Chapter III

Transformation Groups

In this chapter G denotes a fixed Lie group with unit element e and Lie algebra E. M and N denote smooth manifolds.

§1. Action of a Lie group

3.1. Definition: A *right action* of G on a manifold M (or a set V) is a smooth map

$$T: M \times G \to M$$

(or a set map $V \times G \to V$), written $(z, a) \mapsto z \cdot a$, and satisfying

$$z \cdot (ab) = (z \cdot a) \cdot b \qquad \text{and} \qquad z \cdot e = z, \qquad a, b \in G, \quad z \in M.$$

The group G is said to act *transitively* on M if, for every two points $z_1, z_2 \in M$, there is an element $a \in G$ such that $z_1 \cdot a = z_2$.

An action T determines the diffeomorphisms T_a ($a \in G$) of M given by

$$T_a(z) = z \cdot a = T(z, a).$$

(Note that $T_a^{-1} = T_{a^{-1}}$.) T_a is called *right translation by a*.

On the other hand, to each $z \in M$, corresponds the smooth map $A_z: G \to M$ given by

$$A_z(a) = z \cdot a, \qquad a \in G.$$

It satisfies the relations

$$T_b \circ A_z = A_z \circ \rho_b \qquad \text{and} \qquad A_{z \cdot b} = A_z \circ \lambda_b = T_b \circ A_z \circ \tau_b, \qquad b \in G, \quad z \in M$$

(τ_b denotes conjugation in G by b).

Now assume $\hat{T}$ is a right action of G on N. Then a smooth map $\varphi\colon M \to N$ is called *equivariant* with respect to T and $\hat{T}$ if the diagram

$$\begin{array}{ccc} M \times G & \xrightarrow{T} & M \\ {\scriptstyle \varphi\times\iota}\downarrow & & \downarrow{\scriptstyle \varphi} \\ N \times G & \xrightarrow[\hat{T}]{} & N \end{array}$$

commutes. This is equivalent to each of the following three conditions

$$\varphi(z \cdot a) = \varphi(z) \cdot a, \qquad z \in M, \quad a \in G,$$

$$\varphi \circ T_a = \hat{T}_a \circ \varphi, \qquad a \in G,$$

and

$$\varphi \circ A_z = \hat{A}_{\varphi(z)}\,, \qquad z \in M.$$

(For $y \in N$, $\hat{A}_y\colon G \to N$ is the map $a \mapsto y \cdot a$.)

A *left action* of G on M is a smooth map

$$T\colon G \times M \to M,$$

written $T(a, z) = a \cdot z$, and such that

$$(ab) \cdot z = a \cdot (b \cdot z) \qquad \text{and} \qquad e \cdot z = z, \qquad a, b \in G, \quad z \in M.$$

The diffeomorphism $T_a\colon z \mapsto a \cdot z$ of M is called *left translation by a*. The smooth maps $A_z\colon G \to M$ $(z \in M)$ given by

$$A_z(a) = a \cdot z$$

satisfy

$$T_b \circ A_z = A_z \circ \lambda_b \qquad \text{and} \qquad A_{b\cdot z} = A_z \circ \rho_b = T_b \circ A_z \circ \tau_b^{-1}.$$

Finally, if $\hat{T}$ is a left action of G on N, then $\varphi\colon M \to N$ is called *equivariant* if

$$\varphi(a \cdot z) = a \cdot \varphi(z), \qquad a \in G, \quad z \in M.$$

3.2. Examples: 1. The multiplication map $\mu\colon G \times G \to G$ of a Lie group G is both a left and right action of G on itself. The left and right translations by $a \in G$ are simply λ_a and ρ_a.

2. The group $G \times G$ acts from the left on G by

$$T((a, b), z) = azb^{-1}, \qquad (a, b) \in G \times G, \quad z \in G.$$

3. A right action, $\tilde{T}$, of G on $M \times G$ (M, any manifold) is given by

$$\tilde{T}((z, a), b) = (z, ab).$$

If T is *any* right action of G on M, then T is equivariant with respect to $\tilde{T}$ and T.

4. A left action of G on G is given by

$$a \cdot z = aza^{-1}.$$

5. A representation, P, of G in a vector space V defines a left action of G on V:

$$a \cdot v = P(a)v, \qquad a \in G, \quad v \in V.$$

6. Assume that a Lie group H acts from the left on a Lie group G. H is said to *act via homomorphisms*, if each map T_a: $G \to G$ ($a \in H$) is a homomorphism (and hence an automorphism) of G. Assuming that H acts on G via homomorphisms, define a multiplication on the product manifold $H \times G$ by

$$\mu((a, x), (b, y)) = (ab, T_b^{-1}(x)y), \qquad a, b \in H, \quad x, y \in G.$$

It is easy to verify that this multiplication makes $H \times G$ into a Lie group. It is called the *semidirect product of H and G* (with respect to the action T) and is denoted by $H \times_T G$. If the action, T, is *trivial*, ($T_a = \iota$, $a \in H$), the semidirect product is simply the direct product. In any case, $H \times e$ is a closed subgroup of $H \times_T G$, while $e \times G$ is a closed *normal* subgroup.

7. If T: $M \times G \to M$ is an action of G on M, then

$$dT: T_M \times T_G \to T_M$$

is an action of the tangent group T_G (cf. Example 5, sec. 1.4) on T_M. In particular, identify G with the zero vectors in T_G to obtain an action

$$T_M \times G \to T_M$$

of G on T_M. It is given explicitly by

$$\xi \cdot a = dT_a(\xi), \qquad \xi \in T_M, \quad a \in G.$$

8. If $M \times G \to M$ is an action of G on M, a subset $N \subset M$ is called *stable* if

$$z \cdot a \in N, \qquad z \in N, \quad a \in G.$$

If N is stable, the action restricts to a set map $N \times G \to N$. In particular, if N is a stable submanifold of M, this map is smooth (cf. Proposition VI, sec. 3.10, volume I) and hence it is a smooth action of G on N.

As an example, suppose $P\colon G \to O(V)$ represents G by isometries in a Euclidean space V. Then the unit sphere S of V is stable, and so the linear action of G in V restricts to an action $G \times S \to S$.

9. A right action, $T_R\colon M \times G \to M$, determines an *associated left action*, T_L, given by

$$T_L(a, z) = T_R(z, a^{-1}), \qquad z \in M, \quad a \in G.$$

3.3. Action on a homogeneous space. Let K be a closed subgroup of G and consider the homogeneous space G/K of left cosets. Then a left action T of G on G/K is given by

$$T(a, \bar{x}) = a \cdot \bar{x}, \qquad a \in G, \quad \bar{x} \in G/K$$

(cf. sec. 2.11). The projection $\pi\colon G \to G/K$ is equivariant with respect to the left action of G on itself, and T. The action of G on G/K is transitive. In fact, let $\bar{x}_1 = \pi x_1$ and $\bar{x}_2 = \pi x_2$ be arbitrary and set $a = x_2 x_1^{-1}$. Then $a \cdot \bar{x}_1 = \bar{x}_2$.

Similarly, a right action of G is defined on the space of right cosets.

Next consider the normalizer N_K of K (cf. Example 4, sec. 2.4). A right action

$$S\colon G/K \times N_K \to G/K$$

is given by

$$S(\bar{x}, a) = \overline{xa}, \qquad x \in G, \quad a \in N_K.$$

(Since $a \in N_K$, this map is well defined.)

To see that it is smooth, observe that the diagram

$$\begin{array}{ccc} G \times N_K & \xrightarrow{\ \mu\ } & G \\ {\scriptstyle \pi \times \iota}\downarrow & & \downarrow{\scriptstyle \pi} \\ G/K \times N_K & \xrightarrow[S]{} & G/K \end{array},$$

commutes and recall that π makes G/K into a quotient manifold of G. The diagram also shows that the projection π is equivariant with respect to the right actions of N_K on G and on G/K.

Finally, since K is a closed normal subgroup of N_K, we can form the factor group N_K/K. The action S factors over the projection

$$\rho\colon N_K \to N_K/K$$

to give a smooth commutative diagram

$$\begin{array}{ccc} G/K \times N_K & \xrightarrow{\quad S \quad} & G/K \\ & {}_{\iota \times \rho}\searrow \quad \nearrow_{\bar{S}} & \\ & G/K \times N_K/K. & \end{array}$$

Thus $\bar{S}$ is a right action of N_K/K on G/K.

§2. Orbits of an action

In this article, $T: M \times G \to M$ denotes a right action of G on M.

3.4. The isotropy subgroup. Every point $z \in M$ determines the closed subgroup $G_z \subset G$ given by

$$G_z = \{a \in G \mid z \cdot a = z\}.$$

Since G_z is closed, it is a Lie subgroup of G (cf. Theorem I, sec. 2.1). It is called the *isotropy subgroup at* z. If $G_z = \{e\}$ (respectively, G_z is discrete), for each $z \in M$, the action is called *free* (respectively, *almost free*).

Proposition I: The Lie algebra E_z of the isotropy group G_z is given by

$$E_z = \ker(dA_z)_e .$$

Proof: Since the restriction of A_z to G_z is constant, it follows that $E_z \subset \ker(dA_z)_e$. Conversely, assume that $h \in \ker(dA_z)_e$. To show that $h \in E_z$ we must prove that $\exp th \in G_z$, $t \in \mathbb{R}$.

But the path in M given by $\beta(t) = z \cdot \exp th$ satisfies

$$\dot{\beta}(t) = (dA_z \circ R_{\exp th})(h) = (dT_{\exp th} \circ dA_z)(h) = 0, \qquad t \in \mathbb{R},$$

(cf. sec. 3.1). It follows that $z \cdot \exp th = z$ and so $\exp th \in G_z$.

