Indeed, $h = \omega - \psi \int\limits_{\mathbb{R}} \omega dx$ is continuous compactly supported with $\int_{\mathbb{R}} h dx = 0$ and thus it has a unique compactly supported primitive.

Hence

$$\int\limits_{\mathbb{R}} f\phi' dx = \int\limits_{R} f(\omega - \psi \int\limits_{\mathbb{R}} \omega dy) dx = 0$$

or

$$\int\limits_{\mathbb{R}} (f - \int\limits_{\mathbb{R}} f\psi dy)\omega dx = 0$$

for any $\omega \in C_0^\infty(\mathbb{R})$ and thus $f = const$ almost everywhere.

Step 3. Next, if $v(x) = \int_{x_0}^x f(y)dy$ for $f \in L_{1,loc}(\mathbb{R})$, then v is continuous and the generalized derivative of v, Dv, equals f. In the proof, we can put $x_0 = 0$. Then

$$\begin{aligned}
\int\limits_{\mathbb{R}} v\phi' dx &= \int\limits_0^\infty (\int\limits_0^x f(y)\phi'(x)dy)dx - \int\limits_{-\infty}^0 (\int\limits_x^0 f(y)\phi'(x)dy)dx \\
&= \int\limits_0^\infty f(y)(\int\limits_y^\infty \phi'(x)dx)dy - \int\limits_{-\infty}^0 f(y)(\int\limits_{-\infty}^0 \phi'(x)dx)dy \\
&= -\int\limits_{\mathbb{R}} f(y)\phi(y)dy.
\end{aligned}$$

With these results, let $u \in L_{1,loc}(\mathbb{R})$ be the distributional derivative $Du \in L_{1,loc}(\mathbb{R})$ and set $\bar{u}(x) = \int\limits_0^x Du(t)dt$. Then $D\bar{u} = Du$ almost everywhere and hence $\bar{u} + C = u$ almost everywhere. Defining $\widetilde{u} = \bar{u} + C$, we see that $\widetilde{u}$ is continuous and has integral representation and thus it is differentiable almost everywhere.

1.2 Fundamental Theorems of Functional Analysis

The foundation of classical functional analysis are the four theorems which we formulate and discuss below.

1.2.1 Hahn–Banach Theorem

Theorem 1.12. *(Hahn–Banach) Let X be a normed space, X_0 a linear subspace of X, and x_1^* a continuous linear functional defined on X_0. Then there exists a continuous linear functional x^* defined on X such that $x^*(x) = x_1^*(x)$ for $x \in X_0$ and $\|x^*\| = \|x_1^*\|$.*

The Hahn–Banach theorem has a multitude of applications. For us, the most important one is in the theory of the dual space to X. The space $\mathcal{L}(X, \mathbb{R})$ (or $\mathcal{L}(X, \mathbb{C})$) of all continuous functionals is denoted by X^* and referred to as the *dual space*. The Hahn–Banach theorem implies that X^* is nonempty (as one can easily construct a continuous linear functional on a one-dimensional space) and, moreover, there are sufficiently many bounded functionals to separate points of x; that is, for any two points $x_1, x_2 \in X$ there is $x^* \in X^*$ such that $x^*(x_1) = 0$ and $x^*(x_2) = 1$. The Banach space $X^{**} = (X^*)^*$ is called the *second dual*. Every element $x \in X$ can be identified with an element of X^{**} by the evaluation formula

$$x(x^*) = x^*(x); \tag{1.21}$$

that is, X can be viewed as a subspace of X^{**}. To indicate that there is some symmetry between X and its dual and second dual we shall often write

$$x^*(x) = <x^*, x>_{X^* \times X},$$

where the subscript $X^* \times X$ is suppressed if no ambiguity is possible.

In general $X \neq X^{**}$. Spaces for which $X = X^{**}$ are called *reflexive*. Examples of reflexive spaces are rendered by Hilbert and L_p spaces with $1 < p < \infty$. However, the spaces L_1 and L_∞, as well as nontrivial spaces of continuous functions, fail to be reflexive.

Example 1.13. If $1 < p < \infty$, then the dual to $L_p(\Omega)$ can be identified with $L_q(\Omega)$ where $1/p + 1/q = 1$, and the duality pairing is given by

$$<f, g> = \int\limits_{\Omega} f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x}, \quad f \in L_p(\Omega),\ g \in L_q(\Omega). \tag{1.22}$$

This shows, in particular, that $L_2(\Omega)$ is a Hilbert space and the above duality pairing gives the scalar product in the real case. If $L_2(\Omega)$ is considered over the complex field, then in order to get a scalar product, (1.22) should be modified by taking the complex adjoint of g.

Moreover, as mentioned above, the spaces $L_p(\Omega)$ with $1 < p < \infty$ are reflexive. On the other hand, if $p = 1$, then $(L_1(\Omega))^* = L_\infty(\Omega)$ with duality pairing given again by (1.22). However, the dual to L_∞ is much larger than $L_1(\Omega)$ and thus $L_1(\Omega)$ is not a reflexive space.

Another important corollary of the Hahn–Banach theorem is that for each $0 \neq x \in X$ there is an element $\bar{x}^* \in X^*$ that satisfies $\|\bar{x}^*\| = \|x\|$ and $<\bar{x}^*, x> = \|x\|$. In general, the correspondence $x \to \bar{x}^*$ is multi-valued: this is the case in L_1-spaces and spaces of continuous functions it becomes, however, single-valued if the unit ball in X is strictly convex (e.g., in Hilbert spaces or L^p-spaces with $1 < p < \infty$; see [82]).

1.2.2 Spanning theorem and its application

A workhorse of analysis is the spanning criterion.

Theorem 1.14. *Let X be a normed space and $\{y_j\} \subset X$. Then $z \in Y := \overline{\mathcal{L}in}\{y_j\}$ if and only if*

$$\forall_{x^* \in X^*} < x^*, y_j >= 0 \quad \text{implies} \quad < x^*, z >= 0.$$

Proof. In one direction it follows easily from linearity and continuity.

