \xleftarrow[(\varphi')^*]{} & H(F) \end{array}$$

commutes.

4.8. The adjoint representation in $\wedge E^*$. Consider the adjoint representation of G (cf. sec. 1.10). The contragredient representation, $\mathrm{Ad}^\natural$, of G in E^* extends to a representation, $\wedge \mathrm{Ad}^\natural = \sum_p \wedge^p \mathrm{Ad}^\natural$, of G in the graded algebra $\wedge E^*$; it will be denoted by $\mathrm{Ad}^\wedge$. Thus

$$\mathrm{Ad}^\wedge(a)(h^{*1} \wedge \cdots \wedge h^{*p}) = (\mathrm{Ad}\, a^{-1})^* h^{*1} \wedge \cdots \wedge (\mathrm{Ad}\, a^{-1})^* h^{*p}.$$

On the other hand, recall that in sec. 4.6 we defined a representation, θ_E, of E in $\wedge E^*$.

Lemma III: θ_E is the derivative of the representation $\mathrm{Ad}^\wedge$.

Proof: Since $\theta(X_h)$ is a derivation in $A_L(G)$, $\theta_E(h)$ is a derivation in $\wedge E^*$. On the other hand, if $\theta^\wedge$ denotes the derivative of $\mathrm{Ad}^\wedge$, then

$$\theta^\wedge(h)(h^{*1} \wedge \cdots \wedge h^{*p}) = \sum_{i=1}^{p} h^{*1} \wedge \cdots \wedge \theta^\wedge(h)\, h^{*i} \wedge \cdots \wedge h^{*p}$$

(cf. Example 2, sec. 1.9). It follows that $\theta^\wedge(h)$ is a derivation in $\wedge E^*$ as well. Hence we need only prove that

$$\theta^\wedge(h)\, h^* = \theta_E(h)\, h^*, \qquad h \in E, \quad h^* \in E^*.$$

But this follows from the example in sec. 4.6.

Q.E.D.

Next, fix $a \in G$. Since $\mathrm{Ad}\, a$ is the derivative of the inner automorphism τ_a, it follows that

$$\mathrm{Ad}^\wedge(a^{-1}) = \wedge(\tau_a')^*.$$

Hence $\mathrm{Ad}^\wedge(a^{-1})$ commutes with δ_E. In particular, the representation, $\mathrm{Ad}^\wedge$, of G in $\wedge E^*$ induces a representation,

$$a \mapsto \mathrm{Ad}^\#(a),$$

of G in $H(E)$.

Lemma IV: If G is connected, then the representation $\mathrm{Ad}^\#$ is trivial,

$$\mathrm{Ad}^\#(a) = \iota, \qquad a \in G.$$

Proof: It follows from Lemma III and Example 4, sec. 1.9, that the derivative of the representation $\mathrm{Ad}^\#$ is given by

$$h \mapsto \theta_E(h)^\#, \qquad h \in E.$$

But by the relations of sec. 4.6

$$\theta_E(h) = i_E(h)\,\delta_E + \delta_E\, i_E(h).$$

Hence $\theta_E(h)^\# = 0$.

Since G is connected, the lemma follows now from Proposition IX, sec. 1.8.

Q.E.D.

Proposition IV: If G is a connected Lie group, then

$$\det(\iota - \mathrm{Ad}\, a) = 0, \qquad a \in G.$$

Proof: Elementary considerations from linear algebra (cf. sec. A.2) show that

$$\det(\iota - \mathrm{Ad}\, a) = \sum_{p=0}^{n} (-1)^p \operatorname{tr} \wedge^p \mathrm{Ad}\, a = \sum_{p=0}^{n} (-1)^p \operatorname{tr} \wedge^p (\mathrm{Ad}\, a)^*.$$

We have seen above that

$$\mathrm{Ad}^\wedge(a^{-1}) = \sum_{p=0}^{n} \wedge^p (\mathrm{Ad}\, a)^*$$

is an automorphism of the graded differential algebra $\wedge E^*$. Hence the algebraic Lefschetz formula (cf. sec. 0.8) yields

$$\sum_{p=0}^{n} (-1)^p \operatorname{tr} \wedge^p (\mathrm{Ad}\, a)^* = \sum_{p=0}^{n} (-1)^p \operatorname{tr} \mathrm{Ad}^{(p)}(a^{-1}),$$

where $\mathrm{Ad}^{(p)}(a^{-1})$ denotes the restriction of $\mathrm{Ad}^\#(a^{-1})$ to $H^p(E)$.

Now Lemma IV yields

$$\sum_{p=0}^{n} (-1)^p \operatorname{tr} \wedge^p (\mathrm{Ad}\, a)^* = \sum_{p=0}^{n} (-1)^p \operatorname{tr} \mathrm{Ad}^{(p)}(e)^*$$

$$= \sum_{p=0}^{n} (-1)^p \operatorname{tr} \wedge^p (\mathrm{Ad}\, e)^*;$$

i.e.,

$$\det(\iota - \operatorname{Ad} a) = \det(\iota - \operatorname{Ad} e) = \det(0) = 0.$$

Q.E.D.

Corollary: Let $a \in G$. Then of the normalizer, N_a, of a (Example 4, sec. 2.4) has at least dimension 1.

Proof: It follows from the proposition that there exists a nonzero vector $h \in E$ such that

$$(\operatorname{Ad} a)h = h.$$

Hence h is in the Lie algebra of N_a (cf. Example 4, sec. 2.4) and so $\dim N_a \geqslant 1$.

Q.E.D.

§3. Invariant cohomology of Lie groups

4.9. Invariant forms. A differential form $\Phi \in A(G)$ will be called *bi-invariant*, or simply *invariant*, if

$$\lambda_a^*\Phi = \Phi \quad \text{and} \quad \rho_a^*\Phi = \Phi, \qquad a \in G.$$

The set of invariant differential forms is a graded subalgebra of $A(G)$ which we denote by $A_I(G)$. Clearly $A_I(G)$ is stable under δ.

Proposition V: The invariant forms on G are closed.

Lemma V: If $\nu: G \to G$ is the inversion map of G, then

$$\nu^*\Phi = (-1)^p\Phi, \qquad \Phi \in A_I^p(G).$$

Proof: We have (cf. sec. 1.1)

$$(d\nu)_a = -R_{a^{-1}} \circ L_a^{-1}, \qquad a \in G.$$

Thus, for $a \in G$, $h_1, \ldots, h_p \in E$,

$$\begin{aligned}(\nu^*\Phi)(a; L_a h_1, \ldots, L_a h_p) &= \Phi(a^{-1}; -R_{a^{-1}} h_1, \ldots, -R_{a^{-1}} h_p) \\ &= (-1)^p (\rho_{a^{-1}}^* \Phi)(e; h_1, \ldots, h_p) \\ &= (-1)^p \Phi(a; L_a h_1, \ldots, L_a h_p).\end{aligned}$$

Q.E.D.

Proof of the proposition: Since $A_I(G)$ is stable under δ, the lemma yields

$$(-1)^{p+1}\,\delta\Phi = \nu^*\,\delta\Phi = \delta\nu^*\Phi = (-1)^p\,\delta\Phi, \qquad \Phi \in A_I^p(G),$$

whence $\delta\Phi = 0$.

Q.E.D.

It follows from Proposition V that the inclusion $A_I(G) \to A_L(G)$ induces a homomorphism of graded algebras

$$A_I(G) \to H_L(G).$$

Next we determine the subalgebra of $\wedge E^*$ corresponding to $A_I(G)$ under τ_L (cf. sec. 4.5).

Lemma VI: If $\Phi \in A_L(G)$, then $\rho_a^*\Phi \in A_L(G)$ and (cf. sec. 4.8)

$$\tau_L(\rho_a^*\Phi) = \mathrm{Ad}^\wedge(a)(\tau_L\Phi), \qquad a \in G.$$

Proof: That $A_L(G)$ is stable under ρ_a^*, $a \in G$, follows (cf. sec. 1.1) from the relation

$$\rho_a^* \circ \lambda_b^* = \lambda_b^* \circ \rho_a^*, \qquad a, b \in G.$$

Moreover, if $\Phi \in A_L(G)$, then

$$\tau_L(\rho_a^*\Phi) = (\rho_a^*\lambda_{a^{-1}}^*\Phi)(e) = (\tau_{a^{-1}}^*\Phi)(e)$$

($\tau_{a^{-1}}$ is conjugation by a^{-1}). Also $\tau_{a^{-1}}(e) = e$ and

$$\wedge(\tau'_{a^{-1}})^* = \mathrm{Ad}^\wedge(a).$$

Thus

$$\tau_L(\rho_a^*\Phi) = (\wedge\tau'_{a^{-1}})^*(\Phi(e)) = \mathrm{Ad}^\wedge(a)(\tau_L\Phi).$$

Q.E.D.

Now let $(\wedge E^*)_I$ denote the subalgebra of $\wedge E^*$ invariant with respect to the representation $\mathrm{Ad}^\wedge$. Lemma VI implies that the isomorphism τ_L restricts to an isomorphism $\tau_I : A_I(G) \xrightarrow{\cong} (\wedge E^*)_I$. Thus the diagram

$$\begin{array}{ccc} A_I(G) & \longrightarrow & A_L(G) \\ {\scriptstyle \tau_I}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \tau_L} \\ (\wedge E^*)_I & \longrightarrow & \wedge E^* \end{array}$$

commutes (the horizontal maps are inclusions).