Q.E.D.

Corollary: The action is almost free if and only if each $(dA_z)_e$ is injective.

3.5. Orbits. For $z \in M$ the set $z \cdot G$ $(= \operatorname{Im} A_z)$ is called the *orbit of G through z.* M is the disjoint union of its orbits. Clearly, if G acts transitively on M, then M consists of a single orbit.

Let z, $z \cdot a$ be points in the same orbit. Then $G_{z \cdot a} = a^{-1}G_z a$. In particular, if the action is transitive, any two isotropy groups are conjugate.

Next observe that the relation $A_z(ab) = A_z(a) \cdot b$ shows that A_z

factors over the projection $\pi\colon G \to G_z\backslash G$ to yield a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{A_z} & M \\ {\scriptstyle\pi}\big\downarrow & \nearrow{\scriptstyle \bar{A}_z} & \\ G_z\backslash G & & \end{array}$$

Since $G_z\backslash G$ is a quotient manifold of G under π, the map $\bar{A}_z$ is smooth. Moreover, $\bar{A}_z$ is equivariant with respect to the right actions of G on $G_z\backslash G$ and M.

Proposition II: $\bar{A}_z$ embeds the homogeneous space $G_z\backslash G$ into M, with image the orbit $z \cdot G$.

Proof: $\bar{A}_z$ is obviously injective, and has image $z \cdot G$. Thus we need only show that the linear maps

$$(d\bar{A}_z)_{\bar{a}}\colon T_{\bar{a}}(G_z\backslash G) \to T_{z\cdot a}(M), \qquad \bar{a} \in G_z\backslash G,$$

are injective. In view of the equivariance of $\bar{A}_z$, it is sufficient to consider the case $\bar{a} = \bar{e}$. But it follows from Proposition I, sec. 3.4, and Corollary I of sec. 2.11, that

$$\ker(dA_z)_e = E_z = \ker(d\pi)_e .$$

Hence $(d\bar{A}_z)_{\bar{e}}$ is injective.

Q.E.D.

Corollary: If G acts transitively on M, then $\bar{A}_z$ is a diffeomorphism of $G_z\backslash G$ onto M.

Proof: Apply Proposition IV, sec. 3.8, volume I.

3.6. Examples. **1.** Consider the right action T of G on itself by conjugation,

$$T(z, a) = a^{-1}za, \qquad z, a \in G.$$

The orbits of G under this action are called the *conjugacy classes of* G. Two elements z_1, z_2 are in the same orbit if and only if for some $a \in G$

$$a^{-1}z_1a = z_2 .$$

In this case they are called *conjugate*.

On the other hand, the isotropy subgroup at $a \in G$ is the normalizer N_a. Thus Proposition II, sec. 3.5, gives an embedding of $N_a \backslash G$ into G, with image the conjugacy class of a.

2. Let V be an n-dimensional Euclidean space. A left action T of $SO(n)$ on V is defined by

$$T(\sigma, z) = \sigma(z), \qquad \sigma \in SO(n), \quad z \in V.$$

The orbit of a point $a \in V$ ($a \neq 0$) is the sphere $\{x \in V \mid |x| = |a|\}$, while the orbit of 0 consists only of 0.

The action T restricts to a transitive action of $SO(n)$ on the unit sphere S^{n-1}. The isotropy subgroup of a point $x \in S^{n-1}$ is the subgroup $SO(x^\perp)$, where $x^\perp$ denotes the orthogonal complement of x. Hence T induces an equivariant diffeomorphism (cf. the corollary to Proposition II, sec. 3.5):

$$SO(n)/SO(n-1) \xrightarrow{\cong} S^{n-1}, \qquad n \geqslant 2.$$

3. By replacing the Euclidean space, V, of Example 2 with a Hermitian space, W, we obtain an action of $U(n)$ with orbits the spheres of W. In particular, this yields an equivariant diffeomorphism

$$U(n)/U(n-1) \xrightarrow{\cong} S^{2n-1}, \qquad n \geqslant 1.$$

The action of $U(n)$ on W induces an action of the special unitary group $SU(n)$ which restricts to a transitive action on S^{2n-1} for $n \geqslant 2$.

Finally, the use of a quaternionic space leads to equivariant diffeomorphisms

$$Q(n)/Q(n-1) \xrightarrow{\cong} S^{4n-1}, \qquad n \geqslant 1.$$

Proposition III: The groups $SO(n)$, $U(n)$, $SU(n)$, and $Q(n)$ are connected.

Proof: $SO(1)\,(= \iota)$ is connected. Assume by induction that $SO(n-1)$ is connected ($n \geqslant 2$). Then, in view of Example 2, $SO(n)/SO(n-1)$ is also connected. Since (cf. sec. 2.13)

$$(SO(n), \pi, SO(n)/SO(n-1), SO(n-1))$$

is a fibre bundle, it follows that $SO(n)$ is connected and the induction is closed.

The same argument, using Example 3 above shows that $U(n)$, $SU(n)$, and $Q(n)$ are connected.

Q.E.D.

Corollary: $O(n)$ has two components (cf. Example 2, sec. 2.5).

3.7. Embedding of orbits. Consider the injective map of sec. 3.5,

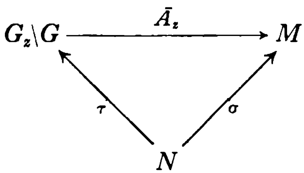
$$\bar{A}_z\colon G_z\backslash G \to M.$$

In general, the pair $(G_z\backslash G, \bar{A}_z)$ is not a submanifold of M as the following example shows: Let $\mathbb{R}$ act on the 2-torus T^2 by setting

$$T_t\pi(x, y) = \pi(x + at, y + bt), \qquad t, x, y \in \mathbb{R},$$

where $\pi\colon \mathbb{R}^2 \to T^2$ denotes the projection and b/a is irrational. Then each orbit is dense in T^2 and so the orbits are not submanifolds of T^2.
Nonetheless we have

Theorem I: With the notation above, let



be a commutative diagram. Then σ is smooth if and only if τ is.

For the proof of this theorem we first establish four lemmas. In view of Corollary II to Theorem II, sec. 2.11, we can find a submanifold W_1 of G such that $e \in W_1$, and the projection $\pi\colon G \to G_z\backslash G$ restricts to a diffeomorphism of W_1 onto a neighbourhood of $\bar{e}$.

Lemma I: There is a submanifold V of M containing z and a connected neighbourhood W of e in W_1, and a neighbourhood U of z in M such that T restricts to a diffeomorphism

$$\psi\colon V \times W \xrightarrow{\cong} U.$$

Proof: Choose a submanifold V_1 of M such that $z \in V_1$ and

$$T_z(M) = T_z(V_1) \oplus \operatorname{Im}(dA_z)_e\,.$$

Write $T_{(z,e)}(V_1 \times W_1) = T_z(V_1) \oplus T_e(W_1)$ and note that

$$dT(\xi, \eta) = \xi + (d\bar{A}_z \circ d\pi)(\eta), \qquad \xi \in T_z(V_1), \quad \eta \in T_e(W_1).$$

In view of Proposition II, sec. 3.5,

$$T_e(W_1) \xrightarrow[\cong]{d\pi} T_{\bar{e}}(G_z \backslash G) \xrightarrow[\cong]{d\bar{A}_z} \operatorname{Im}(dA_z)_e .$$

It follows that dT maps $T_{(z,e)}(V_1 \times W_1)$ isomorphically onto $T_z(M)$. The lemma follows (cf. Theorem I, sec. 3.8, volume I) for suitably small neighbourhoods $V \subset V_1$ and $W \subset W_1$.

Q.E.D.

Lemma II: Suppose that, in the notation of Lemma I, $\psi(y, b) = z \cdot a$ for some $y \in V$, $b \in W$, $a \in G$. Then

$$(d\psi)_{(y,b)}(T_b(W)) = (dA_z)(T_a(G)).$$

Proof: Set $c = ab^{-1}$. Since $\psi(y, b) = y \cdot b$, we have

$$y = z \cdot ab^{-1} = z \cdot c.$$

Since the restriction of ψ to $\{y\} \times W$ is simply A_y $(= A_{z \cdot c})$, it follows that

$$\begin{aligned}(d\psi)_{(y,b)}(T_b(W)) &= (dA_{z \cdot c})(T_b(W)) \subset (dA_{z \cdot c})(T_b(G)) \\ &= (dA_z)(T_a(G)).\end{aligned}$$

Moreover, combining Proposition II, sec. 3.5, with Lemma I, we obtain

$$\dim(d\psi)_{(y,b)}(T_b(W)) = \dim W = \dim(G_z \backslash G) = \dim dA_z(T_a(G)).$$

The lemma follows.

Q.E.D.

Lemma III: Let S denote the subset of V given by

$$S = \{y \in V \mid \psi(y, b) \in z \cdot G \text{ for some } b \in W\}.$$

Then S is countable.

Proof: Consider the open subset $O \subset G_z \backslash G$ given by (cf. Lemma I for U)

$$O = \bar{A}_z^{-1}(U).$$

Let $\varphi\colon O \to V$ be the composite given by

$$O \xrightarrow{\bar{A}_z} U \xrightarrow[\cong]{\psi^{-1}} V \times W \xrightarrow{\pi_V} V.$$

We show that $d\varphi = 0$.

In fact, let $\bar{a} \in O$ and let $\xi \in T_{\bar{a}}(O)$. Then we can write $z \cdot a = \psi(y, b)$ for some $y \in V$, $b \in W$. By Lemma II there exists an $\eta \in T_b(W)$ such that

$$(d\bar{A}_z)(\xi) = (d\psi)_{(y,b)}(\eta).$$

This yields

$$(d\varphi)\xi = (d\pi_V)(d\psi)^{-1}(d\bar{A}_z)\xi = (d\pi_V)\eta = 0,$$

whence $d\varphi = 0$.