Conversely, assume $< x^*, z >= 0$ for all x^* annihilating Y and $z \neq Y$. Thus, $\inf_{y \in Y} \|z - y\| = d > 0$ (from closedness). Define $Z = \mathcal{L}in\{Y, z\}$ and define a functional y^* on Z by $< y*, \xi >=< y^*, y + az >= a$. We have

$$\|y + az\| = |a| \| \frac{y}{a} + z \| \geq |a| d$$

hence

$$| < y^*, \xi >= |a| \leq \frac{\|y + az\|}{d} = d^{-1} \|\xi\|$$

and y^* is bounded. By H.-B. theorem, we extend it to $\widetilde{y}^*$ on X with $< \widetilde{y}^*, x >= 0$ on Y and $< \widetilde{y}^*, z >= 1 \neq 0$.

Next we consider the Müntz theorem.

Theorem 1.15. *Let $(\lambda_j)_{j \in \mathbb{N}}$ be a sequence of positive numbers tending to ∞. The functions $\{t^{\lambda_j}\}_{j \in \mathbb{N}}$ span the space of all continuous functions on $[0, 1]$ that vanish at $t = 0$ if and only if*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty.$$

Proof. We prove the 'sufficient' part. Let x^* be a bounded linear functional that vanishes on all t^{λ_j}:

$\{t^n\}_{n=\cancel{0}, 1 \ldots}$

$$< x^*, t^{\lambda_j} >= 0, \qquad j \in \mathbb{N}.$$

For $\zeta \in \mathbb{C}$ such that $\Re\zeta > 0$, the functions $\zeta \to t^\zeta$ are analytic functions with values in $C([0, 1])$ This can be proved by showing that

$\zeta \to t^\zeta \in C([0,1])$

$$\lim_{\mathbb{C} \ni h \to 0} \frac{t^{\zeta + h} - t^\zeta}{h} = (\ln t) t^\zeta$$

uniformly in $t \in [0, 1]$. Then

$$f(\zeta) =< x^*, t^\zeta >$$

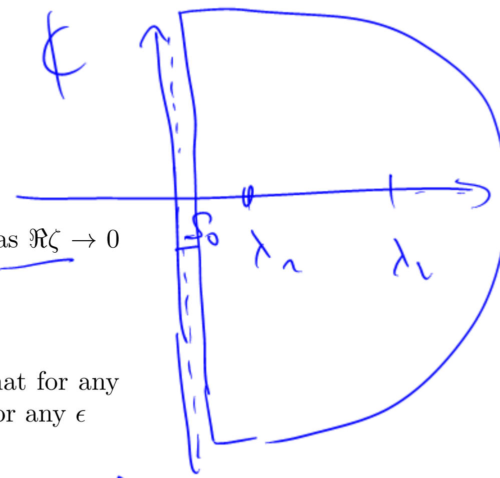
is a scalar analytic function of ζ with $\Re\zeta > 0$. We can assume that $\|x^*\| \leq 1$. Then

$\leq \|x^*\| \cdot \|t^\zeta\|$

$$|f(\zeta)| \leq 1$$

for $\Re\zeta > 0$ and $f(\lambda_j) = 0$ for any $j \in \mathbb{N}$.

Next, for a given N, we define a Blaschke product by



$$B_N(\zeta) = \prod_{j=1}^{N} \frac{\zeta - \lambda_j}{\zeta + \lambda_j}.$$

We see that $B_N(\zeta) = 0$ if and only if $\zeta = \lambda_j$, $|B_N(\zeta)| \to 1$ both as $\Re\zeta \to 0$ and $|\zeta| \to \infty$. Hence

lim

$$g_N(\zeta) = \frac{f(\zeta)}{B_N(\zeta)}$$

is analytic in $\Re\zeta > 0$. Moreover, for any ϵ' there is $\delta_0 > 0$ such that for any $\delta > \delta_0$ we have $|B_N(\zeta)| \geq 1 - \epsilon'$ on $\Re\zeta = \delta$ and $|\zeta| = \delta^{-1}$. Hence for any ϵ

$$|g_N(\zeta)| \leq 1 + \epsilon$$

$|g_N(\zeta)| \le \frac{1}{|B_N(\zeta)|} \le \frac{1}{1-\varepsilon'} = 1 + \varepsilon' + (\varepsilon')^2 + \dots = 1 + \varepsilon'(1+\varepsilon' + \dots)$

$|B_N(\zeta)| \to 1$ on $\Re\zeta \to 0$

there and by the maximum principle the inequality extends to the interior of the domain. Taking $\epsilon \to 0$ we obtain $|g_N(\zeta)| \leq 1$ on $\Re\zeta > 0$.

Assume now there is $k > 0$ for which $f(k) \neq 0$. Then we have

$$\prod_{j=1}^{N} \left| \frac{\lambda_j + k}{\lambda_j - k} \right| \leq \frac{1}{f(k)}.$$

Note that such k cannot equal λ_j for any k's

$|g_N(\zeta)| \le \frac{|f(\zeta)|}{|B_N(\zeta)|} \le 1$, $\frac{1}{|B_N(\zeta)|} \le \frac{1}{|f(\zeta)|}$, $\prod_{j=1}^{N} \left|\frac{\lambda_j+k}{\lambda_j-k}\right| \le \frac{1}{f(k)}$

Note, that this estimate is uniform in N. If we write

$$\frac{\lambda_j + k}{\lambda_j - k} = 1 + \frac{2k}{\lambda_j - k}$$

then, by $\lambda_j \to \infty$ almost all terms bigger then 1. Remembering that boundedness of the product is equivalent to the boundedness of the sum

$\prod (1+a_n) \le C$

$\sum_1 \ln(1+a_n) \le \ln C$

$\ln(1+a_n) \sim a_n$

since $\frac{\ln(1+x)}{x} \xrightarrow[x\to 0]{} 1$

$$\sum_{j=1}^{N} \frac{1}{\lambda_j - k}$$

we see that we arrived at contradiction with the assumption. Hence, we must have $f(k) = 0$ for any $k > 0$. This means, however, that any functional that vanishes on $\{t^{\lambda_j}\}$ vanishes also on t^k for any k. But, by the Stone- Weierstrass theorem, it must vanish on any continuous function (taking value 0 at zero). Hence, by the spanning criterion, any such continuous function belongs to the closed linear span of $\{t^{\lambda_j}\}$.