In particular the elements of $(\wedge E^*)_I$ are in the kernel of δ_E, and we have the commutative diagram

$$\begin{array}{ccc} A_I(G) & \longrightarrow & H_L(G) \\ {\scriptstyle \tau_I}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle (\tau_L)_*} \\ (\wedge E^*)_I & \longrightarrow & H(E). \end{array}$$

Finally, if G is connected, then $(\wedge E^*)_I = (\wedge E^*)_{\theta_E=0}$ (cf. Lemma III, sec. 4.8, and Proposition IX, sec. 1.8).

4.10. Compact connected groups. Suppose G is connected. The inclusion $A_L(G) \to A(G)$ induces a homomorphism $H_L(G) \to H(G)$. Combining this with the diagram just above yields the commutative diagram

$$\begin{array}{ccccc} A_I(G) & \longrightarrow & H_L(G) & \longrightarrow & H(G) \\ \tau_I \big\downarrow \cong & & \cong \big\downarrow (\tau_L)_* & & \\ (\wedge E^*)_{\theta=0} & \longrightarrow & H(E). & & \end{array}$$

Theorem III: If G is compact and connected, all the above maps are isomorphisms (of algebras).

Proof: It is sufficient to show that the inclusions

$$A_I(G) \to A(G), \qquad A_L(G) \to A(G)$$

induce isomorphisms $A_I(G) \xrightarrow{\cong} H(G)$, $H_L(G) \xrightarrow{\cong} H(G)$.

Since $A_L(G)$ is the algebra of differential forms invariant under the left action of G on itself, Theorem I, sec. 4.3. implies that $H_L(G) \to H(G)$ is an isomorphism.

On the other hand, consider the right action, T, of the compact connected group $G \times G$ on G given by

$$T_{(a,b)}(x) = a^{-1}xb, \qquad a, b, x \in G.$$

$A_I(G)$ is the algebra of differential forms on G which are invariant under this action. Since the forms in $A_I(G)$ are closed, Theorem I, sec. 4.3, implies that $A_I(G) \to H(G)$ is an isomorphism.

Q.E.D.

Corollary: The Poincaré polynomial, $f_G(t)$, of a compact connected Lie group G is given by

$$f_G(t) = \int_G \det(\mathrm{Ad}\, a + t\iota)\, da.$$

Proof: By definition (cf. sec. 0.14)

$$f_G(t) = \sum_{p=0}^{n} b_p t^p$$

($n = \dim G$ and $b_p = \dim H^p(G)$).

It follows from Theorem III that

$$b_p = \dim(\wedge^p E^*)_I, \qquad p = 0, 1, \ldots, n.$$

Hence Corollary III to Proposition XV, sec. 1.16 (applied with $P = \mathrm{Ad}$) yields

$$\sum_{p=0}^{n} b_p t^p = \int_G \det(\mathrm{Ad}\, a + t\iota)\, da.$$

Q.E.D.

4.11. Noncompact groups. It can be shown that every connected Lie group contains a compact subgroup as deformation retract (cf. [9, p. 180]). Thus the computation of the cohomology of any Lie group is reduced to the compact case. In particular, as is shown in the example below, the group $SO(n)$ is a deformation retract of $GL^+(n; \mathbb{R})$ (the 1-component of $GL(n; \mathbb{R})$) and hence the cohomology algebras of these groups are isomorphic.

It will be shown in volume III, that the map,

$$A_I(G) \to H_L(G),$$

which is induced by the inclusion map is still an isomorphism if the Lie algebra of G is *reductive*. On the other hand, the homomorphism,

$$H_L(G) \to H(G),$$

is *not* in general an isomorphism if the group is not compact.

In fact, as will be shown in volume III, if the adjoint representation of G is semisimple, then

$$\dim H_L^n(G) = 1, \qquad n = \dim G.$$

(This holds in particular for $G = GL^+(n; \mathbb{R})$.) On the other hand, if G contains a compact subgroup K of lower dimension which is a deformation retract of G (for example $G = GL^+(n; \mathbb{R})$, $K = SO(n)$), we have

$$H^n(G) \cong H^n(K) = 0.$$

Thus $H^n(G)$ and $H_L^n(G)$ are not isomorphic.

Examples: 1. Let V be an n-dimensional Euclidean space, and denote the space of self-adjoint transformations of V by $S(V)$. Then the map,

$$\alpha\colon SO(V) \times S(V) \to GL^+(V),$$

given by $\alpha(\varphi, \psi) = \varphi \exp \psi$ is a diffeomorphism. In particular, $SO(V)$ is a deformation retract of $GL^+(V)$.

In fact, it was shown in Example 11, sec. 1.5, volume I, that exp maps $S(V)$ diffeomorphically onto the open subset $S^+(V) \subset S(V)$ of self-adjoint transformations with strictly positive eigenvalues. In particular, since $\exp 2\varphi = (\exp \varphi)^2$, $\varphi \in S(V)$, it follows that the map

$$\sigma \mapsto \sigma^2$$

is a diffeomorphism of $S^+(V)$. Denote its inverse by

$$\sigma \mapsto \sigma^{1/2}$$

and write $(\sigma^{-1})^{1/2} = \sigma^{-1/2}$.

Then a smooth map $\beta: GL^+(V) \to SO(V) \times S(V)$ is given by

$$\beta(\varphi) = (\varphi \circ (\varphi^* \circ \varphi)^{-1/2}, \exp^{-1}(\varphi^* \circ \varphi)^{1/2})$$

and β is inverse to α. Thus α is a diffeomorphism.

2. Similarly, if W is an complex n-dimensional Hermitian space, then the map

$$\alpha: U(W) \times S(W) \to GL(W)$$

given by

$$\alpha(\varphi, \psi) = \varphi \exp \psi$$

is a diffeomorphism. ($S(W)$ is the space of self-adjoint transformations of W.)

§4. Cohomology of compact connected Lie groups

In this article G denotes a compact connected Lie group.

4.12. The primitive space and the main theorem. Since G is compact, we have the Künneth isomorphism (cf. sec. 0.14)

$$\kappa_{\#}: H(G) \otimes H(G) \xrightarrow{\cong} H(G \times G).$$

Henceforth we shall identify $H(G \times G)$ and $H(G) \otimes H(G)$ under this isomorphism. Thus, if $\mu: G \times G \to G$ denotes the multiplication map, $\mu^{\#}$ becomes a homomorphism

$$\mu^{\#}: H(G) \otimes H(G) \leftarrow H(G).$$

Let $j_1: G \to G \times G$ and $j_2: G \to G \times G$ be the inclusion maps given by

$$j_1(a) = (a, e) \qquad \text{and} \qquad j_2(a) = (e, a).$$

In view of Example 2, sec. 5.17, volume I, if $\gamma \in H^+(G \times G)$, then

$$\gamma = j_1^{\#}\gamma \otimes 1 + \beta + 1 \otimes j_2^{\#}\gamma, \tag{4.3}$$

where $\beta \in H^+(G) \otimes H^+(G)$. Observing that $\mu \circ j_1 = \mu \circ j_2 = \iota$, we obtain

$$\mu^{\#}\alpha = \alpha \otimes 1 + \beta + 1 \otimes \alpha, \qquad \alpha \in H^+(G), \quad \beta \in H^+(G) \otimes H^+(G). \tag{4.4}$$

Definition: An element $\alpha \in H^+(G)$ is called *primitive* if

$$\mu^{\#}\alpha = \alpha \otimes 1 + 1 \otimes \alpha.$$

The primitive elements form a graded subspace, P_G, of $H(G)$ (i.e., $P_G = \sum_{p=0}^{n} P_G \cap H^p(G)$) and $P_G \cap H^p(G) = 0$, if p is even. To see the latter, assume that α is primitive, and has even degree. Then the elements $\alpha \otimes 1$ and $1 \otimes \alpha$ commute; whence

$$\mu^{\#}(\alpha^m) = (\alpha \otimes 1 + 1 \otimes \alpha)^m = \sum_{k=0}^{m} \binom{m}{k} \alpha^k \otimes \alpha^{m-k}, \qquad m \geq 0.$$

Now choose m to be the least integer such that $\alpha^m = 0$. Then

$$\sum_{k=0}^{m} \binom{m}{k} \alpha^k \otimes \alpha^{m-k} = 0.$$

It follows that

$$\alpha^k \otimes \alpha^{m-k} = 0, \qquad k = 0, ..., m.$$

In particular, $\alpha \otimes \alpha^{m-1} = 0$ and so $\alpha = 0$.

Since every homogeneous element of P_G has odd degree, it follows that the square of a primitive element in $H(G)$ is zero. Thus the inclusion map $P_G \to H(G)$ extends to a homomorphism

$$\lambda_G: \wedge P_G \to H(G)$$

of graded algebras. The purpose of this article is to establish

Theorem IV: Let G be a compact connected Lie group. Then λ_G is an isomorphism. Moreover, if r is the dimension of a maximal torus, then

$$\dim P_G = r \qquad \text{and} \qquad \dim H(G) = 2^r.$$

Definition: The number r is called the *rank of G*.

Although the actual proof of Theorem IV does not come till sec. 4.17, the key steps are established in the preceding section (Propositions VIII, IX, and X). These in turn depend on the preliminary results on power maps which are proved in sec. 4.14 and sec. 4.15.