Thus φ must be constant on each of the (countably many) components of O. Since $S = \operatorname{Im} \varphi$, S is a countable set.

Q.E.D.

Lemma IV: Give $(z \cdot G) \cap U$ the subspace topology induced from U. Then

$$\psi(\{z\} \times W) = z \cdot W$$

is a component of $(z \cdot G) \cap U$.

Proof: It is sufficient to show that $\{z\} \times W$ is a component of $\psi^{-1}((z \cdot G) \cap U)$. But

$$\psi^{-1}((z \cdot G) \cap U) = S \times W.$$

Moreover, in view of Lemma III,

$$S \times W = \bigcup_{i=0}^{\infty} (\{y_i\} \times W),$$

with $y_0 = z$. Since W is connected, the lemma follows.

Q.E.D.

3.8. Proof of Theorem I: If τ is smooth, then so is $\sigma = \bar{A}_z \circ \tau$. Conversely, assume that σ is smooth. Translating by elements of G allows us to restrict ourselves to proving that τ is smooth near those points $q \in N$ such that

$$\tau(q) = \bar{e} \qquad \text{and} \qquad \sigma(q) = z.$$

Choose U, V, W, and ψ as in sec. 3.7. Let Q be a connected neighbourhood of q such that

$$Q \subset \sigma^{-1}(U).$$

Restrict σ to a continuous map

$$\sigma_1 : Q \to (z \cdot G) \cap U,$$

where $(z \cdot G) \cap U$ is given the subspace topology. Since Q is connected, so is $\sigma_1(Q)$. Moreover,

$$\sigma_1(q) = z \in \psi(\{z\} \times W).$$

Thus Lemma IV yields

$$\operatorname{Im} \sigma_1 \subset \psi(\{z\} \times W).$$

In particular, the map $\psi^{-1} \circ \sigma : Q \to V \times W$ has the form

$$(\psi^{-1} \circ \sigma)(x) = (z, \chi(x)),$$

where $\chi : Q \to W$ is a (necessarily) smooth map. Moreover, the smooth map $\bar{\chi} : Q \to G_z \backslash G$ given by $\bar{\chi} = \pi \circ \chi$ satisfies

$$\begin{aligned}(\bar{A}_z \circ \bar{\chi})(x) &= (A_z \circ \chi)(x) = z \cdot \chi(x) \\ &= \psi(z, \chi(x)) = \sigma(x) = (\bar{A}_z \circ \tau)(x), \qquad x \in Q.\end{aligned}$$

Since $\bar{A}_z$ is injective, we obtain $\bar{\chi} = \tau$. It follows that τ is smooth in Q.
Q.E.D.

§3. Vector fields

In this article $T: M \times G \to M$ denotes a right action of G on M.

3.9. Fundamental vector fields. The action T determines the strong bundle map,

$$\alpha: M \times E \to T_M,$$

given by

$$\alpha(z, h) = (dT)_{(z,e)}(0_z, h) = dA_z(h).$$

Differentiating the relation $T_a \circ A_z = A_{z \cdot a} \circ \tau_a^{-1}$ (τ_a denotes conjugation by a) yields the commutative diagram

$$\begin{array}{ccc} M \times E & \xrightarrow{\alpha} & T_M \\ {\scriptstyle T_a \times \mathrm{Ad}\, a^{-1}} \downarrow & & \downarrow {\scriptstyle dT_a} \\ M \times E & \xrightarrow[\alpha]{} & T_M \end{array}, \qquad a \in G. \tag{3.1}$$

Now fix $h \in E$. The constant map $M \to \{h\}$ corresponds, under α, to the vector field Z_h on M given by

$$Z_h(z) = dA_z(h), \qquad z \in M.$$

It is called the *fundamental vector field generated by h*. The orbits of Z_h are the paths in M given by

$$t \mapsto z \cdot \exp th.$$

More generally, α induces the homomorphism

$$\alpha_*: \mathscr{S}(M; E) \to \mathscr{X}(M),$$

given by

$$(\alpha_* f)(z) = \alpha(z, f(z)) = dA_z(f(z)), \qquad z \in M. \qquad f \in \mathscr{S}(M; E)$$

We denote $\alpha_* f$ by Z_f and call it the *vector field generated by the function f*. Thus

$$Z_f(z) = Z_{f(z)}(z), \qquad z \in M.$$

Now let $\hat{T}$: $N \times G \to N$ be a right action of G on N and let φ: $M \to N$ be a smooth equivariant map. Then the diagram,

$$\begin{array}{ccc} M \times E & \xrightarrow{\alpha} & T_M \\ {\scriptstyle \varphi \times \iota}\downarrow & & \downarrow{\scriptstyle d\varphi} \\ N \times E & \xrightarrow[\hat{\alpha}]{} & T_N \end{array} \quad ,$$

commutes. In particular, the fundamental fields on M and on N, generated by a vector $h \in E$, are φ-related.

Example: Consider the action of G on itself by ***right*** translations. The fundamental vector fields are precisely the *left* invariant vector fields (cf. sec. 1.2).

To see this, observe that in this case $A_z = \lambda_z$, $z \in G$. It follows that

$$dA_z(h) = L_z(h) = X_h(z), \qquad z \in G, \quad h \in E,$$

whence $Z_h = X_h$.

More generally, if G acts on $M \times G$ (M, any manifold) by right translations of G, then the fundamental fields are given by

$$Z_h(y, x) = X_h(x), \qquad h \in E, \quad y \in M, \quad x \in G.$$

Proposition IV: The map $E \to \mathscr{X}(M)$ given by $h \mapsto Z_h$ is a homomorphism of Lie algebras:

$$[Z_h, Z_k] = Z_{[h,k]}, \qquad h, k \in E.$$

Proof: Consider first the right action $\tilde{T}$ of G on $M \times G$ given by

$$\tilde{T}((z, a), b) = (z, ab).$$

In view of the example above, the fundamental vector fields for this action are given by

$$\tilde{Z}_h(y, x) = X_h(x).$$

It follows now from sec. 1.3, that

$$[\tilde{Z}_h, \tilde{Z}_k] = \tilde{Z}_{[h,k]}. \tag{3.2}$$

Next recall that T: $M \times G \to M$ is equivariant with respect to $\tilde{T}$ and T (Example 3, sec. 3.2). It follows that

$$\tilde{Z}_h \underset{T}{\sim} Z_h, \qquad \tilde{Z}_k \underset{T}{\sim} Z_k, \qquad \tilde{Z}_{[h,k]} \underset{T}{\sim} Z_{[h,k]}.$$

Thus formula (3.2) and Proposition VIII, sec. 3.13, volume I, yield

$$\tilde{Z}_{[h,k]} \underset{T}{\sim} [Z_h , Z_k]$$

and so, since T is surjective,

$$Z_{[h,k]} = [Z_h , Z_k].$$

Q.E.D.

3.10. Invariant vector fields. We saw in Example 7 of sec. 3.2 that a right action of G in M induces an action in T_M. Define an action of G in $\mathscr{X}(M)$ by setting

$$X \cdot a = (T_a)_* X, \qquad a \in G, \quad X \in \mathscr{X}(M).$$

Then

$$[X, Y] \cdot a = [X \cdot a, Y \cdot a], \qquad X, Y \in \mathscr{X}(M), \quad a \in G.$$

A vector field X on M is called *invariant* if $X \cdot a = X$ $(a \in G)$; i.e., if

$$X \underset{T_a}{\sim} X, \qquad a \in G.$$

The subalgebra of $\mathscr{X}(M)$ that consists of invariant vector fields is denoted by $\mathscr{X}^I(M)$.

Examples: 1. If $M = G$ and if G acts on itself by right translations, then the algebra $\mathscr{X}^I(M)$ consists of the right invariant vector fields (sec. 1.2).

2. It follows from diagram (3.1), sec. 3.9, that the fundamental fields satisfy

$$Z_h \cdot a = Z_{(\mathrm{Ad}\, a^{-1})h}, \qquad h \in E, \quad a \in G.$$

Thus Z_h is invariant if $(\mathrm{Ad}\, a)h = h$, $a \in G$. If G is connected, this is equivalent to

$$[h, k] = 0, \qquad k \in E;$$

i.e., Z_h is invariant if h is in the centre of E (cf. Example 4, sec. 2.4).

3. Let $f \in \mathscr{S}(M; E)$ and $a \in G$. Define $a \cdot f \in \mathscr{S}(M; E)$ by

$$(a \cdot f)(z) = (\mathrm{Ad}\, a)(f(z \cdot a)), \qquad z \in M.$$

Then $Z_{a \cdot f} = Z_f \cdot a^{-1}$. Thus Z_f is invariant if

$$(\mathrm{Ad}\, a^{-1})(f(z)) = f(z \cdot a), \qquad z \in M, \quad a \in G.$$

Proposition V: The Lie bracket of a fundamental field Z_h and an invariant vector field X is zero.

Proof: Let $\tilde{X}$ be the vector field on $M \times G$ given by $\tilde{X}(z, a) = X(z)$. Then

$$dT(\tilde{X}(z, a)) = (X \cdot a)(z \cdot a)$$

and hence, since X is invariant, $\tilde{X} \underset{T}{\sim} X$.

On the other hand, as we saw in the proof of Proposition IV, sec. 3.9, the left invariant vector field X_h on G, regarded as a vector field $\tilde{Z}_h$ on $M \times G$, is T-related to Z_h. Thus

$$0 = [\tilde{Z}_h, \tilde{X}] \underset{T}{\sim} [Z_h, X].$$

Since T is surjective, it follows that $[Z_h, X] = 0$.