Non-reflexiveness of $C([-1,1])$

Consider the Banach space $X = C([-1,1])$ normed with the sup norm. If X was reflexive, then we could identify X^{**} with X and thus, for every $x^* \in X^*$ there would be $x \in X$ such that

$$\|x\| = 1, \quad < x^*, x >= \|x^*\|. \tag{1.23}$$

x^* there is $x^{**} \in X^{**}$ $\|x^{**}\| = 1$

$\langle x^{**}, x^* \rangle = \|x^*\|$

$\{t^{\lambda_j}\}$

$f(\zeta) = \langle x^*, t^{\zeta} \rangle$

$f(\lambda_j) = 0$

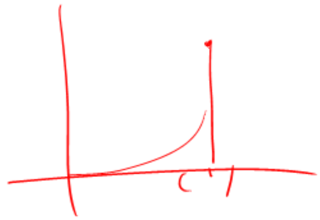
Consider an arbitrary k. Assume that $f(k) \neq 0$

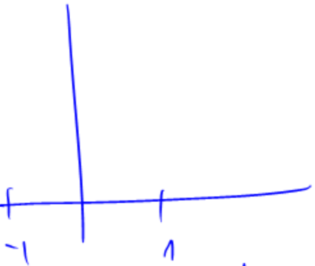
Such a $k \neq \lambda_j$ for any λ_j

$t^n \qquad t \in [0,1]$

$t^n \to 0 \qquad 0 \leq t < 1$

$t^n \to 1$





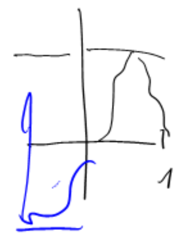
Let us define $x^* \in X^*$ by

$$< x^*, x >= \int_{-1}^{1} \operatorname{sign} t x(t) dt. = -\int_{-1}^{0} x(t)dt + \int_{0}^{1} x(t)dt$$

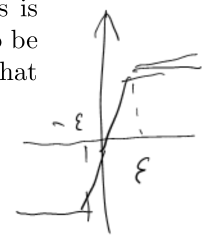
$-\int_{-1}^{0} x(t)dt + \int_{0}^{1} x(t)dx$

Then

$$| < x^*, x > | < 2\|x\|. \tag{1.24}$$



Indeed, restrict our attention to $\|x\| = 1$. We see then that $| < x^*, x > | \le 2$. Clearly, for the integral to attain maximum possible values, the integral should be of opposite values. We can focus on the case when the integral over $(-1, 0)$ is negative and over $(0, 1)$ is positive and then for the best values, $x(t)$ must negative on $(-1, 0)$ and positive on $(0, 1)$. Then, each term is at most 1 and for this $x(t) = 1$ for $t \in (0, 1)$ and $x(t) = -1$ for $t \in (-1, 0)$. But this is impossible as g is continuous at 0. On the other hand, by choosing $x(t)$ to be -1 for $-1 < t < -\epsilon$, 1 for $\epsilon < t < 1$ and linear between $-\epsilon$ and ϵ we see that



$$< x^*, x >= 2 - \epsilon$$

with $\|x\| = 1$. Hence, $\|x^*\| = 2$. However, this is impossible by (1.24).

Norms of functionals

Example 1.16. The existence of an element $\bar{x}^*$ satisfying $<\bar{x}^*, x>= \|x\|$ has an important consequence for the relation between X and X^{**} in a nonreflexive case. Let B, B^*, B^{**} denote the unit balls in X, X^*, X^{**}, respectively. Because $x^* \in X^*$ is an operator over X, the definition of the operator norm gives

real spaces

$\|A\| = \sup_{\|x\|\le 1} \|Ax\|$

$$\|x^*\|_{X^*} = \sup_{x\in B} | <x^*, x> | = \sup_{x\in B} <x^*, x>, \tag{1.25}$$

$\langle x^*, x\rangle_{X^* \times X} = \langle K(x), x^*\rangle_{X^{**} \times X^*}$

and similarly, for $x \in X$ considered as an element of X^{**} according to (1.21), we have

$$\|x\|_{X^{**}} = \sup_{x^*\in B^*} | <x^*, x> | = \sup_{x^*\in B^*} <x^*, x> . \tag{1.26}$$

$\le \sup_{x^* \in B^*} \|x^*\| \cdot \|x\| \le \|x\|$

Thus, $\|x\|_{X^{**}} \le \|x\|_X$. On the other hand,

$$\|x\|_X =<\bar{x}^*, x>\le \sup_{x^*\in B^*} <x^*, x>= \|x\|_{X^{**}}$$

and

$$\|x\|_{X^{**}} = \|x\|_X. \tag{1.27}$$

Hence, in particular, the identification given by (1.21) is an isometry and X is a closed subspace of X^{**}.

$X \overset{K}{\hookrightarrow} X^{**}$

First comments on weak convergence

The existence of a large number of functionals over X allows us to introduce new types of convergence. Apart from the standard *norm (or strong) convergence* where $(x_n)_{n\in\mathbb{N}} \subset X$ converges to x if

$$\lim_{n\to\infty} \|x_n - x\| = 0,$$

$\lim \langle x^*, x_n\rangle = l_{x^*}$

we define *weak convergence* by saying that $(x_n)_{n\in\mathbb{N}}$ weakly converges to x, if for any $x^* \in X^*$,

$$\lim_{n\to\infty} <x^*, x_n>=<x^*, x> .$$

In a similar manner, we say that $(x_n^*)_{n\in\mathbb{N}} \subset X^*$ converges $*$-*weakly* to x^* if, for any $x \in X$,

$$\lim_{n\to\infty} <x_n^*, x>=<x^*, x> .$$

This is like the pointwise convergence of functions

Let x, y be weak limits of (x_n)

Remark 1.17. It is worthwhile to note that we have a concept of a *weakly convergent* or *weakly Cauchy* sequence if the finite limit $\lim_{n\to\infty} <x^*, x_n>$ exists for any $x^* \in X^*$. In general, in this case we do not have a limit element. If every weakly convergent sequence converges weakly to an element of X, the Banach space is said to be *weakly sequentially complete.* It can be proved that reflexive spaces and L_1 spaces are weakly sequentially complete. On the other hand, no space containing a subspace isomorphic to the space c_0 (of sequences that converge to 0) is weakly sequentially complete (see, e.g., [6]).

$\forall_{x^*\in X^*} \lim \langle x^*, x_n\rangle = \langle x^*, x\rangle = \langle x^*, y\rangle \Rightarrow \forall_{x^*\in X^*} \langle x^*, x-y\rangle = 0$

$x_n = x_n^1 e_1 + \dots x_n^k e_k$

$\langle e_i^*, x_n\rangle = x_n^i$

Remark 1.18. In finite dimensional spaces weak and strong convergence is equivalent which can be seen by taking x^* being the coordinate vectors. Then weak convergence reduces to coordinate-wise convergence.