However, before proceeding with the proof, we consider the case that our group is a torus.

4.13. Cohomology of a torus. Let T be an r-dimensional torus with Lie algebra F. Since T is abelian, the adjoint representation is trivial, and hence $(\wedge F^*)_I = \wedge F^*$. Thus Theorem III, sec. 4.10, yields an isomorphism

$$\alpha_T : \wedge F^* \xrightarrow{\cong} H(T).$$

Moreover, if S is a second torus with Lie algebra L and $\varphi: S \to T$ is a homomorphism, then the diagram

$$\begin{array}{ccc} H(S) & \xleftarrow[\cong]{\alpha_S} & \wedge L^* \\ {\scriptstyle \varphi^*}\uparrow & & \uparrow{\scriptstyle \wedge(\varphi')^*} \\ H(T) & \xleftarrow[\alpha_T]{\cong} & \wedge F^* \end{array}$$

commutes.

Now, since T is abelian, the multiplication map $\mu: T \times T \to T$ is a homomorphism.

The derivative of μ at e is the linear map, $\mu': F \oplus F \to F$, given by $\mu'(h, k) = h + k$. Hence $(\mu')^* (h^*) = h^* \otimes 1 + 1 \otimes h^*$, $h^* \in F^*$. Thus the diagram above reads

$$\begin{array}{ccc} H(T) \otimes H(T) & \xleftarrow[\cong]{\alpha_T \otimes \alpha_T} & \wedge F^* \otimes \wedge F^* \\ \uparrow{\scriptstyle \mu^*} & & \uparrow{\scriptstyle \wedge(\mu')^*} \\ H(T) & \xleftarrow[\alpha_T]{\cong} & \wedge F^*. \end{array}$$

It follows at once that α_T restricts to a linear isomorphism from F^* onto the primitive subspace of $H(T)$. (This proves Theorem IV for tori.)

4.14. The power maps. The kth *power map* $P_k: G \to G$ is defined by

$$P_k(x) = x^k, \qquad P_0(x) = e, \qquad P_{-k}(x) = (x^{-1})^k, \qquad k \geqslant 1.$$

In particular, P_1 is the identity and P_{-1} is the inversion map, ν.

Example: The power maps, P_k, for an r-dimensional torus, T, are homomorphisms. Moreover P_k' is simply scalar multiplication by k. Thus it follows from sec. 4.13 that $P_k^\#$ is given by

$$P_k^\# \alpha = k^p \alpha, \qquad \alpha \in H^p(T).$$

In particular, the degree and Lefschetz number of P_k are given by

$$\deg P_k = k^r \qquad \text{and} \qquad L(P_k) = \sum_{p=0}^{r} (-1)^p \binom{r}{p} k^p = (1 - k)^r.$$

(cf. sec. 0.14, and note from sec. 4.13 that $\dim H^p(T) = \binom{r}{p}$.)

In the next sections we generalize these results to arbitrary compact connected Lie groups G.

Let $\mu: G \times G \to G$ and $\sigma: G \times G \to G \times G$ denote, respectively, the multiplication map and the interchange map $\sigma(x, y) = (y, x)$.

Proposition VI: With the notation above

(1) $\nu^\#(\alpha) = (-1)^p \alpha, \qquad \alpha \in H^p(G)$.

(2) The diagram

$$\begin{array}{ccc} H(G)\otimes H(G) & & \\ {\scriptstyle\sigma^*}\uparrow & \nwarrow{\scriptstyle\mu^*} & \\ & & H(G) \\ & \swarrow{\scriptstyle\mu^*} & \\ H(G)\otimes H(G) & & \end{array}$$

commutes.

(3) The diagrams

$$\begin{array}{ccc} H(G)\otimes H(G) & \xleftarrow{\mu^*} & H(G) \\ {\scriptstyle P_k^\#\otimes P_k^\#}\uparrow & & \uparrow{\scriptstyle P_k^*} \\ H(G)\otimes H(G) & \xleftarrow[\mu^*]{} & H(G), \end{array} \qquad k\in\mathbb{Z},$$

commute.

Proof: (1) This follows from Lemma V, sec. 4.9, and Theorem III, sec. 4.10.

(2) Since the inversion map $\nu_{G\times G}$ for the Lie group $G \times G$ is given by

$$\nu_{G\times G}(a, b) = (a^{-1}, b^{-1}),$$

it follows that $\mu \circ \sigma \circ \nu_{G\times G} = \nu \circ \mu$. Hence (1) yields, for $\alpha \in H^p(G)$,

$$(-1)^p\mu^\#\alpha = \mu^\#\nu^\#\alpha = (\nu^\#_{G\times G} \circ \sigma^\# \circ \mu^\#)\alpha = (-1)^p\sigma^\#\mu^\#\alpha.$$

(3) Let

$$\mu_p\colon \underbrace{G \times \cdots \times G}_{(p \text{ factors})} \to G$$

be the multiplication map and let σ_i be the diffeomorphism of $G \times \cdots \times G$ that interchanges the ith and the $(i+1)$-th component:

$$\sigma_i(x_1, \ldots, x_p) = (x_1, \ldots, x_{i+1}, x_i, \ldots, x_p).$$

Then (2) implies that $\sigma_i^\# \circ \mu_p^\# = \mu_p^\#$. It follows that if τ is any permutation of the elements $(1, \ldots, p)$ and if τ also denotes the diffeomorphism

$$\tau\colon (x_1, \ldots, x_p) \mapsto (x_{\tau(1)}, \ldots, x_{\tau(p)}),$$

then

$$\tau^\# \circ \mu_p^\# = \mu_p^\#.$$

Now fix $k \geqslant 1$ and define maps

$$\Delta_1, \Delta_2 : G \times G \to \underset{(2k \text{ factors})}{G \times \cdots \times G}$$

by

$$\Delta_1(x, y) = (x, ..., x, y, ..., y) \qquad \text{and} \qquad \Delta_2(x, y) = (x, y, ..., x, y).$$

Then

$$\mu_{2k} \circ \Delta_1 = \mu \circ (P_k \times P_k) \qquad \text{and} \qquad \mu_{2k} \circ \Delta_2 = P_k \circ \mu.$$

Since, for a suitable permutation τ, $\Delta_1 = \tau \circ \Delta_2$, and since $\tau^\# \circ \mu_{2k}^\# = \mu_{2k}^\#$, it follows that

$$\mu^\# \circ P_k^\# = \Delta_2^\# \circ \mu_{2k}^\# = \Delta_1^\# \circ \mu_{2k}^\# = (P_k \times P_k)^\# \circ \mu^\#.$$

The case $k \leqslant -1$ can be treated in the same way and the case $k = 0$ is obvious.

Q.E.D.

4.15. The Lefschetz class. In this section we assume that G is oriented. Denote its orientation class by $\omega_G \in H^n(G)$ (cf. sec. 0.14). Define the *quotient map*

$$q: G \times G \to G$$

by $q(a, b) = a^{-1}b$.

Proposition VII: The Lefschetz class, Λ_G, for G is given by

$$\Lambda_G = q^\# \omega_G$$

(cf. sec. 10.3, volume I).

Proof: Let π_L, π_R: $G \times G \to G$ be the left and right projections. It has to be shown (cf. Corollary I to Proposition I, sec. 10.3, volume I) that

$$\oint_G^\# \pi_R^\# \alpha \cdot q^\# \omega_G = \alpha, \qquad \alpha \in H(G).$$

Let φ be the diffeomorphism of $G \times G$ given by

$$\varphi(a, b) = (a, ab).$$

φ is a fibre preserving and orientation preserving map of the trivial bundle $(G \times G, \pi_L, G, G)$. Moreover, it induces the identity map in the base. Hence Proposition VIII, sec. 7.12, volume I, yields

$$\oint_G \circ \varphi^* = \oint_G ,$$

whence

$$\oint_G^{\#} \varphi^{\#}(\pi_R^{\#}\alpha \cdot q^{\#}\omega_G) = \oint_G^{\#} \pi_R^{\#}\alpha \cdot q^{\#}\omega_G , \qquad \alpha \in H(G).$$

But $q \circ \varphi = \pi_R$ and so this relation becomes

$$\oint_G^{\#} \varphi^{\#}\pi_R^{\#}\alpha \cdot \pi_R^{\#}\omega_G = \oint_G^{\#} \pi_R^{\#}\alpha \cdot q^{\#}\omega_G , \qquad \alpha \in H(G).$$

It remains to prove that

$$\alpha = \oint_G^{\#} \varphi^{\#}\pi_R^{\#}\alpha \cdot \pi_R^{\#}\omega_G , \qquad \alpha \in H(G).$$

Recall that we identify $H(G) \otimes H(G)$ with $H(G \times G)$ via the Künneth isomorphism $\kappa_{\#}$ (cf. sec. 0.14). It follows from Example 2, sec. 5.17, volume I, that if $\gamma \in H(G \times G)$, then

$$\gamma - j_1^{\#}\gamma \otimes 1 \in H(G) \otimes H^+(G),$$

where $j_1 : G \to G \times G$ is given by $j_1(a) = (a, e)$. Since $\omega_G \cdot H^+(G) = 0$, this yields

$$\gamma \cdot \pi_R^{\#}\omega_G = j_1^{\#}\gamma \otimes \omega_G = \pi_L^{\#} j_1^{\#}\gamma \cdot \pi_R^{\#}\omega_G .$$

Now set $\gamma = \varphi^{\#}\pi_R^{\#}\alpha$. Observing that $\pi_R \circ \varphi \circ j_1 = \iota$ we find that

$$\oint_G^{\#} \varphi^{\#}\pi_R^{\#}\alpha \cdot \pi_R^{\#}\omega_G = \oint_G^{\#} \pi_L^{\#}\alpha \cdot \pi_R^{\#}\omega_G = \alpha \cdot \int_G^{\#} \omega_G = \alpha$$

(cf. Example 2, sec. 7.12, volume I).