Q.E.D.

3.11. Fundamental subbundle. Recall from sec. 3.4 that T is called almost free if each isotropy subgroup G_z is discrete. In view of the corollary to Proposition I, sec. 3.4, this is equivalent to each of the following conditions:

(1) The Lie algebras E_z are zero.
(2) The fundamental vector fields Z_h $(h \neq 0)$ have no zeros.
(3) The bundle map $\alpha\colon M \times E \to T_M$ of sec. 3.9 restricts to linear injections in the fibres.

In this case $\operatorname{Im} \alpha$ is a subbundle of T_M, called the *fundamental subbundle* F_M. The rank of F_M is the dimension of G. Diagram (3.1), sec. 3.9, shows that is F_M stable under the action dT of G in T_M. Moreover, α is a strong isomorphism,

$$\alpha\colon M \times E \xrightarrow{\cong} F_M,$$

and so F_M is trivial. Thus the correspondence $f \to Z_f$ defines an isomorphism

$$\mathscr{S}(M; E) \xrightarrow{\cong} \operatorname{Sec} F_M.$$

§4. Differential forms

In this article $T: M \times G \to M$ denotes a right action of G on M.

3.12. Invariant differential forms. The right translations T_a of M ($a \in G$) induce automorphisms T_a^* of the graded algebra $A(M)$ of differential forms on M. Evidently,

$$T_{ab}^* = T_a^* \circ T_b^* \quad \text{and} \quad T_e^* = \iota \quad a, b \in G.$$

Since, for $X \in \mathscr{X}(M)$, $a \in G$ (cf. sec. 3.10),

$$(X \cdot a)(z) = dT_a(X(z \cdot a^{-1})),$$

it follows that (cf. sec. 0.13)

$$i(X) \circ T_a^* = T_a^* \circ i(X \cdot a) \quad \text{and} \quad \theta(X) \circ T_a^* = T_a^* \circ \theta(X \cdot a).$$

Moreover, clearly

$$T_a^* \circ \delta = \delta \circ T_a^*.$$

A differential form Φ on M is called *invariant under the action of* G if it satisfies

$$T_a^*\Phi = \Phi, \quad a \in G.$$

The invariant differential forms are a graded subalgebra of $A(M)$, which will be denoted by $A_I(M)$. In particular, the invariant functions form a subalgebra of $\mathscr{S}(M)$ which we denote by $\mathscr{S}_I(M)$. (The invariant vector fields on M are a module over $\mathscr{S}_I(M)$.)

Since T_a^* commutes with δ, it follows that the subalgebra $A_I(M)$ is stable under δ. The other commutation relations above show that the subalgebra $A_I(M)$ is stable under $i(X)$ and $\theta(X)$ provided that X is an invariant vector field on M.

3.13. The operators $i(h)$ and $\theta(h)$. Consider the fundamental vector field Z_h generated by $h \in E$ (cf. sec. 3.9). The operators $i(Z_h)$ and $\theta(Z_h)$ in $A(M)$ will often be denoted simply by $i(h)$ and $\theta(h)$. Proposition I, sec. 4.2, and Proposition II, sec. 4.3, both of volume I, together with the relation $Z_{[h,k]} = [Z_h, Z_k]$ $(h, k \in E)$, imply that

$$i([h, k]) = \theta(h) \circ i(k) - i(k) \circ \theta(h),$$

$$\theta([h, k]) = \theta(h) \circ \theta(k) - \theta(k) \circ \theta(h),$$

and

$$\theta(h) = i(h) \circ \delta + \delta \circ i(h), \qquad h, k \in E.$$

A differential form $\Phi \in A(M)$ is called *horizontal with respect to the action of* G if it satisfies

$$i(h)\Phi = 0, \qquad h \in E.$$

Since each $i(h)$ is an antiderivation, the horizontal forms are a graded subalgebra of $A(M)$. This subalgebra will be denoted by $A(M)_{i=0}$. The first identity above shows that the horizontal subalgebra is stable under the operators $\theta(h)$. However, in general it is *not* stable under δ.

Similarly, the differential forms satisfying

$$\theta(h)\Phi = 0, \qquad h \in E,$$

form a graded subalgebra, denoted by $A(M)_{\theta=0}$. Since δ commutes with $\theta(h)$, the subalgebra $A(M)_{\theta=0}$ is stable under δ.

The intersection of the subalgebras $A(M)_{i=0}$ and $A(M)_{\theta=0}$ will be denoted by $A(M)_{i=0,\theta=0}$. This subalgebra is stable under δ. In fact, if $\theta(h)\Phi = 0$ and $i(h)\Phi = 0$, $h \in E$, it follows that

$$\theta(h)\,\delta\Phi = \delta\theta(h)\Phi = 0 \qquad \text{and} \qquad i(h)\,\delta\Phi = \theta(h)\Phi - \delta i(h)\Phi = 0, \qquad h \in E.$$

Proposition VI: $A_I(M) \subset A(M)_{\theta=0}$. If G is connected, then

$$A_I(M) = A(M)_{\theta=0}.$$

Proof: Recall from sec. 3.9 that the orbits of a fundamental vector field Z_h are given by

$$\beta_z(t) = z \cdot \exp th, \qquad z \in M, \quad t \in \mathbb{R}.$$

It follows (cf. the corollary to Proposition X, sec. 4.11, volume I) that, if $\Phi \in A(M)$, the conditions

$$\theta(h)\Phi = 0 \qquad \text{and} \qquad T^*_{\exp th}\Phi = \Phi, \qquad t \in \mathbb{R},$$

are equivalent. Thus $A_I(M) \subset A(M)_{\theta=0}$. If G is connected, $\exp E$ generates G, and so

$$A_I(M) = A(M)_{\theta=0}.$$

Q.E.D.

3.14. Equivariant maps. Suppose $\hat{T}$ is a right action of G on N, and let $\varphi: M \to N$ be a smooth equivariant map. Then every pair of fundamental vector fields $Z_h \in \mathscr{X}(M)$ and $\hat{Z}_h \in \mathscr{X}(N)$ are φ-related (cf. sec. 3.9). Hence (cf. Proposition III, sec. 4.4, volume I or sec. 0.13)

$$\varphi^* \circ i_N(h) = i_M(h) \circ \varphi^* \qquad \text{and} \qquad \varphi^* \circ \theta_N(h) = \theta_M(h) \circ \varphi^*, \qquad h \in E,$$

where $i_N(h)$, $\theta_N(h)$, $i_M(h)$, and $\theta_M(h)$ denote the obvious operators on $A(N)$ and $A(M)$. In particular, φ^* restricts to homomorphisms

$$\varphi^*_{i=0}: A(M)_{i=0} \leftarrow A(N)_{i=0}$$

$$\varphi^*_{\theta=0}: A(M)_{\theta=0} \leftarrow A(N)_{\theta=0}$$

and

$$\varphi^*_{i=0,\theta=0}: A(M)_{i=0,\theta=0} \leftarrow A(N)_{i=0,\theta=0}.$$

Finally, the relation

$$\varphi \circ T_a = \hat{T}_a \circ \varphi, \qquad a \in G,$$

implies that

$$T_a^* \circ \varphi^* = \varphi^* \circ \hat{T}_a^*, \qquad a \in G,$$

and so φ restricts to a homomorphism

$$\varphi_I^*: A_I(M) \leftarrow A_I(N).$$

3.15. Equivariant differential forms. Suppose P is a representation of G in a vector space W. Then each $a \in G$ determines the operator $P(a)_*$ in the space $A(M; W)$ of W-valued differential forms given by

$$(P(a)_*\Omega)(z; \zeta_1, \ldots, \zeta_p) = P(a)(\Omega(z; \zeta_1, \ldots, \zeta_p)), \qquad z \in M, \quad \zeta_i \in T_z(M).$$

We denote $P(a)_*$ simply by $P(a)$.

A left action of G in the set $A(M; W)$ is given by

$$a \cdot \Omega = (P(a) \circ T_a^*)\Omega = (T_a^* \otimes P(a))\Omega, \qquad \Omega \in A(M; W), \quad a \in G,$$

where (as in sec. 0.13) we write $A(M; W) = A(M) \otimes W$. Evidently

$$\delta(a \cdot \Omega) = a \cdot \delta\Omega.$$

A W-valued form Ω is called *equivariant* with respect to P if

$$a \cdot \Omega = \Omega, \qquad a \in G.$$

This is equivalent to the condition

$$T_a^*\Omega = P(a)^{-1}\Omega, \qquad a \in G.$$

The space of equivariant forms is denoted by $A_I(M; W)$. It is a module over the algebra $A_I(M)$, and is stable under δ.

Now consider the induced representation P' of E in W. For each $h \in E$, $P'(h)$ determines the operator $P'(h)_*$ in $A(M; W)$; it is denoted simply by $P'(h)$. The following relations are immediate from the definitions:

$$P'([h, k]) = P'(h) \circ P'(k) - P'(k) \circ P'(h), \qquad P'(h) \circ T_a^* = T_a^* \circ P'(h)$$

and

$$P'(h) \circ \delta = \delta \circ P'(h), \qquad h, k \in E, \quad a \in G.$$

Now recall that the operators $i(h)$ and $\theta(h)$ in $A(M)$ extend to operators in $A(M; W)$ (cf. sec. 0.13). The extensions will also be denoted by $i(h)$ and $\theta(h)$.

Proposition VII: An equivariant differential form Ω satisfies the relation

$$\theta(h)\Omega = -P'(h)\Omega, \qquad h \in E.$$

If G is connected, this condition is equivalent to equivariance.