However, the weak convergence is indeed weaker than the convergence in norm. For example, consider any orthonormal basis $\{e_n\}_{n\geq 1}$ of a separable Hilbert space X. Then $\|e_n\| = 1$ but for any $f \in X$ we know that the series

$X \approx X^*$

$$\sum_{n=1}^{\infty} < f, e_n > e_n$$

$\cong X^*$ $(x_n) \subset \mathbb{R}^N$ $x_n \to x$ $\mathbb{R}^N$

$\|e_n - e_k\| \geq \sqrt{2}$

converges in X and, equivalently,

$x \in \mathbb{R}^N$

$$\sum_{n=1}^{\infty} | < f, e_n > |^2 < \infty.$$

$\|e_n - e_k\|^2 = (e_n - e_k, e_n - e_k) = \|e_n\|^2 + \|e_k\|^2 - (e_n, e_k) - (e_k, e_n)$, $\|e_n\|^2 = 1$, $\|e_k\|^2 = 1$, $(e_n, e_k) = 0$, $(e_k, e_n) = 0$

$\leftarrow (x, x_n) = x_1 x_n^1 + \dots + x_N x_n^N$

Thus

$$\lim_{n\to\infty} < f, e_n >= 0$$

for any $f \in X(= X^*)$ and so $(e_n)_{n\geq 0}$ weakly converges to zero.

1.2.3 Banach–Steinhaus Theorem

Another fundamental theorem of functional analysis is the Banach–Steinhaus theorem, or the Uniform Boundedness Principle. It is based on a fundamental topological results known as the Baire Category Theorem.

Theorem 1.19. *Let X be a complete metric space and let $\{X_n\}_{n\geq 1}$ be a sequence of closed subsets in X. If $Int\, X_n = \emptyset$ for any $n \geq 1$, then $Int \bigcup\limits_{n=1}^{\infty} X_n = \emptyset$. Equivalently, taking complements, we can state that a countable intersection of open dense sets is dense.*

Remark 1.20. Baire's theorem is often used in the following equivalent form: if X is a complete metric space and $\{X_n\}_{n\geq 1}$ is a countable family of closed sets such that $\bigcup\limits_{n=1}^{\infty} X_n = X$, then $Int X_n \neq \emptyset$ at least for one n.

Chaotic dynamical systems

We assume that X is a complete metric space, called the state space. In general, a *dynamical system* on X is just a family of states $(\mathbf{x}(t))_{t\in\mathbb{T}}$ parametrized by some parameter t (time). Two main types of dynamical systems occur in applications: those for which the time variable is discrete (like the observation times) and those for which it is continuous.

Theories for discrete and continuous dynamical systems are to some extent parallel. In what follows mainly we will be concerned with continuous dynamical systems. Also, to fix attention we shall discuss only systems defined for $t \geq 0$, that are sometimes called *semidynamical systems*. Thus by a *continuous dynamical system* we will understand a family of functions (operators) $(\mathbf{x}(t,\cdot))_{t\geq 0}$ such that for each t, $\mathbf{x}(t,\cdot) : X \to X$ is a continuous function, for each $\mathbf{x}_0$ the function $t \to \mathbf{x}(t, \mathbf{x}_0)$ is continuous with $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$. Moreover, typically it is required that the following semigroup property is satisfied (both in discrete and continuous case)

In discrete time

$x_n(x_0) =$

$x(x(x \ldots x(x_0)))$

$$\mathbf{x}(t+s, \mathbf{x}_0) = \mathbf{x}(t, \mathbf{x}(s, \mathbf{x}_0)), \qquad t, s \geq 0, \tag{1.28}$$

which expresses the fact that the final state of the system can be obtained as the superposition of intermediate states.

Often discrete dynamical systems arise from iterations of a function

$$\mathbf{x}(t+1, \mathbf{x}_0) = f(\mathbf{x}(t, \mathbf{x}_0)), \qquad t \in \mathbb{N}, \tag{1.29}$$

while when t is continuous, the dynamics are usually described by a differential equation

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = A(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0 \qquad t \in \mathbb{R}_+. \tag{1.30}$$

Let (X, d) be a metric space where, to avoid non-degeneracy, we assume that $X \neq \{\mathbf{x}(t, \boldsymbol{p})\}_{t\geq 0}$ for any $\boldsymbol{p} \in X$, that is, the space does not degenerates to a single orbit). We say that the dynamical system $(\mathbf{x}(t))_{t\geq 0}$ on (X, d) is *topologically transitive* if for any two non-empty open sets $U, V \subset X$ there is $t_0 \geq 0$ such that $\mathbf{x}(t, U) \cap V \neq \emptyset$. A *periodic point* of $(\mathbf{x}(t))_{t\geq 0}$ is any point $\boldsymbol{p} \in X$ satisfying

$$\mathbf{x}(T, \boldsymbol{p}) = \boldsymbol{p},$$

for some $T > 0$. The smallest such T is called the period of $\boldsymbol{p}$. We say that the system has *sensitive dependence on initial conditions*, abbreviated as sdic, if there exists $\delta > 0$ such that for every $\boldsymbol{p} \in X$ and a neighbourhood N_p of $\boldsymbol{p}$ there exists a point $\mathbf{y} \in N_p$ and $t_0 > 0$ such that the distance between $\mathbf{x}(t_0, \boldsymbol{p})$ and $\mathbf{x}(t_0, \mathbf{y})$ is larger than δ. This property captures the idea that in chaotic systems minute errors in experimental readings eventually lead to large scale divergence, and is widely understood to be the central idea in chaos.

With this preliminaries we are able to state Devaney's definition of chaos (as applied to continuous dynamical systems).

Definition 1.21. *Let X be a metric space. A dynamical system $(\mathbf{x}(t))_{t\geq 0}$ in X is said to be chaotic in X if*

1. *$(\mathbf{x}(t))_{t\geq 0}$ is transitive,*
2. *the set of periodic points of $(\mathbf{x}(t))_{t\geq 0}$ is dense in X,*
3. *$(\mathbf{x}(t))_{t\geq 0}$ has sdic.*

$(x, Ax, A^2x, \ldots)$

$(x(t,p))$

$(A^n)_{n\geq 1}$

Ax

A^2x

To summarize, chaotic systems have three ingredients: indecomposability (property 1), unpredictability (property 3), and an element of regularity (property 2).

It is then a remarkable observation that properties 1. and 2 together imply sdic.