Q.E.D.

Corollary I: Let M be a compact connected oriented manifold and let $\varphi, \psi: M \to G$ ($\dim M = \dim G = n$) be smooth maps. Then the coincidence number (cf. sec. 0.14) for φ and ψ is given by

$$L(\varphi, \psi) = \deg(\varphi^{-1} \cdot \psi),$$

where $\varphi^{-1} \cdot \psi \colon M \to G$ is given by

$$(\varphi^{-1} \cdot \psi)(x) = \varphi(x)^{-1} \cdot \psi(x), \qquad x \in M.$$

Proof: Apply Proposition VII, sec. 10.7, volume I, noting that

$$\varphi^{-1} \cdot \psi = q \circ (\varphi \times \psi) \circ \Delta_M,$$

where $\Delta_M : M \to M \times M$ is the diagonal map.

Q.E.D.

Corollary II: Let $\varphi \colon G \to G$ be a smooth map and denote by $\varphi^{(p)}$ the restriction of φ^* to $H^p(G)$. Then the Lefschetz number of φ is given by

$$L(\varphi) = \sum_{p=0}^{n} (-1)^p \operatorname{tr} \varphi^{(p)} = \deg \varphi_1 ,$$

where $\varphi_1 = \varphi^{-1} \cdot \iota$.

Corollary III: Let $k \in \mathbb{Z}$. Then the Lefschetz number of the power map P_k is given by

$$L(P_k) = \deg P_{1-k} .$$

In particular, the Euler–Poincaré characteristic of G is 0 (set $k = 1$).

4.16. The spaces $H_p(G)$. Let T be a maximal torus in G and let $r = \dim T$. Recall that a smooth map $\psi \colon G/T \times T \to G$ is given by

$$\psi(\pi a, y) = aya^{-1}, \qquad a \in G, \quad y \in T,$$

where $\pi \colon G \to G/T$ denotes the projection (cf. sec. 2.17).

Clearly, the diagrams

$$\begin{array}{ccc} G/T \times T & \xrightarrow{\psi} & G \\ {\scriptstyle \iota \times \hat{P}_k} \downarrow & & \downarrow {\scriptstyle P_k} \\ G/T \times T & \xrightarrow[\psi]{} & G, \end{array} \qquad k \in \mathbb{Z}, \tag{4.5}$$

commute, where $\hat{P}_k$ denotes the power map for T.

These yield the commutative diagrams

$$\begin{array}{ccc} H(G/T)\otimes H(T) & \xleftarrow{\psi^*} & H(G) \\ \uparrow{\scriptstyle \iota\otimes\hat{P}_k^*} & & \uparrow{\scriptstyle P_k^*} \\ H(G/T)\otimes H(T) & \xleftarrow[\psi^*]{} & H(G), \end{array} \qquad k\in\mathbb{Z}. \tag{4.6}$$

Proposition VIII: Let $H_p(G)$ denote the eigenspace of the linear map $P_2^\# : H(G) \to H(G)$ corresponding to the eigenvalue 2^p. Then

(1) $H(G) = \sum_{p=0}^r H_p(G)$.

(2) For every $k \neq 0$, $H_p(G)$ is an eigenspace of the linear map, $P_k^\#$, corresponding to the eigenvalue k^p.

Proof: Recall from the example of sec. 4.14 that, for $\alpha \in H^p(T)$,

$$\hat{P}_k^\#(\alpha) = k^p \cdot \alpha.$$

Thus $H(G/T) \otimes H(T)$ is the direct sum of the eigenspaces $H(G/T) \otimes H^p(T)$ of $\iota \otimes \hat{P}_k^\#$ corresponding to the eigenvalues k^p $(p = 0, ..., r)$.

In view of the diagram above, Im $\psi^\#$ is stable under the map $\iota \otimes \hat{P}_k^\#$. This implies that

$$\operatorname{Im}\psi^\# = \sum_{p=0}^r \operatorname{Im}\psi^\# \cap [H(G/T) \otimes H^p(T)].$$

Next observe that, according to Proposition IV, sec. 2.18, $\deg \psi \neq 0$ and so $\psi^\#$ is injective (cf. Corollary I to Proposition III, sec. 6.5, volume I). Hence the relation above shows that

$$H(G) = \sum_{p=0}^r (\psi^\#)^{-1}(H(G/T) \otimes H^p(T))$$

and that $P_k^\#$ restricts to $k^p \cdot \iota$ in $(\psi^\#)^{-1}(H(G/T) \otimes H^p(T))$. In particular, it follows that

$$H_p(G) = (\psi^\#)^{-1}(H(G/T) \otimes H^p(T))$$

(consider the case $k = 2$), and so both parts of the proposition are obvious.

Q.E.D.

Corollary: $\mu^\#$ restricts to linear maps

$$\mu^\#: H_p(G) \to \sum_{i+j=p} H_i(G) \otimes H_j(G).$$

Proof: Apply Proposition VI (3), sec. 4.14.

Q.E.D.

Lemma VII: Each space $H_p(G)$ is graded,

$$H_p(G) = \sum_{q=0}^{n} H_p^q(G),$$

where $H_p^q(G) = H_p(G) \cap H^q(G)$. Moreover, if $p \not\equiv q \pmod 2$, then $H_p^q(G) = 0$.

Proof: The first part of the lemma is obvious. Now assume that $\alpha \in H_p^q(G)$. Then Proposition VI, (1), sec. 4.14, yields

$$(-1)^q\alpha = \nu^\#\alpha = P_{-1}^\#\alpha = (-1)^p\alpha.$$

Thus, if $p \not\equiv q \pmod 2$, $\alpha = 0$.

Q.E.D.

Proposition IX: The dimension of $H_p(G)$ is given by

$$\dim H_p(G) = \binom{r}{p}, \qquad 0 \leqslant p \leqslant r,$$

where $r = \dim T$.

Proof: First observe that, in view of the commutative diagram (4.5),

$$\deg \psi \cdot \deg(\iota \times \hat{P}_k) = \deg P_k \cdot \deg \psi$$

so that (cf. Proposition IV, sec. 2.18)

$$\deg P_k = \deg \hat{P}_k = k^r, \qquad k \in \mathbb{Z}.$$

Thus in view of Corollary III to Proposition VII, sec. 4.15, we have

$$L(P_k) = \deg P_{1-k} = (1-k)^r = \sum_{p=0}^{r} (-k)^p \binom{r}{p}.$$

On the other hand, Proposition VIII and Lemma VII give

$$L(P_k) = \sum_{p,q} (-1)^q k^p \dim H_p^q(G)$$

$$= \sum_{p,q} (-1)^p k^p \dim H_p^q(G) = \sum_{p=0}^{r} (-k)^p \dim H_p(G), \qquad k \in \mathbb{Z}.$$

These relations yield

$$\sum_{p=0}^{r} k^p \left(\dim H_p(G) - \binom{r}{p}\right) = 0, \qquad k = 1, 2, \dots .$$

Since the Vandermonde matrix (k^p) $(k = 1, \dots, r + 1, p = 0, \dots, r)$ has nonzero determinant, we obtain

$$\dim H_p(G) = \binom{r}{p}.$$

Q.E.D.

Corollary: $H_0(G) = H^0(G) = \mathbb{R}$.

Proposition X: The spaces $H_1(G)$ and P_G coincide.

Proof: Let $\alpha \in P_G$. Then, if $\Delta : G \to G \times G$ is the diagonal map,

$$P_2^\#(\alpha) = \Delta^\# \mu^\# \alpha = \Delta^\#(\alpha \otimes 1 + 1 \otimes \alpha) = 2\alpha,$$

whence $\alpha \in H_1(G)$.

On the other hand, if $\alpha \in H_1(G)$, the corollary to Proposition VIII implies that

$$\mu^\# \alpha \in H_1(G) \otimes \mathbb{R} + \mathbb{R} \otimes H_1(G).$$

Hence, by formula (4.4), sec. 4.12, $\alpha \in P_G$.

Q.E.D.

4.17. Proof and consequences of Theorem IV. Lemma VIII: If $\alpha_1, \dots, \alpha_k \in P_G$ are homogeneous and linearly independent, then

$$\prod_{i=1}^{k} \alpha_i \neq 0.$$

Proof: Suppose $\deg \alpha_i = p_i$ with $p_1 \leqslant \cdots \leqslant p_k$. Then the component of $\mu^\#(\alpha_1 \cdot \cdots \cdot \alpha_k)$ in $H^{p_1}(G) \otimes H^{p_2+\cdots+p_k}(G)$ is given by

$$\sum_i (-1)^{i-1} \alpha_i \otimes (\alpha_1 \cdot \cdots \hat{\alpha}_i \cdots \cdot \alpha_k),$$

where the sum ranges over those indices i such that $\deg \alpha_i = p_1$. We may assume by induction that $\alpha_1 \cdot \cdots \hat{\alpha}_i \cdots \cdot \alpha_k \neq 0$ ($i = 1, ..., k$). Since the α_i are linearly independent, it follows that

$$\sum_i (-1)^{i-1} \alpha_i \otimes (\alpha_1 \cdot \cdots \hat{\alpha}_i \cdots \cdot \alpha_k) \neq 0,$$

whence $\mu^\#(\alpha_1 \cdot \cdots \cdot \alpha_k) \neq 0$. In particular, $\alpha_1 \cdot \cdots \cdot \alpha_k \neq 0$.