Proof: Recall, from sec. 0.13, that the decomposition,

$$\tau_{M \times W^*} = \tau_M \times \tau_{W^*},$$

leads to a bigradation of $A(M \times W^*)$; $A^{p,q}(M \times W^*)$ consists of those forms which depend on p vectors tangent to M and q vectors tangent to W^*. Define a linear *injection*

$$\lambda: A^p(M; W) \to A^{p,0}(M \times W^*)$$

by setting

$$(\lambda\Omega)(z, w^*; \zeta_1, \dots, \zeta_p) = \langle w^*, \Omega(z; \zeta_1, \dots, \zeta_p)\rangle,$$
$$z \in M, \quad w^* \in W^*, \quad \zeta_i \in T_z(M).$$

Let $\hat{T}$ be the right action of G on $M \times W^*$ given by

$$\hat{T}_a(z, w^*) = (z \cdot a, P(a)^* w^*), \qquad a \in G, \quad z \in M, \quad w^* \in W^*,$$

and let $\hat{Z}_h$ denote the corresponding fundamental vector field generated by h ($h \in E$). A simple computation shows that

$$\lambda \circ P(a) \circ T_a^* = \hat{T}_a^* \circ \lambda \qquad \text{and} \qquad \lambda \circ (P'(h) + \theta(h)) = \theta(\hat{Z}_h) \circ \lambda.$$

Since λ is injective, the proposition follows from Proposition VI, sec. 3.13, with M replaced by $M \times W^*$.

Q.E.D.

3.16. Examples: **1.** Suppose $W = \mathbb{R}$ and $P(a) = \iota$, $a \in G$. Then the equivariant forms in $A(M)$ are precisely the invariant forms (cf. sec. 3.12), and Proposition VII coincides in this case with Proposition VI.

2. Suppose $W = E$ and $P = \mathrm{Ad}$. An equivariant E-valued form Ω is a form satisfying

$$T_a^*\Omega = (\mathrm{Ad}\, a^{-1})\Omega, \qquad a \in G.$$

If G is connected, this is equivalent to (cf. Proposition VII, sec 3.15)

$$\theta(h)\Omega = -(\mathrm{ad}\, h)\Omega, \qquad h \in E.$$

In particular, recall that each E-valued function f on M determines the vector field Z_f on M (cf. sec. 3.9). Moreover, Example 3 of sec. 3.10 states that if f is equivariant, then Z_f is invariant. Finally, recall from sec. 3.11 that if the action of G is almost free, then $f \mapsto Z_f$ is injective. Thus, in this case, Z_f is invariant if and only if f is equivariant.

3. *Scalar products*: Define bilinear maps,

$$\langle\!\langle\, ,\, \rangle\!\rangle: A^p(M; W^*) \times A^q(M; W) \to A^{p+q}(M),$$

by

$$\langle\!\langle \Phi, \Psi \rangle\!\rangle(z; \zeta_1, \ldots, \zeta_{p+q})$$
$$= \frac{1}{p!\, q!} \sum_{\sigma \in S^{p+q}} \epsilon_\sigma \langle \Phi(z; \zeta_{\sigma(1)}, \ldots, \zeta_{\sigma(p)}), \Psi(z; \zeta_{\sigma(p+1)}, \ldots, \zeta_{\sigma(p+q)}) \rangle,$$

$$\Phi \in A^p(M; W^*), \qquad \Psi \in A^q(M; W), \qquad z \in M, \quad \zeta_i \in T_z(M).$$

Thus if $\Phi_1, \Psi_1 \in A(M)$, $w \in W$, $w^* \in W^*$, then

$$\langle\!\langle \Phi_1 \otimes w^*, \Psi_1 \otimes w \rangle\!\rangle = \langle w^*, w \rangle\, \Phi_1 \wedge \Psi_1 .$$

The contragredient representation, $P^\natural$, of G in W^* determines the left action $a \mapsto P(a)^\natural \circ T_a^*$ of G in $A(M; W^*)$, denoted by $\Phi \mapsto a \cdot \Phi$. Since $P(a)^\natural = (P(a)^*)^{-1}$, it follows that

$$T_a^*\langle \Phi, \Psi \rangle = \langle a \cdot \Phi, a \cdot \Psi \rangle, \qquad a \in G, \quad \Phi \in A(M; W^*), \quad \Psi \in A(M; W).$$

In particular, if Φ and Ψ are equivariant, then $\langle \Phi, \Psi \rangle$ is an invariant differential form.

4. *Action of G on a bundle:* Let $\mathscr{B} = (M, \pi, B, F)$ be a smooth fibre bundle. Assume that right actions

$$T: M \times G \to M, \qquad \hat{T}: B \times G \to B,$$

are given such that π is equivariant. In this case, the diffeomorphisms T_a are all fibre preserving and G is said to *act on the bundle*.

Since π is equivariant the fundamental fields Z_h on M and $\hat{Z}_h$ on B are π-related. Thus (cf. sec. 3.14)

$$\pi^* \circ i(h) = i(h) \circ \pi^* \qquad \text{and} \qquad \pi^* \circ \theta(h) = \theta(h) \circ \pi^*.$$

Moreover, if $\mathscr{B}$ is oriented, then Proposition X, sec. 7.13, volume I, gives

$$\oint_F \circ\, i(h) = i(h) \circ \oint_F \qquad \text{and} \qquad \oint_F \circ\, \theta(h) = \theta(h) \circ \oint_F.$$

Now assume that G is connected. We shall show that each T_a preserves the bundle orientations, so that (cf. Proposition VIII, sec. 7.12, volume I).

$$\oint_F \circ\, T_a^* = \hat{T}_a^* \circ \oint_F, \qquad a \in G.$$

To see that G preserves the bundle orientations observe first that the components of M are stable under G (because G is connected). Thus we may assume that M is connected. In this case each T_a either preserves or reverses the bundle orientations. Since

$$T_{\exp h} = (T_{\exp(h/2)})^2, \qquad h \in E,$$

it follows that $T_{\exp h}$ preserves the orientation. But, because G is connected, $\exp E$ generates G; hence each T_a preserves the orientation.

§5. Invariant cross-sections

In this article $\xi = (N, \pi, B, F)$ denotes a fixed vector bundle.

3.17. Action of G on ξ. *A right action of G on ξ* consists of right actions

$$T: N \times G \to N, \qquad \hat{T}: B \times G \to B$$

subject to the conditions:

(1) π is equivariant

(2) The right translations T_a are bundle maps (i.e., linear in each fibre).

A *left action of G on ξ* is defined analogously.

Assume that T, $\hat{T}$ define a right action of G on ξ. Define a right action of G on Sec ξ, $(\sigma, a) \mapsto \sigma \cdot a$, by setting

$$(\sigma \cdot a)(x) = T_a(\sigma(x \cdot a^{-1})), \qquad \sigma \in \operatorname{Sec} \xi,\ a \in G,\ x \in B.$$

A cross-section σ is called *invariant* if

$$\sigma \cdot a = \sigma, \qquad a \in G.$$

Thus σ is invariant if and only if the map $\sigma: B \to E$ is equivariant. The set of invariant cross-sections forms a subspace of the vector space Sec ξ which we denote by $\operatorname{Sec}^I(\xi)$. $\operatorname{Sec}^I(\xi)$ is a module over $\mathscr{S}_I(B)$ (cf. sec. 3.12).

Example: A right action $\hat{T}$ of G on M induces a right action of G on the tangent bundle, T_M, with T: $T_M \times G \to T_M$ given by

$$T(z, a) = (d\hat{T}_a)z$$

(cf. Example 7, sec. 3.2). As usual, denote $T(z, a)$ by $z \cdot a$.

If X is a vector field on M, then $X \cdot a = (T_a)_* X$, and so the definition above coincides with that of sec. 3.10. Thus the definitions of invariant vector field and of $\mathscr{X}^I(M)$ given in sec. 3.10 agree with the definitions above.

3.18. Integration of cross-sections. Assume that G is compact. Give G a left orientation, and let $\Delta \in A_L^n(G)$ ($n = \dim G$) be the unique left invariant n-form such that $\int_G \Delta = 1$ (cf. sec. 1.15). We write (as in sec. 1.15)

$$\int_G f(a)\, da = \int_G f \cdot \Delta,$$

if f is a vector-valued function on G.

Now suppose G acts on ξ and fix $\sigma \in \operatorname{Sec} \xi$ and $x \in B$. Then a smooth F_x-valued function f_x on G is given by

$$f_x(a) = (\sigma \cdot a)(x).$$

Hence a map $\tau: B \to N$ is defined by

$$\tau(x) = \int_G f_x(a)\, da = \int_G (\sigma \cdot a)(x)\, da.$$

It is denoted by $\int_G \sigma$ and is called the *integral of σ over G*.

τ is a cross-section in ξ. Indeed, this follows from Proposition VII, sec. 7.11, volume I, once we observe that $\tau = \oint_G \Phi$, where

$$\Phi: B \times \wedge^n T_G \to N$$

is the bundle map given by $\Phi(x, a; \eta_1, \ldots, \eta_n) = \Delta(a; \eta_1, \ldots, \eta_n)(\sigma \cdot a)(x)$. (Observe that $B \times T_G$ is the vertical bundle of the trivial bundle $(B \times G, \pi_B, B, G)$.)

Proposition VIII: (1) For any $\sigma \in \operatorname{Sec} \xi$, $\int_G \sigma$ is invariant.

(2) If τ is invariant, then $\int_G \tau = \tau$.

(3) The correspondence $\sigma \mapsto \int_G \sigma$ is linear (over $\mathbb{R}$).