Theorem 1.22. *If $(\mathbf{x}(t))_{t\geq 0}$ is topologically transitive and has dense set of periodic points, then it has sdic.*

We say that X is non-degenerate, if continuous images of a compact intervals are nowhere dense in X.

Lemma 1.23. *Let X be a non-degenerate metric space. If the orbit $O(\boldsymbol{p}) = \{\mathbf{x}(t, \boldsymbol{p})\}_{t\geq 0}$ is dense in X, then also the orbit $O(\mathbf{x}(s, \boldsymbol{p})) = \{\mathbf{x}(t, \boldsymbol{p})\}_{t>s}$ is dense in X, for any $s > 0$.*

p

$\{x(t,p)\}_{t\geq 0}$

$O(p) = \{x(t,p)\}_{t\geq 0}$

Proof. Assume that $O(\mathbf{x}(s, \boldsymbol{p}))$ is not dense in X, then there is an open ball B such that $B \cap \overline{O(\mathbf{x}(s, \boldsymbol{p}))} = \emptyset$. However, each point of the ball is a limit point of the whole orbit $O(\boldsymbol{p})$, thus we must have $\overline{\{\mathbf{x}(t, \boldsymbol{p})\}_{0\leq t\leq s}} = \overline{\{\mathbf{x}(t, \boldsymbol{p})\}_{0\leq t<s}} \supset B$ which contradicts the assumption of nondegeneracy. ■

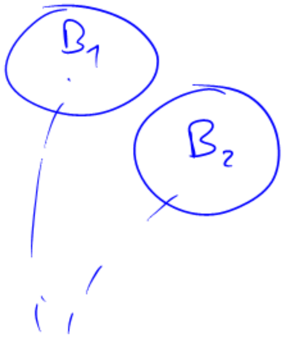
To fix terminology we say that a semigroup having a dense trajectory is called *hypercyclic*. We note that by continuity $\overline{O(\boldsymbol{p})} = \overline{\{\mathbf{x}(t, \boldsymbol{p})\}_{t\in\mathbb{Q}}}$, where $\mathbb{Q}$ is the set of positive rational numbers, therefore hypercyclic semigroups can exist only in separable spaces.

$A^k(A^l) = A^{k+l}$

e^t. $(e^t x)_{t\geq 0}$

By X_h we denote the set of hypercyclic vectors, that is,

$$X_h = \{\boldsymbol{p} \in X;\ O(\boldsymbol{p}) \text{ is dense in } X\}$$



Note that if $(\mathbf{x}(t))_{t\geq 0}$ has one hypercyclic vector, then it has a dense set of hypercyclic vectors as each of the point on the orbit $O(\boldsymbol{p})$ is hypercyclic (by the first part of the proof above).

Theorem 1.24. *Let $(\mathbf{x}(t))_{t\geq 0}$ be a strongly continuous semigroup of continuous operators (possibly nonlinear) on a complete (separable) metric space X. The following conditions are equivalent:*

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1. *X_h is dense in X,*
2. *$(\mathbf{x}(t))_{t\geq 0}$ is topologically transitive.*

Proof. Let as take the set of nonegative rational numbers and enumerate them as $\{t_1, t_2, \ldots\}$. Consider now the family $\{\mathbf{x}(t_n)\}_{n\in\mathbb{N}}$. Clearly, the orbit of $\boldsymbol{p}$ through $(\mathbf{x}(t))_{t\geq 0}$ is dense in X if and only if the set $\{\mathbf{x}(t_n)\boldsymbol{p}\}_{n\in\mathbb{N}}$ is dense.

$x^{-1}(t_1, B_1) \cap x^{-1}(t_2, B_2)$

Consider now the covering of X by the enumerated sequence of balls B_m centered at points of a countable subset of X with rational radii. Since each $\mathbf{x}(t_m)$ is continuous, the sets

$$G_m = \bigcup_{n\in\mathbb{N}} \mathbf{x}^{-1}(t_n, B_m)$$

are open. Next we claim that

$$\boxed{X_h = \bigcap_{m\in\mathbb{N}} G_m.} \ ?$$

In fact, let $\boldsymbol{p} \in X_h$, that is, $\boldsymbol{p}$ is hypercyclic. It means that $\mathbf{x}(t_n, \boldsymbol{p})$ visits each neigbourhood of each point of X for some n. In particular, for each m there must be n such that $\mathbf{x}(t_n, \boldsymbol{p}) \in B_m$ or $\boldsymbol{p} \in \mathbf{x}^{-1}(t_n, B_m)$ which means $\boldsymbol{p} \in \bigcap\limits_{m\in\mathbb{N}} G_m$.

Conversely, if $\boldsymbol{p} \in \bigcap\limits_{m\in\mathbb{N}} G_m$, then for each m there is n such that $\boldsymbol{p} \in \mathbf{x}^{-1}(t_n, B_m)$, that is, $\mathbf{x}(t_n, \boldsymbol{p}) \in B_m$. This means that $\{\mathbf{x}(t_n, \boldsymbol{p})\}_{n\in\mathbb{N}}$ is dense.

The next claim is condition 2. is equivalent to each set G_m being dense in X. If G_m were not dense, then for some B_r, $B_r \cap \mathbf{x}^{-1}(t_n, B_m) = \emptyset$ for any n. But then $\mathbf{x}(t_n, B_r) \cap B_m = \emptyset$ for any n. Since the continuous semigroup is topologically transitive, we know that there is $\mathbf{y} \in B_r$ such that $\mathbf{x}(t_0, \mathbf{y}) \in B_m$ for some t_0. Since B_m is open, $\mathbf{x}(t, \mathbf{y}) \in B_m$ for t from some neighbourhood of t_0 and this neighbourhood must contain rational numbers.

The converse is immediate as for given open U and V we find $B_m \subset V$ and since G_m is dense $U \cap G_m \neq \emptyset$. Thus $U \cap \mathbf{x}^{-1}(t_n, B_m) \neq \emptyset$ for some n, hence $\mathbf{x}(t_n, U) \cap B_m \neq \emptyset$.

So, if $(\mathbf{x}(t))_{t\geq 0}$ is topologically transitive, then X_h is the intersection of a countable collection of open dense sets, and by Baire Theorem in a complete space such an intersection must be still dense, thus X_h is dense.