Q.E.D.

Proof of Theorem IV: Lemma VIII implies that λ_G is injective. On the other hand, by Propositions IX and X, sec. 4.16,

$$\dim P_G = \dim H_1(G) = r \qquad \text{and} \qquad \dim H(G) = 2^r.$$

Thus $\dim \wedge P_G = \dim H(G)$, and so λ_G is an isomorphism.

Q.E.D.

Corollary I: The cohomology algebra of G is isomorphic to the cohomology algebra of the product of r spheres ($r = \dim T$) each of which has odd dimension.

Proof: Choose a homogeneous basis of P_G and denote by P_j the subspace generated by the jth basis vector. Then P_j is a graded one-dimensional vector space whose elements are homogeneous of degree g_j (g_j odd). Hence (cf. sec. 5.6 and sec. 5.20, volume I)

$$\begin{aligned} H(G) &\cong \wedge P_G \cong \wedge P_1 \otimes \cdots \otimes \wedge P_r \cong H(S^{g_1}) \otimes \cdots \otimes H(S^{g_r}) \\ &\cong H(S^{g_1} \times \cdots \times S^{g_r}). \end{aligned}$$

Q.E.D.

Corollary II: The Poincaré polynomial of G is of the form

$$f_G(t) = (1 + t^{g_1}) \cdots (1 + t^{g_r}), \qquad g_i \text{ odd}.$$

In particular

$$\sum_{q=1}^{n} (-1)^q \dim H^q(G) = 0, \quad \sum_{\mu=1}^{r} g_\mu = n, \quad \text{and} \quad n \equiv r \pmod 2 \qquad (n = \dim G).$$

Corollary III: The exponents g_i are all equal to 1 if and only if G is a torus.

Corollary IV: The isomorphism λ_G restricts to isomorphisms

$$\lambda_G^p\colon \wedge^p P_G \xrightarrow{\cong} H_p(G).$$

Proof: Since $P_2^\#$ is a homomorphism, it follows that

$$H_i(G) \cdot H_j(G) \subset H_{i+j}(G),$$

Thus Proposition X, sec. 4.16, implies that $\lambda_G(\wedge^p P_G) \subset H_p(G)$. But, in view of Proposition IX, sec. 4.16,

$$\dim H_p(G) = \binom{r}{p} = \dim \wedge^p P_G .$$

Q.E.D.

Corollary V: Let $\varphi\colon G \to K$ be a smooth map between compact connected Lie groups such that

$$(\varphi x)^2 = \varphi(x^2), \qquad x \in G.$$

Then $\varphi^\#$ restricts to a linear map $\varphi_P : P_G \leftarrow P_K$ and the diagram

$$\begin{array}{ccc} \wedge P_G & \xrightarrow[\cong]{\lambda_G} & H(G) \\ {\scriptstyle \wedge\varphi_P}\uparrow & & \uparrow{\scriptstyle \varphi^\#} \\ \wedge P_K & \xrightarrow[\lambda_K]{\cong} & H(K) \end{array}$$

commutes. In particular, if $K = G$, then

$$L(\varphi) = \det(\iota - \varphi_P).$$

Proof: Observe that $P_G = H_1(G)$ is the eigenspace of the map $P_2^\#$ corresponding to the eigenvalue 2, and conclude that $\varphi^\#(P_K) \subset P_G$. The commutativity of the diagram follows immediately. In view of Lemma VII, sec. 4.16, this implies that

$$L(\varphi) = \sum_{p=0}^{r} (-1)^p \operatorname{tr} \wedge^p \varphi_P = \det(\iota - \varphi_P).$$

Q.E.D.

§5. Homogeneous spaces

In this article K denotes a closed q-dimensional subgroup of G with Lie algebra F. The left action of G on G/K is denoted by $T: G \times G/K \to G/K$.

4.18. The representation $\mathrm{Ad}^{\perp}$. Since K is a subgroup of G, its Lie algebra F is stable under the operators $\mathrm{Ad}\, y$ $(y \in K)$. Thus the orthogonal complement, $F^{\perp}$, of F in E^* is stable under $\mathrm{Ad}^{\natural}(y)$ $(y \in K)$. The restrictions of these operators to $F^{\perp}$ define a representation

$$\mathrm{Ad}^{\perp}: K \to GL(F^{\perp}).$$

It extends to a representation, $\wedge \mathrm{Ad}^{\perp}$, of K in the exterior algebra $\wedge F^{\perp}$.

Now consider the projection, $\pi: G \to G/K$, and recall (Corollary I, sec. 2.11) that $(d\pi)_e$ induces a linear isomorphism

$$E/F \xrightarrow{\cong} T_{\bar{e}}(G/K).$$

Hence the dual map can be regarded as a linear isomorphism

$$(d\pi)_e^*: T_{\bar{e}}(G/K)^* \xrightarrow{\cong} F^{\perp}$$

and $\wedge(d\pi)_e^*$ is an isomorphism

$$\wedge T_{\bar{e}}(G/K)^* \xrightarrow{\cong} \wedge F^{\perp}.$$

Since $\pi(yxy^{-1}) = \pi(yx) = y \cdot \pi(x)$, $y \in K$, $x \in G$, we have

$$(d\pi)_e \circ \mathrm{Ad}\ y = (dT_y)_{\bar{e}} \circ (d\pi)_e ,$$

whence

$$\wedge \mathrm{Ad}^{\perp}(y^{-1}) \circ \wedge(d\pi)_e^* = \wedge(d\pi)_e^* \circ \wedge(dT_y)_{\bar{e}}^*, \qquad y \in K. \tag{4.7}$$

Next denote by $A_I(G/K)$ the algebra of differential forms on G/K invariant under the action of G. Since π is equivariant with respect to the left action of G on itself, π^* restricts to a homomorphism

$$\pi_I^*: A_I(G/K) \to A_L(G).$$

On the other hand, let $(\wedge F^{\perp})_I$ denote the invariant subalgebra of the representation $\wedge \mathrm{Ad}^{\perp}$. Relation (4.7) shows that if $\Phi \in A_I(G/K)$, then

$$\wedge (d\pi)_e^*(\Phi(\bar{e})) \in (\wedge F^{\perp})_I .$$

Thus a homomorphism, σ: $A_I(G/K) \to (\wedge F^{\perp})_I$, is defined by

$$\sigma(\Phi) = \wedge (d\pi)_e^*(\Phi(\bar{e})).$$

Recall the isomorphism, τ_L, of sec. 4.5.

Proposition XI: σ is an isomorphism of graded algebras which makes the diagram,

$$\begin{array}{ccc} A_L(G) & \xrightarrow[\cong]{\tau_L} & \wedge E^* \\ \uparrow{\scriptstyle \pi_I^*} & & \uparrow{\scriptstyle i} \\ A_I(G/K) & \xrightarrow[\sigma]{\cong} & (\wedge F^{\perp})_I \end{array},$$

commute (where i is the inclusion).

Proof: Evidently,

$$\sigma(\Phi) = (\pi^*\Phi)(e) = (\tau_L \pi_I^*)(\Phi), \qquad \Phi \in A_I(G/K),$$

and so the diagram commutes. Since π is a submersion, π^* is injective; it follows that σ is injective. It remains to prove that σ is surjective.

Fix $\alpha \in (\wedge F^{\perp})_I$ and let $\beta \in \wedge T_{\bar{e}}(G/K)^*$ be the unique element satisfying

$$\wedge (d\pi)_e^*(\beta) = \alpha.$$

Since α is invariant we have, for $y \in K$,

$$\wedge (dT_y)_{\bar{e}}^*(\beta) = \beta.$$

Thus a set map, Ψ: $G/K \to \wedge T_{G/K}^*$, is defined by

$$\Psi(\pi(x)) = \wedge (dT_{x^{-1}})_{\bar{e}}^*(\beta), \qquad x \in G.$$

To check that Ψ is a differential form, let $\Phi \in A_L(G)$ be the unique left invariant form such that $\Phi(e) = \alpha$. Then, for $x \in G$,

$$\Phi(x) = \wedge (d\pi)_x^* \Psi(\pi(x)).$$

Fix $\bar{x} \in G/K$ and let φ: $U \to G$ be a local cross-section, where U is a neighbourhood of $\bar{x}$ (cf. Corollary II, sec. 2.11). The relation just obtained implies that, in U, $\varphi^*\Phi = \Psi$. Hence Ψ is a differential form.

Ψ is clearly invariant and satisfies $\sigma(\Psi) = \alpha$. It follows that σ is surjective.

Q.E.D.

Corollary: Assume that K is compact and connected. Then G/K can be oriented by an invariant $(n - q)$-form.

Proof: Since K is compact and connected, $\det \mathrm{Ad}^{\perp}(y) = 1$, for $y \in K$ (cf. the example of sec. 1.13). It follows that

$$\dim A_I^{n-q}(G/K) = \dim(\wedge^{n-q}F^{\perp})_I = 1.$$

Every nonzero element of this space orients G/K.