Proof: (1) Let $\sigma \in \operatorname{Sec} \xi$, $b \in G$. It is immediate from the definitions that

$$\left[\left(\int_G \sigma\right) \cdot b\right](x) = T_b \left(\int_G (\sigma \cdot a)(x \cdot b^{-1})\, da\right).$$

Since $T_b: F_{x \cdot b^{-1}} \to F_x$ is linear, it commutes with $\int_G$. Thus, by formula (1.2), sec. 1.15,

$$\left[\left(\int_G \sigma\right) \cdot b\right](x) = \int_G (\sigma \cdot ab)(x)\, da = \int_G (\sigma \cdot a)(x)\, da = \left(\int_G \sigma\right)(x).$$

This proves (1).

(2) follows from the relation,

$$\left(\int_G \tau\right)(x) = \int_G (\tau \cdot a)(x)\, da = \left(\int_G da\right)\tau(x) = \tau(x),$$

and (3) is obvious.

Q.E.D.

Examples: 1. If G is a compact Lie group that acts on a vector bundle $\xi = (N, \pi, B, F)$ via T, $\hat{T}$, then there exists a Riemannian metric in ξ with respect to which the translations $T_a: N \to N$ $(a \in G)$ are isometries.

In fact, the action T determines the (right) action of G in $\vee^2\xi^*$ given by

$$(\Phi \cdot a)(u, v) = \Phi(u \cdot a^{-1}, v \cdot a^{-1}) \qquad \Phi \in \vee^2 F_x^*, \quad x \in B, \quad a \in G, \quad u, v \in F_{x \cdot a}.$$

Now let g be any Riemannian metric in ξ and regard g as a cross-section in the vector the bundle $\vee^2\xi^*$. Then

$$g_0 = \int_G g$$

is a metric with the desired properties.

Suppose now that η is a subbundle of ξ which is stable under the action of G on ξ. Then there is a G-stable subbundle, ζ, of ξ such that $\eta \oplus \zeta = \xi$ (Whitney sum).

In fact, choose a Riemannian metric in ξ such that the translations by G are isometries, as above, and then let ζ be the bundle $\eta^\perp$ whose fibres are the orthogonal complements of those of η (cf. Proposition VII, sec. 2.18, volume I).

2. Suppose G acts on B and consider the induced action,

$$T: (B \times \mathbb{R}) \times G \to B \times \mathbb{R},$$

given by $T((x, t), a) = (x \cdot a, t)$. This is an action of G on the trivial bundle $\xi = (B \times \mathbb{R}, \pi, B, \mathbb{R})$.

The cross-sections of ξ are simply the smooth functions on B. If $f \in \mathscr{S}(B)$, then the integral over G of f is the invariant function f_I given by

$$f_I(x) = \int_G f(x \cdot a)\, da.$$

3.19. Remark. All of the results of this chapter have analogues if right actions are replaced by left actions, $T: G \times M \to M$. Among the notational differences in formulae, recall that $A_z(a)$ becomes $a \cdot z$ so that

$$A_{a\cdot z} = T_a \circ A_z \circ \tau_a^{-1}$$

(cf. sec. 3.1). This in turn implies that the (left) fundamental vector field Z_h generated by $h \in E$ is T_a-related to $Z_{(\mathrm{Ad}\,a)h}$ (cf. sec. 3.9). A form $\Omega \in A(M; W)$ will be called *equivariant* (cf. sec. 3.15) if

$$T_a^*\Omega = P(a)\Omega, \qquad a \in G.$$

Problems

G denotes a Lie group with Lie algebra E and M denotes a manifold.

1. Let $T: M \times G \to M$ be a right action of G on M. Show that

$$i(h) \circ T_a^* = T_a^* \circ i(\mathrm{Ad}(a^{-1})h), \qquad a \in G, \quad h \in E.$$

2. Suppose G is connected, and let G act on M. Show that a horizontal form Φ is invariant if and only if $\delta\Phi$ is horizontal.

3. Let G act on M and consider the induced action on T_M. Show how the fundamental vector fields on M determine the fundamental vector fields on T_M.

4. Construct an almost free action of S^1 on a 3-manifold such that every finite subgroup of S^1 appears as the isotropy subgroup for some point.

5. **Proper actions, I.** A left (right) action of G on M is called *proper*, if for all compact subsets $A, B \subset M$, the subset S of G given by $S = \{a \in G \mid (a \cdot A) \cap B \neq \varnothing\}$ is compact.

(i) Show that the isotropy subgroups of a proper action are all compact. Show that the orbits of a proper action are all closed submanifolds of M.

(ii) Construct an action of $\mathbb{R}$ on $S^1 \times \mathbb{R}$ subject to the following conditions: (a) $S^1 \times \mathbb{R}$ is covered by stable open subsets, each of which is equivariantly diffeomorphic to $(0, 1) \times \mathbb{R}$; (b) the action is not proper. Show, nonetheless, that the action is free and that the orbits are all closed submanifolds.

6. **Orbit space.** Let G act from the left on M. Let M/G denote the set of orbits of G, endowed with the quotient topology via the canonical projection $\pi: M \to M/G$. It is called the *orbit space* of the action.

(i) Show that π is an open map, and that M/G is second countable.

(ii) If the action is proper, show that M/G is Hausdorff and locally compact. Find examples of actions where M/G is not Hausdorff (cf. problem 5,(ii)).

(iii) Assume that the action is proper and free. Fix $z \in M$. Find a submanifold N_z of M and an open subset U_z of M such that $z \in N_z$ and the action restricts to an equivariant diffeomorphism $G \times N_z \xrightarrow{\cong} U_z$.

(iv) (Gleason) Show, if the action is proper and free, that M/G possesses a unique smooth structure for which π is a submersion. Construct a smooth bundle $(M, \pi, M/G, G)$.

7. Bundles over a homogeneous space, I. Let G act from the left on a bundle $\mathscr{B} = (M, \rho, G/K, Q)$, where the action of G on G/K is defined as in sec. 2.11. Identify Q with $Q_{\bar{e}}$ ($Q_{\bar{e}}$, the fibre over $\bar{e}$).

(i) Obtain an action of K on Q.

(ii) Define a right action of K on $G \times Q$ by setting

$$(a, y) \cdot b = (ab, b^{-1}y), \qquad a \in G, \quad y \in Q, \quad b \in K.$$

Show that this action is free. Use the bundle $(G, \pi, G/K, K)$ (cf. sec. 2.13) to make the orbit space $(G \times Q)/K$ into a manifold; denote this manifold by $G \times_K Q$.

(iii) Construct a smooth bundle $\xi = (G \times_K Q, p, G/K, Q)$ and an action of G on ξ. Construct a G-equivariant fibre preserving diffeomorphism $G \times_K Q \xrightarrow{\cong} M$.

(iv) Show that every K-stable submanifold Q_1 of Q leads to a bundle $G \times_K Q_1$ and a smooth fibre preserving map $G \times_K Q_1 \to M$.

(v) Let K_y denote the isotropy subgroup at $y \in Q$ for the action of K on Q. Show that every isotropy subgroup G_x for the action of G on M is conjugate to one of the K_y. Show that the inclusion $Q \to M$ induces a homeomorphism $Q/K \xrightarrow{\cong} M/G$ of orbit spaces.

(vi) Show that G acts properly on M if and only if the action of K on Q is proper. Conclude that if K is compact, then the action of G is proper.

8. Bundles over a homogeneous space, II. Adopt the hypotheses of problem 7.

(i) Show that the vertical subbundle V_M of τ_M is stable under the action of G. If K is compact, construct a G-stable horizontal subbundle.

(ii) Assume that K is compact. Denote the Lie algebra of K by F. Use the adjoint representation to obtain a representation of K

in $\wedge(E/F)^*$. Denote by $A_I(M)$ the algebra of G-invariant differential forms on M and by $A_I(Q; \wedge(E/F)^*)$ the algebra of K-equivariant differential forms on Q with values in $\wedge(E/F)^*$. Obtain an isomorphism

$$A_I(M) \xrightarrow{\cong} A_I(Q; \wedge(E/F)^*).$$

(iii) If K is compact, show that the inclusion map $i: Q \to M$ induces an isomorphism

$$i^*_{i=0,I}: A_I(Q)_{i=0} \xleftarrow{\cong} A_I(M)_{i=0},$$

where $A_I(Q)_{i=0}$ (respectively, $A_I(M)_{i=0}$) denotes the algebra of differential forms on Q that are K-horizontal and K-invariant (respectively, the algebra of differential forms on M that are G-horizontal and G-invariant).

9. Vector bundles over a homogeneous space. Let $\xi = (M, \rho, G/K, F)$ be a vector bundle acted on by G so as to induce the standard action on G/K. Identify F with F_ε.

(i) Show that the induced action of K on F is a representation (cf. problem 7, (i)). Show that $G \times_K F \to G/K$ is a vector bundle. Construct a strong equivariant isomorphism $G \times_K F \xrightarrow{\cong} \xi$.

(ii) Obtain a bijection between direct decompositions of F into K-stable subspaces and decompositions of ξ as a Whitney sum of G-stable subbundles.

(iii) Construct a bijection between K-invariant Euclidean metrics in F and G-invariant Riemannian metrics in ξ.

(iv) Assume K compact and fix a G-invariant Riemannian metric in ξ. Show that the action of G restricts to actions on the unit sphere bundle and on the open disc bundle of vectors of length $< r$. Identify these bundles with $G \times_K S$ and $G \times_K F_r$, respectively, where S (respectively, F_r) denotes the unit sphere (respectively, the open disc of radius r) in F.

(v) With the hypotheses and notation of (iv), let M_r denote the open disc bundle of radius r. Construct a G-equivariant, fibre preserving diffeomorphism $M \xrightarrow{\cong} M_r$ inducing the identity map in G/K.

(vi) With the hypotheses and notation of (iv), construct a smooth G-invariant function f on M such that: (a) $0 \leqslant f(z) \leqslant 1$, $z \in M$; (b) $f(0_x) = 1$, $x \in G/K$; (c) f has fibre compact carrier.