Conversely, if X_h is dense, then each term of the intersection must be dense, thus each G_m is dense which yields the transitivity. ■

Back to the Banach–Steinhaus Theorem

To understand its importance, let us reflect for a moment on possible types of convergence of sequences of operators. Because the space $\mathcal{L}(X,Y)$ can be made a normed space by introducing the norm (1.11), the most natural concept of convergence of a sequence $(A_n)_{n\in\mathbb{N}}$ would be with respect to this norm. Such a convergence is referred to as the *uniform operator convergence*. However, for many purposes this notion is too strong and we work with the pointwise or *strong convergence*: the sequence $(A_n)_{n\in\mathbb{N}}$ is said to converge strongly if, for each $x \in X$, the sequence $(A_n x)_{n\in\mathbb{N}}$ converges in the norm of Y. In the same way we define uniform and strong boundedness of a subset of $\mathcal{L}(X,Y)$.

Note that if $Y = \mathbb{R}$ (or $\mathbb{C}$), then strong convergence coincides with $*$-weak convergence.

After these preliminaries we can formulate the Banach–Steinhaus theorem.

Theorem 1.25. *Assume that X is a Banach space and Y is a normed space. Then a subset of $\mathcal{L}(X,Y)$ is uniformly bounded if and only if it is strongly bounded.*

One of the most important consequences of the Banach–Steinhaus theorem is that a strongly converging sequence of bounded operators is always converging to a linear bounded operator. That is, if for each x there is y_x such that

$$\lim_{n\to\infty} A_n x = y_x,$$

then there is $A \in \mathcal{L}(X,Y)$ satisfying $Ax = y_x$.

Further comments on weak convergence

Example 1.26. We can use the above result to get a better understanding of the concept of weak convergence and, in particular, to clarify the relation between reflexive and weakly sequentially complete spaces. First, by considering elements of X^* as operators in $\mathcal{L}(X,\mathbb{C})$, we see that every $*$-weakly converging sequence of functionals converges to an element of X^* in $*$-weak topology. On the other hand, for a weakly converging sequence $(x_n)_{n\in\mathbb{N}} \subset X$, such an approach requires that $x_n, n \in \mathbb{N}$, be identified with elements of X^{**} and thus, by the Banach–Steinhaus theorem, a weakly converging sequence always has a limit $x \in X^{**}$. If X is reflexive, then $x \in X$ and X is weakly sequentially complete. However, for nonreflexive X we might have $x \in X^{**} \setminus X$ and then $(x_n)_{n\in\mathbb{N}}$ does not converge weakly to any element of X.

$A_n \to A$ if $\forall A_n x \to Ax$, $x \in X$ strong

$\forall \exists M \|A_n x\|$ x $M_x \le M_x$

$\|A\| = \sup_{\|x\|\le 1} \|Ax\|$

$\|A_n\| \le M$

$\|A_n - A\|$ $\downarrow_{n\to\infty}$ 0

$\|A\|$

$\sup \|A\|$ $\|x\| \le 1$

Zbiór $M \subset \mathcal{L}(X,Y)$ jedn. ogr.

$\sup_{A\in M} \|Ax\| \le C_x$ $\forall_x \exists C_x$

$\{A_n(x+z)\} - A_n x + A_n z$

y_{x+z} $\quad y_x + y_z$

$\lim \langle x_n^*, x\rangle$ $= \exists x^* \langle x^*, x\rangle$

Weak Cauchy sequence (x_n)

$\forall_{x^* \in X^*} \langle x^*, x_n\rangle$ is a scalar Cauchy sequence $\Rightarrow$

$\forall_{x^*} \langle x^*, x_n\rangle \to \ell_{x^*}$

$\langle x^*, x_n\rangle = \langle x_n, x^*\rangle \to \ell_{x^*}$

$\langle x^*, x\rangle$ $\quad x^{**} \in X^{**}$ $\quad \ell_{x^*} = \langle x^{**}, x^*\rangle$

Is there x such that $\ell_{x^*} = \langle x^*, x\rangle$?

On the other hand, (1.27) implies that a weakly convergent sequence in a normed space is norm bounded. Indeed, we consider $(x_n)_{n\in\mathbb{N}}$ such that for each $x^* \in X^*$, $< x^*, x_n >$ converges. Treating x_n as elements of X^{**}, we see that the numerical sequences $< x_n, x^* >$ are bounded for each $x^* \in X^*$. X^* is a Banach space (even if X is not). Then $(\|x_n\|)_{n\geq 0}$ is bounded by the Banach-Steinhaus theorem.

$\|x_n\|_{X^{**}}$ is bounded by B-S and thus $\|x_n\|_X$ is bounded by (1.27)

We can also prove ~~the~~ a partial reverse of this inequality: if $(x_n)_{n\in\mathbb{N}}$ is a sequence in a normed space X weakly converging to x, then

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|. \tag{1.31}$$

To prove this, there is $x^* \in X^*$ such that

$$\|x^*\| = 1, \qquad | < x^*, x > | = \|x\|.$$

Hence

$$\|x\| = | < x^*, x > | = | \lim_{n\to\infty} < x^*, x_n > | \leq \liminf_{n\to\infty} | \underbrace{< x^*, x_n >}_{\leq \|x^*\| \|x_n\|} | \leq \liminf_{n\to\infty} \|x_n\|.$$

closed

However, we point out that a theorem proved by Mazur (e.g., see [172], p. 120) says that if $x_n \to x$ weakly, then there is a sequence of convex combinations of elements of $(x_n)_{n\in\mathbb{N}}$ that converges to x in norm. To prove this result, let us introduce the concept of the support function of a set. For a set M we define

$$S_M(x^*) = \sup_{x\in M} < x^*, x > .$$

A crucial result is

Lemma 1.27. *If X is a normed space over $\mathbb{R}$ and M is a closed convex subset of X, then $z \in M$ if and only if $< x^*, z > \leq S_M(x^*)$ for any $x^* \in X^*$.*

$M = \{x_0\}$

$S_M(x^*) = \langle x^*, x_0 \rangle$

$M = B(0, r)$

Proof. If $z \in M$, then $< x^*, z > \leq \sup_{x\in M} < x^*, x > = S_M(x^*)$ by definition.