Q.E.D.

4.19. Invariant cohomology. It is an immediate consequence of Proposition XI, sec. 4.18, that $(\wedge F^{\perp})_I$ is stable under the operator δ_E defined in sec. 4.6. Thus we have the commutative diagram

$$\begin{array}{ccc} H_L(G) & \xrightarrow[\cong]{(\tau_L)_*} & H(E) \\ {\scriptstyle \pi_I^*}\uparrow & & \uparrow{\scriptstyle i_*} \\ H_I(G/K) & \xrightarrow[\sigma_*]{\cong} & H((\wedge F^{\perp})_I, \delta_E). \end{array}$$

Applying Theorem I, sec. 4.3, we obtain

Theorem V: Suppose that G and K are compact and connected. Then, in the commutative diagram,

$$\begin{array}{ccccc} H(G) & \xleftarrow{\cong} & H_L(G) & \xrightarrow[\cong]{(\tau_L)_*} & H(E) \\ {\scriptstyle \pi^*}\uparrow & & {\scriptstyle \pi_I^*}\uparrow & & \uparrow{\scriptstyle i_*} \\ H(G/K) & \xleftarrow{\cong} & H_I(G/K) & \xrightarrow[\sigma_*]{\cong} & H((\wedge F^{\perp})_I), \end{array}$$

all horizontal maps are isomorphisms.

4.20. Invariant Euler-Poincaré characteristic. Suppose again that K is an arbitrary closed subgroup of the Lie group G.

Then the *invariant Euler–Poincaré characteristic* of G/K is defined by

$$\chi_I(G/K) = \sum_{p=0}^{n-q} (-1)^p \dim H_I^p(G/K) = \sum_{p=0}^{n-q} (-1)^p \dim H^p((\wedge F^\perp)_I).$$

Now assume that K is compact. Then there exists an inner product $\langle\ ,\ \rangle$ in E, invariant under the transformations $\operatorname{Ad} y$, $y \in K$ (cf. Proposition XVI, sec. 1.17). If we identify E^* with E under this inner product, then $F^\perp$ becomes the orthogonal complement of F in E, and we have the direct decomposition $E = F^\perp \oplus F$. Moreover, in this case $\operatorname{Ad}^\perp(y)$ is simply the restriction of $\operatorname{Ad} y$ to $F^\perp$.

Proposition XII: If K is compact, then

$$\chi_I(G/K) = \int_K \det(\iota - \operatorname{Ad}^\perp(y))\, dy.$$

Proof: It follows from Corollary III to Proposition XV, sec. 1.16, that

$$\int_K \det(\iota - \operatorname{Ad}^\perp(y))\, dy = \sum_{p=0}^{n-q} (-1)^p \dim(\wedge^p F^\perp)_I .$$

On the other hand, the algebraic Lefschetz formula (sec. 0.8) yields

$$\sum_{p=0}^{n-q} (-1)^p \dim(\wedge^p F^\perp)_I = \sum_{p=0}^{n-q} (-1)^p \dim H^p((\wedge F^\perp)_I) = \chi_I(G/K).$$

Q.E.D.

Corollary: If K is connected, then $\chi_I(G/K) \geqslant 0$. Equality holds if and only if, for every $y \in K$,

$$F^\perp \cap T_e(N_y) \neq 0,$$

where N_y denotes the normalizer of y.

Proof: Since K is compact and connected, $\operatorname{Ad}^\perp(y)$ is a proper rotation with respect to a suitable Euclidean inner product in $F^\perp$. Hence,

$$\det(\iota - \operatorname{Ad}^\perp(y)) \geqslant 0, \qquad y \in K.$$

Now the proposition shows that $\chi_I(G/K) \geqslant 0$ and $\chi_I(G/K) = 0$ if and only if

$$\det(\iota - \operatorname{Ad}^\perp(y)) = 0, \qquad y \in K;$$

i.e., if and only if, for every $y \in K$, there exists a nonzero vector $h \in F^{\perp}$ satisfying $\mathrm{Ad}^{\perp}(y)h = h$. But these are precisely the vectors of $F^{\perp} \cap T_e(N_y)$ (cf. Example 4, sec. 2.4).

Q.E.D.

4.21. Euler–Poincaré characteristic. Proposition XIII: Let K be a closed connected subgroup of a compact connected Lie group G. Then the Euler–Poincaré characteristic of G/K is given by

$$\chi_{G/K} = \int_K \det(\iota - \mathrm{Ad}^{\perp}(x))\, dx.$$

In particular, $\chi_{G/K} \geqslant 0$. Moreover, $\chi_{G/K} > 0$ if and only if G and K have the same rank, and in this case

$$\chi_{G/K} = |W_G|/|W_K|.$$

Proof: The first formula follows from Proposition XII and Theorem V, and shows that $\chi_{G/K} \geqslant 0$.

Let S be a maximal torus in K, and let L, F, and E denote the Lie algebras of S, K, and G. Write

$$E = F \oplus F^{\perp} = L \oplus (L^{\perp} \cap F) \oplus F^{\perp}.$$

Let $\mathrm{Ad}_K^{\perp}$, $\mathrm{Ad}^{\perp}$, and $\mathrm{Ad}_G^{\perp}$ denote the representations of S in $L^{\perp} \cap F$, $F^{\perp}$ and $(L^{\perp} \cap F) \oplus F^{\perp}$, induced by the adjoint representation of G; thus

$$\mathrm{Ad}_G^{\perp}(y) = \mathrm{Ad}_K^{\perp}(y) \oplus \mathrm{Ad}^{\perp}(y), \qquad y \in S.$$

The Weyl integration formula (cf. Theorem IV, sec. 2.19) yields

$$\int_K \det(\iota - \mathrm{Ad}^{\perp}(x))\, dx = |W_K|^{-1} \int_S \det(\iota - \mathrm{Ad}^{\perp}(y)) \cdot \det(\iota - \mathrm{Ad}_K^{\perp}(y))\, dy.$$

On the other hand, since $\mathrm{Ad}_G^{\perp}(y) = \mathrm{Ad}_K^{\perp}(y) \oplus \mathrm{Ad}^{\perp}(y)$, it follows that

$$\det(\iota - \mathrm{Ad}^{\perp}(y)) \cdot \det(\iota - \mathrm{Ad}_K^{\perp}(y)) = \det(\iota - \mathrm{Ad}_G^{\perp}(y)), \qquad y \in S.$$

Thus the first formula in the proposition applied to both G/S and G/K gives

$$\chi_{G/K} = |W_K|^{-1} \int_S \det(\iota - \mathrm{Ad}_G^{\perp}(y))\, dy = |W_K|^{-1} \chi_{G/S}.$$

Now assume that rank $K <$ rank G. Then S is not a maximal torus in G. Hence the corollary to Proposition XII implies that $\chi_{G/S} = 0$; it follows that $\chi_{G/K} = 0$.

On the other hand, if rank K = rank G, then S is a maximal torus in G. Now the first formula in the proposition (applied when $K = S$) together with the corollary to Theorem IV, sec. 2.20, yields $\chi_{G/S} = |W_G|$. This shows that

$$\chi_{G/K} = |W_G|/|W_K|.$$

Q.E.D.

Problems

1. **Left invariant p-vector fields.** A *p-vector* field on an n-manifold M is a cross-section in the vector bundle $\wedge^p \tau_M$. Denote the space of p-vector fields on M by $A_p(M)$ $(p = 0, \ldots, n)$.

(i) Given a Lie group G, use left multiplication to define a left action of G on τ_G, $\wedge^p \tau_G$ and on $A_p(G)$. A p-vector field Φ is called *left invariant*, if it is invariant under this action. The space of left invariant p-vector fields on G is denoted by $A_p^L(G)$.

(ii) Show that the map, $\tau^L: A_p^L(G) \to \wedge^p E$, given by evaluation at e, is an isomorphism.

(iii) Consider the space $D_p(G)$ of p-densities on G $(0 \leqslant p \leqslant n)$ and let $\partial: D_p(G) \to D_{p-1}(G)$ denote the divergence operator (cf. problem 8, Chap. IV, volume I). Show that an isomorphism,

$$\mu: A_p(G) \xrightarrow{\cong} D_p(G) \qquad (p = 0, \ldots, n),$$

is defined by $\mu(\Phi) = \Phi \otimes \Delta$, where Δ is a fixed nonzero left invariant n-form on G. Define an operator, $\partial_G : A_p(G) \to A_{p-1}(G)$, by

$$\partial_G = \mu^{-1} \circ \partial \circ \mu.$$

Show that ∂_G is independent of the choice of Δ. Show that ∂_G restricts to an operator in the space of left invariant multivector fields.

2. Let G be a Lie group with Lie algebra E.

(i) Use ∂_G (cf. problem 1) to obtain an operator ∂_E in $\wedge E$.

(ii) Show that ∂_E is explicitly given by

$$\partial_E(h_1 \wedge \cdots \wedge h_p) = \sum_{i<j} (-1)^{i+j+1} [h_i, h_j] \wedge h_1 \wedge \cdots \hat{h}_i \cdots \hat{h}_j \cdots h_p .$$

(iii) Show that the operators ∂_E and $-\delta_E$ are dual, where δ_E is the operator in $\wedge E^*$ defined in sec. 4.6.