10. Affine sprays. Assume G is compact, and acts on M. Recall, from the Appendix of volume I, the definition of an affine spray as a vector field on T_M. It is called *complete*, if it generates a global 1-parameter group of transformations $\varphi_t : T_M \to T_M$ ($t \in \mathbb{R}$).

(i) Show that M admits a complete G-invariant affine spray. Show that the corresponding map exp: $T_M \to M$ is G-equivariant.

(ii) Show that the map $\exp_G$: $E \to G$ is the restriction of the exponential map of a certain affine spray.

11. Isotropy representation and stable tubular neighbourhoods. Let G act from the left on M.

(i) Use the action to define a representation of the isotropy subgroup G_x in $T_x(M)$ ($x \in M$). This is called the *isotropy representation.*

(ii) Let $\mathrm{Ad}_x^{\perp}$ denote the representation of G_x in E/E_x (E_x, the Lie algebra of G_x). Construct a representation of G_x in a space N_x and a G_x-linear short exact sequence

$$0 \to E/E_x \to T_x(M) \to N_x \to 0.$$

(iii) Let G act on a manifold P and let φ: $P \to M$ be an equivariant immersion. Obtain an action of G on the normal bundle of P in M. In the case that φ is the inclusion map $\bar{A}_x$: $G/G_x \to M$, show that the normal bundle is the vector bundle $G \times_{G_x} N_x$ (cf. problem 9).

(iv) Suppose that G_x is compact. Use a complete G_x-invariant affine spray (cf. problem 10) to construct a G_x-equivariant smooth map φ: $N_x \to M$ satisfying $\varphi(0_x) = x$. Show that the smooth map $G \times N_x \to M$ given by $(a, y) \mapsto a \cdot \varphi(y)$ factors to yield a smooth G-equivariant map ψ: $G \times_{G_x} N_x \to M$ (cf. problem 7).

(v) Assume G_x is compact and let $O_x(r)$ denote the open disc of radius r in N_x with respect to a G_x-invariant Euclidean inner product. Show that, for sufficiently small r, ψ restricts to an equivariant local diffeomorphism $G \times_{G_x} O_x(r) \xrightarrow{\cong} V_x$, where V_x is a G-stable neighbourhood of the orbit $G \cdot x$.

(vi) Assume that the action of G is proper. Construct a G-equivariant diffeomorphism σ of the normal bundle of $G \cdot x$ onto a neighbourhood of $G \cdot x$, such that $\sigma(0_x) = x$ (cf. problem 5).

12. Proper actions, II. Assume that G acts properly from the left on M. Use problems 5–11 to establish the following properties:

(i) Every covering of M by G-stable open sets admits a subordinate G-invariant partition of unity.

(ii) M admits a complete G-invariant affine spray. If N is a closed G-stable submanifold of M, then there is a G-equivariant diffeomorphism from the normal bundle of N onto a neighbourhood of N.

(iii) M admits a G-invariant Riemannian metric.

(iv) If G acts *effectively* on M (i.e., $\bigcap_{x \in M} G_x = e$), the isotropy representations are all faithful.

13. Orbit types. Let G act on M. The conjugacy classes (G_x) are called *orbit types* for the action (cf. problem 24, Chap. II).

(i) If the action is proper and $x \in M$, find a neighbourhood U_x of G_x such that $(G_y) \leqslant (G_x)$, for $y \in U_x$.

(ii) If the action is proper and M/G is compact, conclude that there are only finitely many orbit types.

(iii) Assume that the action is proper. Show that there is a unique orbit type (H) such that $(H) \leqslant (G_x)$ for every $x \in M$. It is called the *principal orbit type*. Show that the set $\{x \in M \mid (G_x) = (H)\}$ is open, connected, and dense in M.

(iv) Show that the principal orbit type for the action of G on T_M is strictly contained in the principal orbit type for the action on M. Show by example that the principal orbit type of T_M need not be (e). Find the principal orbit type of the adjoint representation of a compact Lie group.

(v) Show that (G_x) is the principal orbit type if and only if the representation of G_x in N_x (cf. problem 11) is trivial. In this case show that the normal bundle is $G/G_x \times N_x$.

(vi) Fix $x_0 \in M$. Show that the union of the points $x \in M$ such that $(G_x) \leqslant (G_{x_0})$ is an open G-stable subset of M. Show that the union of the points $x \in M$ such that $(G_x) = (G_{x_0})$ is a closed submanifold of this open set.

14. Fixed point sets. Let G act properly on M and let F be the set of points $x \in M$ such that $G_x = G$.

(i) Show that each component of F is a closed submanifold of M.

(ii) Let F_0 be a component of F and suppose $\dim F_0 = p$. Construct a representation of G in $\mathbb{R}^{n-p}$ ($n = \dim M$) and a G-equivariant coordinate representation $\psi_\alpha \colon U_\alpha \times \mathbb{R}^{n-p} \xrightarrow{\cong} \rho^{-1}(U_\alpha)$ for the normal bundle $(N_0, \rho, F_0, \mathbb{R}^{n-p})$.

(iii) If G is a torus, give the normal bundle of a component, F_λ, of F an invariant complex structure. Conclude that $\dim M - \dim F_\lambda \equiv 0 \pmod 2$, and that the normal bundle is orientable.

15. Actions on vector bundles. Suppose G acts on a vector bundle $\xi = (M, \rho, B, F)$. Assume the action on B is proper.

(i) Show that the action on M is proper.

(ii) Construct a G-invariant Riemannian metric in ξ. Conclude that the fundamental fields on M are tangent to the sphere bundles.

16. Actions of Lie algebras. Suppose G is connected, and that G is its own universal covering group. Assume $\Phi: E \to \mathscr{X}(M)$ is a Lie algebra homomorphism such that each vector field $\Phi(h)$ can be integrated to produce a 1-parameter group of diffeomorphisms of M.

Prove that there is a unique smooth action of G on M such that $\Phi(h)$ is the fundamental field generated by h (cf. problem 20, Chap. I).

17. Let X_ν ($\nu = 1, ..., n$) be vector fields on a connected n-manifold M such that

(1) $[X_i, X_j] = \sum_k c_{ij}^k X_k$ ($c_{ij}^k \in \mathbb{R}$).

(2) For each $x \in M$, $X_\nu(x)$ ($\nu = 1, ..., n$) is a basis of $T_x(M)$.

(3) Every real linear combination of the X_i generates a global flow φ_t ($t \in \mathbb{R}$).

Show that M is the quotient of a Lie group by a closed discrete subgroup. If the c_{ij}^k are all zero, show that M is an abelian Lie group.

18. Let $f_1, ..., f_p$ be smooth functions on a manifold M. Fix real constants $c_1, ..., c_p$ and set $N = f_1^{-1}(c_1) \cap \cdots \cap f_p^{-1}(c_p)$. Assume that, for each $x \in N$, $(\delta f_1 \wedge \cdots \wedge \delta f_p)(x) \neq 0$.

(i) Show that N is a closed submanifold of M. Let U be a tubular neighbourhood of N; identify U with the normal bundle ν and let $\rho: U \to N$ be the projection. Show that, for suitable U,

$$x \mapsto (\rho(x), f_1(x), ..., f_p(x)), \qquad x \in U,$$

is a diffeomorphism from U to $N \times V$ (V, a neighbourhood of 0 in $\mathbb{R}^p$).

(ii) Assume that the dimension of M is even and that M admits a closed 2-form Φ such that $\Phi \wedge \cdots \wedge \Phi$ orients M. Show that vector fields Y_j on M are determined by the equations $i(Y_j)\Phi = \delta f_j$. Show that the Y_j restrict to vector fields X_j on N. Show that $[Y_{j_1}, Y_{j_2}] = 0$ and conclude that, if N is compact and connected, it is a torus.

(iii) Let Y be defined by $i(Y)\Phi = \delta f$, where $f \in \mathscr{S}(M)$ satisfies $Y_j(f) = 0$ $(j = 1, \ldots, p)$. Show that Y restricts to an invariant vector field on N.

19. **Non-Euclidean geometry in the unit disc.** Define a Riemannian metric in the open unit disc Ω of the complex plane by

$$g(z; \zeta_1, \zeta_2) = \frac{\operatorname{Re}(\zeta_1 \bar{\zeta}_2)}{(1 - |z|^2)^2}, \qquad |z| < 1, \quad \zeta_1, \zeta_2 \in \mathbb{C}.$$

(cf. problem 16, Chap. II.) Define $\rho: \Omega \times \Omega \to \mathbb{R}$ by

$$\rho(z_1, z_2) = \log \frac{1 + r}{1 - r}, \qquad \text{where} \quad r = \left| \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} \right|.$$

Let G be the subgroup of the Möbius group consisting of the fractional linear transformations which map Ω onto itself.

(i) Show that the angles with respect to g coincide with the Euclidean angles.

(ii) Show that $\rho(z_1, z_2) = \rho(Tz_1, Tz_2)$, $T \in G$.

(iii) A *hyperbolic straight line* in Ω is a circle orthogonal to the unit circle. Show that, for any two distinct points z_1 and z_2 of Ω, there is a unique straight line joining z_1 to z_2.

(iv) The *hyperbolic* length of a smooth curve in Ω: c: $t \mapsto z(t)$, $0 \leqslant t \leqslant 1$ is defined by

$$l(c) = \int_0^1 \frac{|\dot{z}|}{1 - |z|^2}\, dt.$$

Show that $\rho(z_1, z_2) = l(s) \leqslant l(c)$, where s is the hyperbolic straight line segment joining z_1 to z_2 and c is any smooth curve in Ω from z_1 to z_2.