If $z \notin M$ then, by closedness, there is a ball $B(z, r)$ not intersecting with M. By the geometric version of the Hahn-Banach theorem, there is a linear functional z^* and a constant c such that for any $x \in M$ and $y \in B(z, r)$ we have

$\forall_{y\in B}\ S_M(z^*) \leq c \leq \langle z^*, y \rangle$

$$\boxed{< z^*, x > \leq c \leq < z^*, y > .}$$

$\forall_{x\in M,\, y\in B} \langle z^*, x \rangle \leq \langle z^*, y \rangle$

Since $y = z + rv$, $\|v\| \leq 1$, we have

$S_M(z^*) \leq \inf_{y\in B} \langle z^*, y \rangle$

$$c \leq < z^*, z + rv > = < z^*, z > + r < z^*, v > .$$

$\inf \langle z^*, z + rv \rangle = \langle z^*, z \rangle + r \inf \langle z^*, v \rangle$

Using the fact that $\inf_{\|v\|\leq 1} < z^*, v > = -\|z^*\|$, we obtain

$$c \underset{\inf}{\leq} < z^*, z + rv > = < z^*, z > - r\|z^*\|.$$

$< c$

$S_M(x^*) = \sup_{x\in B(0,r)} \langle x^*, x \rangle = r \sup_{x\in B(0,1)} \langle x^*, x \rangle = r\|x^*\|$

On the other hand

$\langle x^*, x \rangle = c$

$\|z^*\| = \sup_{x\in B(0,1)} \langle z^*, x \rangle$

$-\|z^*\| = \inf_{x\in B(0,1)} \langle z^*, x \rangle$

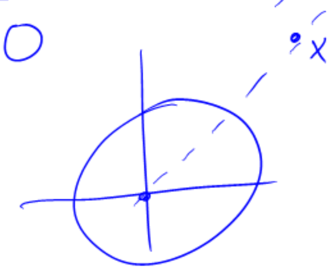
$> c$

Geometric formulation of the Hahn-Banach theorem

1. Minkowski functional (gauge function)

C - open, convex, containing 0

$p_C(x) = \inf\{\alpha > 0;\ \frac{x}{\alpha} \in C\}$



1) $C = \{x \in X;\ p_C(x) < 1\}$
2) $p(\lambda x) = \lambda p(x) \quad \lambda > 0$
3) $p(x+y) \le p(x) + p(y)$
4) $0 \le p(x) \le M\|x\|$ for some M

$p(x) < 1$ there is $\alpha < 1$ such that $\frac{x}{\alpha} \in C$

$x = \alpha \frac{x}{\alpha} + (1-\alpha)\, 0 \in C$

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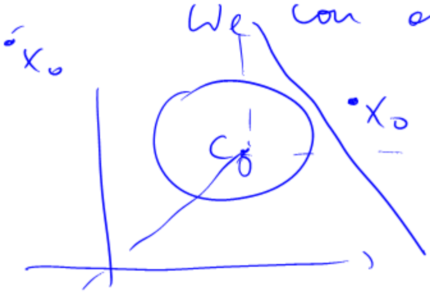
2. Separation of a point and an open convex set C

We can assume $0 \in C$



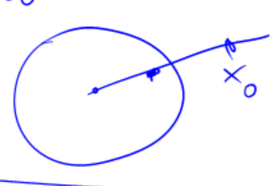
Statement of the theorem: There is $x^* \in X^*$ such that

$$\forall_{x \in C} \quad \langle x^*, x\rangle > \langle x^*, x_0\rangle$$

$p_C(x)$ - Mink. functi. for C

Linear subspace $X_0 = \mathbb{R}\cdot x_0 = \{x;\ x = \alpha x_0,\ \alpha \in \mathbb{R}\}$ and linear functional on X_0 defined by $\langle y^*, x\rangle = \alpha$. It follows that

$$\langle y^*, x\rangle \le p_C(x)$$



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3. If M is closed convex, C is open convex then there is a linear function y^* such that $M \cap C = \emptyset$

$\forall \quad \langle y^*, x\rangle < \langle y^*, z\rangle$
$x \in M$
$z \in C$

This version is reduced to 2 by considering

$$C - M = \{x - z \;;\; x \in C, z \in M\}$$

(it is an open, convex set)

$$C - M = \bigcup_{z \in M} C - z$$

and $x_0 = 0$.

$\left.\begin{matrix} r \\ s \end{matrix}\right\} \in C - M \Rightarrow \quad r = x_1 - z_1$
$\quad s = x_2 - z_2$

$\alpha, \beta \quad \alpha + \beta = 1$

$$\alpha r + \beta s = \alpha x_1 - \alpha z_1 + (\beta x_2 - \beta z_2)$$
$$= \underbrace{\alpha x_1 + \beta x_2}_{\in C} - (\underbrace{\alpha z_1 + \beta z_2}_{\in Z})$$

$$\inf_{y \in B(z,r)} \langle z^*, y\rangle \overset{y = z + rv}{=} \inf_{\|v\| \le 1} (\langle z^*, z\rangle + \langle z^*, rv\rangle)$$
$$= \langle z^*, z\rangle + \inf_{\|v\| \le 1} \langle z^*, rv\rangle$$
$$\langle z^*, z\rangle + r \inf_{\|v\| \le 1} \langle z^*, v\rangle$$

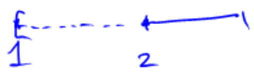
$$-\sup_{\|v\| \le 1} \langle z^*, v\rangle = -\|z^*\|$$
$$\|$$
$$\inf_{\|v\| \le 1} \langle z^*, -v\rangle = -\|z^*\|$$
$$\inf_{\|-v\| \le 1} \langle z^*, v\rangle = -\|z^*\|$$
$$\|v\| \le 1$$

What we proved? That if K is convex & str. closed and $(x_n) \subset K$, $x_n \rightharpoonup x \Rightarrow x \in K$

Proof of the final version. Let $x_n \rightharpoonup x$

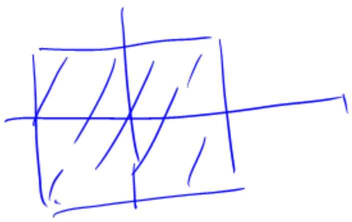
$K = \overline{\operatorname{conv}\{x_1 \ldots x_n \ldots\}}$ $\leftarrow$ norm closure $\Rightarrow K$ is strongly closed, convex

$x \in K$, from the previous proof.