(iv) Establish the Koszul formula

$$\delta_E = \frac{1}{2} \sum_\nu \mu(e^{*\nu}) \circ \theta_E(e_\nu),$$

where $\{e^{\nu}\}$, $\{e^{*\nu}\}$ is a pair of dual bases for E and E^*. *Hint*: Show that both sides are antiderivations in $\wedge E^*$.

(v) Find an analogous formula for ∂_E.

3. Define $\Phi \in A^1(GL^+(n; \mathbb{R}))$ by $\Phi(\alpha; \varphi) = \operatorname{tr}(\alpha^{-1} \circ \varphi)$, $\alpha \in GL^+(n; \mathbb{R})$, $\varphi \in L_{\mathbb{R}^n}$.

(i) Show that Φ is biinvariant.

(ii) Construct a scalar function f on $GL^+(n; \mathbb{R})$ such that $\delta f = \Phi$.

4. Let $T: M \times G \to M$ be a right action of a Lie group on a manifold M. Denote by $\theta(M)$ the subspace of $A(M)$ that is linearly generated by the differential forms $\theta(h)\Phi$, $h \in E$, $\Phi \in A(M)$.

(i) Prove the formula

$$\theta(h) \circ T_a^* - T_a^* \circ \theta(h) = \theta(h - \operatorname{Ad}(a)\, h) \circ T_a^*, \qquad a \in G, \quad h \in E.$$

Conclude that $\theta(M)$ is stable under T_a^*, $a \in G$.

(ii) If G is connected, show that

$$T_a^*\Phi - \Phi \in \theta(M), \qquad \Phi \in A(M).$$

(iii) Assume that G is compact and connected and let $\rho: A(M) \to A_I(M)$ denote the projection defined by

$$\rho(\Phi) = \int_G T_a^*\Phi \, da.$$

Prove that $\ker \rho = \theta(M)$, so that

$$A(M) = A_I(M) \oplus \theta(M) = A(M)_{\theta=0} \oplus \theta(M).$$

5. Let G_1, G_2 be Lie groups with Lie algebras E_1, E_2.

(i) Establish a canonical isomorphism $H(E_1 \oplus E_2) \cong H(E_1) \otimes H(E_2)$.

(ii) If E_1 is unimodular, show that multiplication in $H(E_1)$ determines nondegenerate scalar products, $H^p(E_1) \times H^{n-p}(E_1) \to \mathbb{R}$, where $n = \dim E_1$ (Poincaré duality).

(iii) Assume that G_1 and G_2 are connected and compact. Show that the Künneth isomorphism and the Poincaré isomorphisms correspond to the isomorphisms (i) and (ii) under the map of Theorem III.

6. Let $H \subset K \subset G$ be a sequence of compact connected Lie groups.

(i) Construct a subgroup $W_{K,G}$ of the Weyl group W_G and a surjective homomorphism $W_{K,G} \to W_K$ (cf. problem 25, Chap. II). If K has the same rank as G, show that this is an isomorphism; i.e., that W_K is a subgroup of W_G.

(ii) Show that $\chi_{G/K}$ is the index of W_K in W_G. Conclude that $\chi_{G/H} = \chi_{G/K} \cdot \chi_{K/H}$.

(iii) If L is a compact subgroup of G with 1-component L^0, show that

$$\chi_{G/L} \cdot |L/L^0| = \chi_{G/L^0}.$$

7. (i) Use the Weyl integration formula and residue calculus to show that the Poincaré polynomial of $U(n)$ is the coefficient of $(z_1 \cdot \cdots \cdot z_n)^{2n-1}$ in the polynomial, P, given by

$$n!\, P(z_1, \ldots, z_n) = \prod_{\nu,\mu=1}^{n} (tz_\nu + z_\mu) \cdot \prod_{1 \leqslant \nu \neq \mu \leqslant n} (z_\nu - z_\mu)$$

(t, a parameter). Show that the Poincaré polynomials of $U(2)$ and $U(3)$ are respectively given by

$$f(t) = (1+t)(1+t^3) \qquad \text{and} \qquad f(t) = (1+t)(1+t^3)(1+t^5).$$

(ii) Compute $H_I(SO(3))$ and verify that it coincides with $H(\mathbb{R}P^3)$.

(iii) Compute $H_L(SL(2;\mathbb{R}))$, $H_I(SL(2;\mathbb{R}))$, and $H(SL(2;\mathbb{R}))$.

8. Let E be the Lie algebra of a compact connected Lie group G.

(i) Show that $H^1(E) = Z_E^*$ and that $H(E) \cong \wedge Z_E^* \otimes H(E')$ (cf. problem 7, Chap. II). Interpret these statements in terms of $H(G)$.

(ii) Show that $H^3(G) = P_G \cap H^3(G)$, if $E = E'$.

(iii) Assume that G is not abelian and let K denote the Killing form of E (cf. problem 7, Chap. II). Define a 3-linear function Φ in E by

$$\Phi(h_1, h_2, h_3) = K(h_1, [h_2, h_3]), \qquad h_j \in E.$$

Show that Φ is skew-symmetric and depends only on the vectors in E'. Show that Φ is invariant, and conclude that it represents nonzero classes $\alpha_E \in H^3(E')$ and $\alpha_G \in H^3(G)$.

(iv) Show that the only spheres which are Lie groups are S^1 and S^3.

9. Conjugation. The set of elements in a Lie group G conjugate to a given element a is called the *conjugacy class of a*. The set of elements in the Lie algebra E of G of the form $(\mathrm{Ad}\, x)h$ (fixed h, all $x \in G$) is called the *conjugacy class of* h.

(i) Show that each conjugacy class is an embedded homogeneous space.

(ii) Show that "exp" maps the conjugacy class of h onto the conjugacy class of $\exp h$. Identify the sets of conjugacy classes in G (respectively, in E) with an orbit space of an action of G. Denote the second orbit space by E/G.

(iii) Assume G is compact and regard the elements of $(\vee E^*)_I$ as functions in E. Show that, for such a function f, $f(h)$ depends only on the conjugacy class of h.

(iv) In volume III it will be shown that, if G is compact and connected, then $(\vee E^*)_I$ is a polynomial algebra over a graded subspace Q_E with $\dim Q_E = \mathrm{rank}\, G$. Use this fact to obtain an embedding of E/G in $\mathbb{R}^r$ ($r = \mathrm{rank}\, G$).
Show that the image of the embedding contains an open set of $\mathbb{R}^r$.

(v) Assume G compact and connected. Show that an automorphism, τ, of G determines a homeomorphism $\bar{\tau}: E/G \to E/G$. Show that, for $f \in (\vee E^*)_I$, $\bar{h} \in E/G$,

$$f(\bar{\tau}\bar{h}) = ((\tau')^\vee f)(\bar{h}).$$

(vi) Let G, τ be as in (v). In volume III we shall construct a linear isomorphism $\lambda: P_G \xrightarrow{\cong} Q_E$ such that $\lambda \circ \tau^\# = (\tau')^\vee \circ \lambda$. Use this fact to conclude that $\tau^\#$ is the identity map of $H(G)$ if and only if $\bar{\tau}$ is the identity map of E/G.

10. Automorphisms. Let τ be an automorphism of a compact connected Lie group G. τ is called *inner* if, for some $a \in G$, $\tau(x) = axa^{-1}$, $x \in G$.

(i) Show that $\tau^\#(\alpha_G) = \alpha_G$, where α_G is the class defined in problem 8, (iii).

(ii) Let $Z(\tau) = \{x \in G \mid \tau(x) = x\}$. Show that $Z(\tau)$ is a compact Lie subgroup of G. If $G' \neq e$, show that $Z(\tau)$ contains a nontrivial 1-parameter subgroup of G'. If S is a maximal torus of $Z(\tau)$ conclude that its centralizer Z_S is a maximal torus of G.

(iii) Suppose $Z(\tau)$ contains a maximal torus of G. Prove that τ is inner. (*Hint*: Use problems 28 and 29, Chap. II).

(iv) Show that the following conditions are equivalent: (a) $\tau^\# = \iota$.

(b) τ is inner. (c) for each $x \in G$ there is some $a_x \in G$ such that $\tau(x) = a_x x a_x^{-1}$. (*Hint*: Use problem 9, (vi).)

11. Toral actions. Let a torus, T, with Lie algebra E act on a manifold M so that the isotropy subgroups are all different from T. Choose $h \in E$ so that the 1-parameter subgroup generated by h is dense in T.

(i) Show that the fundamental vector field Z_h has no zeros.

(ii) Give M a T-invariant Riemannian metric. Define a 1-form ω on M by

$$\omega(X) = \langle Z_h, Z_h\rangle^{-1}\langle Z_h, X\rangle, \qquad X \in \mathscr{X}(M).$$

Show that $i(h)\omega = 1$ and, for $a \in T$, $k \in E$,

$$T_a^*\omega = \omega, \qquad \theta(k)\omega = 0.$$

(iii) Set

$$A(M)_{i(h)=0} = \ker i(h), \qquad A(M)_{\theta(h)=0} = \ker \theta(h)$$

and

$$A(M)_{i(h)=0,\theta(h)=0} = A(M)_{i(h)=0} \cap A(M)_{\theta(h)=0}\,.$$

Show that the multiplication induces an isomorphism,

$$A(M)_{i(h)=0,\theta(h)=0} \otimes \wedge\omega \xrightarrow{\cong} A_I(M),$$

where $\wedge\omega$ denotes the exterior algebra over the one-dimensional space spanned by ω.