(v) Show that ρ makes Ω into a metric space with the standard topology.

(vi) Let τ_Ω be the tangent bundle of Ω and let M be the total space of the corresponding sphere bundle (with respect to g). Show that the left action of G on Ω induces an action on τ_Ω which restricts to an action on M.

(vii) Let $x \in M$. Show that the map $T \mapsto T \cdot x$ defines a diffeomorphism $G \cong M$. Conclude that G is diffeomorphic to $\Omega \times S^1$.

Conclude also that, given any $z_1, z_2 \in \Omega$, $\zeta_1, \zeta_2 \in S^1$, there is a unique $T \in G$ such that

$$T(z_1) = z_2 \qquad \text{and} \qquad T'(z_1; \zeta_1) = (z_2, \zeta_2).$$

(viii) Show that, if $z_1, w_1, z_2, w_2 \in \Omega$ are given such that $\rho(z_1, z_2) = \rho(w_1, w_2)$, there is a unique $T \in G$ such that $T(z_1) = w_1$ and $T(z_2) = w_2$.

20. Convex polygons in Ω. A subset $A \subset \Omega$ is called *hyperbolic convex* if, whenever $z_1 \in A$ and $z_2 \in A$, then the hyperbolic straight line segment between z_1 and z_2 is contained in A. A *convex polygon* is a closed convex set Δ in Ω whose boundary consists of a finite number of hyperbolic straight line segments, called its *sides*. If the boundary of Δ consists of n sides, Δ will be called *an n-polygon.*

Let Δ be an n-polygon. Show that

$$\sum_{\nu=1}^{n} \alpha_\nu + \tfrac{1}{4} \int_\Delta \Phi = (n-2)\pi,$$

where the α_ν denote the interior angles of Δ and Φ is the 2-form given by

$$\Phi(z; \zeta_1, \zeta_2) = \frac{\operatorname{Im}(\bar{\zeta}_1 \zeta_2)}{(1 - |z|^2)^2}, \qquad z \in \Omega, \qquad \zeta_1, \zeta_2 \in \mathbb{C}.$$

21. Discontinuous actions. An effective action (cf. problem 12) of a group Γ on a manifold M is called *discontinuous*, if every point $x \in M$ has a neighbourhood U such that no two distinct points of U are in the same orbit of Γ. A *fundamental domain* is an open subset F of M such that

(1) any two distinct points of F are in different orbits,

and

(2) every point $x \in M$ is in the orbit of some point of the closure $\bar{F}$.

(i) If Γ acts discontinuously on M, show that Γ is finite or countable, and that the action is free. Is the action necessarily proper ?

(ii) Let $M = \Omega$ and let Γ be a group of fractional linear transformations of Ω that acts discontinuously on Ω. Set $z_\nu = T_{\gamma_\nu}(0)$, where the γ_ν are the elements of Γ. Show that the set given by

$$\{z \in \Omega \mid \rho(z, 0) < \rho(z, z_\nu), \quad \nu = 1, 2, \ldots,\}$$

is a convex fundamental domain for the action.

22. Poincaré polygons. We adopt the notation of problems 19–21. A convex $4p$-polygon Δ in Ω with consecutive sides a_1, b_1, a_1', b_1', ..., a_p, b_p, a_p', b_p' is called a *Poincaré polygon* if it satisfies the following conditions:

(a) $l(a_i) = l(a_i')$ and $l(b_i) = l(b_i')$ $(i = 1, ..., p)$.
(b) The sum of interior angles is 2π.

(i) Construct Poincaré polygons for each $p \geqslant 2$.
(ii) Show that if Δ is a Poincaré polygon, then the 2-form Φ of problem 20 satisfies

$$\tfrac{1}{4}\iint_\Delta \Phi = 4(p-1)\pi.$$

Conclude that there is no Poincaré polygon for $p = 1$.

23. Fuchsian groups. Adopt the notation of problems 19–22. Fix a Poincaré $4p$-polygon, with consecutive vertices $z_0, ..., z_{4p}$ $(z_0 = z_{4p})$. Let $\vec{a}_i$, $\vec{b}_i$, $\vec{a}_i'$, $\vec{b}_i'$ denote the sides as defined above, directed from the lower to higher vertex (e.g. $\vec{a}_i$ is the side from z_{4i} to z_{4i+1}) and let $\overleftarrow{a}_i$, $\overleftarrow{b}_i$, $\overleftarrow{a}_i'$, $\overleftarrow{b}_i'$ be the same sides with opposite orientation.

(i) Show that there are unique elements α_i, β_i, $(i = 1, ..., p)$ in G such that $\alpha_i^{-1}(\vec{a}_i) = \overleftarrow{a}_i'$ and $\beta_i(\vec{b}_i) = \overleftarrow{b}_i'$. Denote the subgroup of G generated by $\alpha_1, ..., \alpha_p, \beta_1, ..., \beta_p$ by Γ. Γ is called the *Fuchsian group associated with* Δ. (cf. problem 19 for the definition of G.)
(ii) Show that $\Delta \cap \alpha_i^{-1}(\Delta) = a_i'$ and $\Delta \cap \beta_i(\Delta) = b_i'$ $(i = 1, ..., p)$
(iii) Consider the sequence

$$\alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, ..., \alpha_p, \beta_p, \alpha_p^{-1}, \beta_p^{-1},$$

of elements in Γ. Denote the product (in the given order) of the first m elements by τ_m $(m = 1, ..., 4p)$. Show that for a suitable permutation σ of $(1, ..., 4p)$,

$$\tau_j(z_{\sigma(j)}) = z_0, \qquad j = 1, ..., 4p.$$

(iv) Set $\tau_j(\Delta) = \Delta_j$. Show that $\Delta_j \cap \Delta_{j+1}$ is a common side having z_0 as an endpoint. Show that

$$\Delta_j \cap \Delta_k = z_0 \qquad \text{if} \quad |k-j| > 1, \quad 1 \leqslant j, k \leqslant 4p.$$

(v) Show that $\Delta_{4p} = \Delta$ and that the polygons $\Delta_1, ..., \Delta_{4p}$ cover a neighbourhood of z_0. Conclude that $\tau_{4p} = \iota$; i.e., Γ has the relation

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_p\beta_p\alpha_p^{-1}\beta_p^{-1} = \iota.$$

24. Poincaré polygons as fundamental domains. Adopt the notation of problem 23.

(i) Set $w_j = \tau_j^{-1}(z_0)$ (i.e., $w_j = z_{\sigma(j)}$, $j = 1, ..., 4p$). Define a symmetric relation in $\Gamma \times \Delta$ as follows:

(a) If z is an interior point of Δ, then $(g, z) \sim (g', z')$ if and only if $g = g'$ and $z = z'$.

(b) If z is an interior point of a_i, then

$$(g, z) \sim (g, z) \qquad \text{and} \qquad (g, z) \sim (g\alpha_i, \alpha_i^{-1}z).$$

(c) If z is an interior point of b_i, then

$$(g, z) \sim (g, z) \qquad \text{and} \qquad (g, z) \sim (g\beta_i^{-1}, \beta_i z).$$

(d) If z is a vertex ($z = w_i$), then

$$(g, w_i) \sim (g\tau_i^{-1}\tau_j, w_j), \qquad j = 1, ..., 4p.$$

Show that this relation is an equivalence relation, and write down the equivalence classes.

(ii) Give Γ the discrete topology. Let X be the quotient space under the equivalence relation above (quotient topology) and let $q: \Gamma \times \Delta \to X$ be the projection. Show that X is second countable, Hausdorff, and pathwise connected.

(iii) Define a map $\varphi: \Gamma \times \Delta \to \Omega$ by $\varphi(g, z) = g \cdot z$. Show that φ factors over the projection q to yield a continuous map $\psi: X \to \Omega$. Show that ψ is a local homeomorphism.

(iv) Let $t \mapsto z(t)$ be a continuous map from $[0, 1]$ into Ω. Let $x_0 \in X$ be any point such that $\psi(x_0) = z(0)$. Show that there is a unique continuous map $t \mapsto x(t)$ from $[0, 1]$ into X such that

$$x(0) = x_0 \qquad \text{and} \qquad \psi(x(t)) = z(t), \qquad 0 \leqslant t \leqslant 1.$$

(*Hint*: Cover the curve $z(t)$ by finitely many Γ-translates of Δ.) Conclude that ψ is a homeomorphism, onto Ω.

(v) Show that Γ acts discontinuously and properly on Ω and that the interior of Δ is a fundamental domain for the action. Conclude that the orbit space ($M = \Omega/\Gamma$) is a smooth compact connected orientable 2-manifold, and that $\pi: \Omega \to M$ is the universal covering projection.

(vi) Compute the cohomology algebra and Euler–Poincaré characteristic of M.

(vii) Generalize to nonconvex polygons.

25. The Möbius group. Consider the action of the Möbius group M on S^2 (cf. problem 14, Chap. II).

(i) Show that this action is transitive and determine the isotropy subgroups.

(ii) Consider the induced action on the tangent bundle τ_{S^2}. Determine the isotropy subgroups. Show that there are exactly two orbits, namely the zero cross-section and the deleted bundle. Thus obtain a smooth bundle $(M, \pi, \dot{\tau}_{S^2}, \mathbb{C})$ (cf. Example 5, sec. 3.10, volume I).

(iii) Show that M is diffeomorphic to $\dot{T}_{S^2} \times \mathbb{C}$. Conclude that M is diffeomorphic to $\mathbb{R}P^3 \times \mathbb{R}^3$. Construct an inclusion $SO(3) \to M$ of Lie groups that is a smooth strong deformation retract.

(iv) Find the fundamental fields for the action of M on S^2 and on T_{S^2}.