(iv) Show that $A(M)_{i(h)=0,\,\theta(h)=0}$ is stable under δ and that

$$\delta\omega \in A^2(M)_{i(h)=0,\theta(h)=0}\,.$$

Show that the differential operator d in the tensor product, induced by δ under the isomorphism of (iii), is given by ($p = \deg \Psi$)

$$d(\Phi \otimes 1 + \Psi \otimes \omega) = \delta\Phi \otimes 1 + \delta\Psi \otimes \omega + (-1)^p\, \delta\omega \wedge \delta\Psi \otimes 1.$$

(v) Obtain a short exact sequence of differential spaces

$$0 \longrightarrow A(M)_{i(h)=0,\theta(h)=0} \xrightarrow{\lambda} A_I(M) \xrightarrow{i(h)} A(M)_{i(h)=0,\theta(h)=0} \longrightarrow 0,$$

where λ is the inclusion map. Derive an exact triangle

$$\begin{array}{ccc} H(A(M)_{i(h)=0,\theta(h)=0}) & \xrightarrow{\quad\lambda_*\quad} & H(M) \\ {\scriptstyle D}\nwarrow & & \swarrow{\scriptstyle i(h)_*} \\ & H(A(M)_{i(h)=0,\theta(h)=0}). & \end{array}$$

If $\tau_h \in H^2(A(M)_{i(h)=0,\,\theta(h)=0})$ is the class represented by $\delta\omega$, show that

$$D(\alpha) = \tau_h \cdot \alpha, \qquad \alpha \in H(A(M)_{i(h)=0,\theta(h)=0}).$$

(vi) Show that $H(M)$ has finite dimension if and only if $H(A(M)_{i(h)=0,\,\theta(h)=0})$ has finite dimension.

(vii) Assume that $H(M)$ has finite dimension. Show that $\chi_M = 0$ (even if M is not compact). Show that the Lefschetz number of an equivariant map is zero.

(viii) If M is compact and dim $M = 4k$, prove that M has signature zero.

(ix) Show that any toral action on $\mathbb{R}^n$ has a fixed point.

12. Action on homogeneous spaces. Let G be a compact connected Lie group and let K be a closed connected subgroup. Let T be the action of G on G/K.

(i) Show that the isotropy subgroups are all conjugate to K. Hence show that each T_a has a fixed point if and only if

$$\bigcup_{a\in G} aKa^{-1} = G.$$

(ii) Let a be a generator of a maximal torus in G. Show that the fixed point set of T_a is finite (possibly empty). Show that the set of elements $a \in G$ such that T_a has only finitely many fixed points, is dense in G.

(iii) Obtain the results of the text and problem 6 on $\chi_{G/K}$ by considering the Lefschetz number of T_a, where a is a generator of a maximal torus.

(iv) If rank $G = r$, rank $K = s$, show that a subtorus of rank $r - s$ can act almost freely on G/K. Show that this is the maximum dimension for such an action.

13. Symmetric spaces. Let τ be an automorphism of a compact connected Lie group G such that $\tau^2 = \iota$. Let K be the 1-component of the subgroup of G left pointwise fixed by τ. Then G/K is called a *symmetric space of compact type with connected fibre.* We refer to it simply as a *symmetric space.* Denote the Lie algebras of G and K by E and F.

(i) Show that a compact connected Lie group is diffeomorphic to a symmetric space.

(ii) Let G/K be a symmetric space. Show that the restriction of δ_E to $(\wedge F^{\perp})_I$ is zero and conclude that

$$H(G/K) \cong (\wedge F^{\perp})_I\,.$$

(iii) Assume G is compact and connected. Show that there are elements $a \in G$ such that $\tau_a \neq \iota$, $\tau_a^2 = \iota$, where τ_a is conjugation by a. Let K be the 1-component of the centralizer of a. Show that $a \in K$ and that $\tau_a' = -\iota$ in $F^{\perp}$. Conclude that $(\wedge F^{\perp})_I$ and $H(G/K)$ are evenly graded (i.e., $(\wedge^p F^{\perp})_I = 0 = H^p(G/K)$ if p is even).

14. The representation of W_G. G is a compact connected Lie group with maximal torus T.

(i) By considering the projection $G/T \to G/N_T$, construct a smooth bundle $(G/T, \pi, G/N_T, W_G)$.

(ii) Show that G/N_T is the orbit space (cf. problem 6, Chap. 3) for a suitable free action of W_G on G/T.

(iii) From the action of W_G on G/T obtain a representation of W_G in $H(G/T) \otimes \mathbb{C}$. In volume III it will be shown that $H^p(G/T) = 0$, p odd. Use this fact to determine the character of this representation (cf. problem 12, Chap. I). Conclude that it is equivalent to the left regular representation of W_G (cf. problem 14, Chap. I).

(iv) Let W_G^+ be the subgroup of W_G that acts in G/T by orientation preserving diffeomorphisms. Show that W_G^+ is a normal subgroup of index 2 in W_G. Is it the only normal subgroup of index 2?

(v) Show that $H^+(G/N_T) = 0$.

15. Let G, T be as in problem 14, and consider the map $\psi: G/T \times T \to G$ of sec. 2.17.

(i) Construct an action of W_G on T (by conjugation). Hence obtain an action of W_G on $G/T \times T$ and construct a smooth bundle $(G/T \times_{W_G} T, \rho, G/N_T, T)$ (cf. problem 7, Chap. III).

(ii) Show that ψ factors to yield the following smooth map: $\bar{\psi}: G/T \times_{W_G} T \to G$. Show that $\deg \bar{\psi} = 1$.

(iii) Show that $H(G/T \times_{W_G} T)$ is isomorphic to the subalgebra of $H(G/T) \otimes H(T)$ whose elements are invariant under the action of W_G. Conclude that $\tilde{\psi}^{\#}: H(G) \to H(G/T \times_{W_G} T)$ is an isomorphism of graded algebras.

(iv) Show that the cohomology algebra of the total space of the bundle in (i) is isomorphic to the tensor product of the cohomology of fibre and base as algebras, but *not* as graded vector spaces.

16. Use the map ψ of problem 15 to obtain a smooth map

$$G/T \times S_F \to S_E$$

(S_F and S_E are the unit spheres in the Lie algebras of T and G). Compute the degree of this map.

17. Let G be a connected Lie group with Lie algebra E.

(i) Assume that G acts on M and N and that $\varphi, \psi: M \to N$ are equivariant smooth maps connected by an equivariant homotopy H. Conclude that the homomorphisms $\varphi^*_{i=0,\theta=0}$ and $\psi^*_{i=0,\theta=0}$ (respectively, $(\varphi^*_{i=0})_I$ and $(\psi^*_{i=0})_I$) are homotopic.

(ii) Let U be a suitable tubular neighbourhood of an orbit G/K of G under a proper action (cf. problem 11, Chap. III). Show that the orbit space U/G is homeomorphic to the cone over an orbit space S/K, where K acts on a sphere S by orthogonal transformations. (The *cone* over a space X is obtained from $X \times [0, 1]$ by identifying the points $(x, 1)$, $x \in X$.)

(iii) Let U be as in (ii). Construct an equivariant retraction ρ of U onto the orbit and show that $i \circ \rho$ is equivariantly homotopic to the identity map of U. Hence find isomorphisms

$$H_I(U) \cong H_I(G/K) \cong H((\wedge F^\perp)_I)$$

and

$$H((A(U)_{i=0})_I) \cong H(\text{point})$$

(F denotes the Lie algebra of K).

(iv) Establish a Mayer–Vietoris axiom and a disjoint union axiom for $H_I(M)$ and $H(A(M)_{i=0,\theta=0})$ (with respect to proper actions of a fixed Lie group).

(v) Assume that G acts properly on M and that, for all isotropy subgroups K, $H(G/K) = H_I(G/K)$. Conclude that $H(M) = H_I(M)$.

18. Čech cohomology. Let G act on M. Establish a bijection between open coverings of M/G and G-stable open coverings of M. If the action is proper, define an isomorphism

$$\check{H}(M/G) \xrightarrow{\cong} H(A_I(M)_{i=0}),$$

where $\check{H}(M/G)$ denotes the Čech cohomology of M/G (cf. problem 25, Chap. V, volume I).

19. Equivariant cohomology of sphere and vector bundles. Generalize as far as possible the results of Chaps. VIII and IX, volume I, to the equivariant case (i.e., invariant cohomology and proper actions). In particular, define equivariant Gysin and Thom classes.

20. Give an elementary example where the orbit space of an action of a compact connected Lie group on a compact connected manifold does not satisfy Poincaré duality.

21. Represent S^1 in $\mathbb{C}^n$ by

$$e^{i\theta} \cdot (z_1, \ldots, z_n) = (e^{ik_1\theta} z_1, \ldots, e^{ik_n\theta} z_n), \qquad z_r \in \mathbb{C}, \quad \theta \in \mathbb{R},$$

where the k_r are integers with greatest common divisor 1. Obtain an action of S^1 on S^{2n-1}. Find the fundamental vector field and determine $H(A(S^{2n-1})_{i=0,\,\theta=0})$. Show that any equivariant smooth map $\varphi\colon S^{2n-1} \to S^{2n-1}$ has degree 1.