$\alpha \cdot 1 + \beta \cdot 2 \qquad \alpha + \beta = 1$

$\alpha, \beta \geq 0$



Uniform convergence $\quad f_n \rightrightarrows f$

$$\forall_{\varepsilon > 0} \ \exists_{n_0} \ \forall_{\substack{n > n_0 \\ x \in \Omega}} \ |f_n(x) - f(x)| < \varepsilon$$

$$\Updownarrow$$

$$\sup_{x \in \Omega} |f_n(x) - f(x)| < \varepsilon \qquad \text{uniform}$$

$$\forall_{\varepsilon > 0} \ \forall_{x \in \Omega} \ \exists_{n_0} \ \forall_{n > n_0} \ |f_n(x) - f(x)| < \varepsilon \qquad \text{pointwise}$$

Convergence of sequence of operators A_n in the operator norm is uniform on the unit ball

$$0 \leftarrow \|A_n - A\| = \sup_{x \in B(0,1)} \|A_n x - Ax\|$$

If $A_n x \to Ax$ for any x, then for any compact set Ω

$$\sup_{x \in \Omega} \|A_n x - Ax\| \to 0$$

z

$$S_M(z^*) \leq c \leq < z^*, z > -r\|z^*\|$$

M

which yields

$$< z^*, z > \geq S_M(z^*) + r\|z^*\| > S_M(z^*)$$

and completes the proof.

With this result we can prove the Mazur theorem.

Let K be a closed convex set and $(x_n)_{n\in\mathbb{N}}$ be a sequence weakly converging to x ~~$\in K$~~. Consider $S_K(x^*)$. We have

$$< x^*, x_n > \leq S_K(x^*)$$

for any $x^* \in X^*$. But this implies

that $x \in K$
$$< x^*, x > \leq S_K(x^*)$$

and the result follows by the above lemma.

Mazur theorem states that there is a sequence of convex combination strongly converged to x

Unit sphere is strongly closed but not convex

The Banach-Steinhaus theorem and convergence on subsets

We note another important corollary of the Banach–Steinhaus theorem which we use in the sequel.

A_n - strong convergent: $\forall_x A_n x \to Ax$ $\Updownarrow$ $A_n K \to AK$ for any compact K in uniform topology

Corollary 1.28. *A sequence of operators $(A_n)_{n\in\mathbb{N}}$ is strongly convergent if and only if it is convergent uniformly on compact sets.*

Proof. It is enough to consider convergence to 0. If $(A_n)_{n\in\mathbb{N}}$ converges strongly, then by the Banach–Steinhaus theorem, $a = \sup_{n\in\mathbb{N}} \|A_n\| < +\infty$. Next, if $\Omega \subset X$ is compact, then for any ϵ we can find a finite set $N_\epsilon = \{x_1, \ldots, x_k\}$ such that for any $x \in \Omega$ there is $x_i \in N_\epsilon$ with $\|x - x_i\| \leq \epsilon/2a$. Because N_ϵ is finite, we can find n_0 such that for all $n > n_0$ and $i = 1, \ldots, k$ we have $\|A_n x_i\| \leq \epsilon/2$ and hence

$\|A_n(x - x_i)\| \leq \|A_n\|\|x - x_i\|$

$$\|A_n x\| \leq \|A_n x_i\| + a\|x - x_i\| \leq \epsilon$$

for any $x \in \Omega$. The converse statement is obvious. □

We conclude this unit by presenting a frequently used result related to the Banach–Steinhaus theorem.

Proposition 1.29. *Let X, Y be Banach spaces and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X, Y)$ be a sequence of operators satisfying $\sup_{n\in\mathbb{N}} \|A_n\| \leq M$ for some $M > 0$. If there is a dense subset $D \subset X$ such that $(A_n x)_{n\in\mathbb{N}}$ is a Cauchy sequence for any $x \in D$, then $(A_n x)_{n\in\mathbb{N}}$ converges for any $x \in X$ to some $A \in \mathcal{L}(X, Y)$.*

X

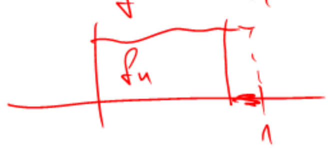
Proof. Let us fix $\epsilon > 0$ and $y \in X$. For this ϵ we find $x \in D$ with $\|x - y\| < \epsilon/M$ and for this x we find n_0 such that $\|A_n x - A_m x\| < \epsilon$ for all $n, m > n_0$. Thus,

$$\|A_n y - A_m y\| \leq \|A_n x - A_m x\| + \|A_n(x - y)\| + \|A_m(x - y)\| \leq 3\epsilon.$$

D

Hence, $(A_n y)_{n\in\mathbb{N}}$ is a Cauchy sequence for any $y \in X$ and, because Y is a Banach space, it converges and an application of the Banach–Steinhaus theorem ends the proof. □

$f \in L_1([0,1])$



$\frac{dm}{dt} = g(m)$

Application–limits of integral expressions

Consider an equation describing growth of, say, cells

$$\frac{\partial N}{\partial t} + \frac{\partial (g(m)N)}{\partial m} = -\mu(m)N(t,m), \quad m \in (0,1), \tag{1.32}$$

with the boundary condition

$$g(0)N(t,0) = 0 \tag{1.33}$$

and with the initial condition

$$N(0,m) = N_0(m) \quad \text{for } m \in [0,1]. \tag{1.34}$$

Here $N(m)$ denotes cells' density with respect to their size/mass and we consider the problem in $L_1([0,1])$.

Consider the 'formal' equation for the stationary version of the equation (the resolvent equation)

$$\lambda N(m) + (g(m)N(m))' + \mu(m)N(m) = f(m) \in L_1([0,1]),$$

$\int_0^\infty e^{-\lambda t} N' dt = e^{-\lambda t} N \Big|_0^\infty + \lambda \int_0^\infty e^{-\lambda N} N dt$

whose solution is given by

$f = \begin{cases} f(s) & s < 1-\delta \\ 0 & 1 > s > 1-\delta \end{cases}$

$$N_\lambda(m) = \frac{e^{-\lambda G(m) - Q(m)}}{g(m)} \int_0^m e^{\lambda G(s) + Q(s)} f(s)\, ds \tag{1.35}$$

$m \to 1^-$

where $G(m) = \int_0^m (1/g(s))\, ds$ and $Q(m) = \int_0^m (\mu(s)/g(s))\, ds$. To shorten notation we denote

$$e_{-\lambda}(m) := e^{-\lambda G(m) - Q(m)}, \qquad e_\lambda(m) := e^{\lambda G(m) + Q(m)}.$$

Our aim is to show that $g(m)N_\lambda(m) \to 0$ as $m \to 1^-$ provided $1/g$ or μ is not integrable close to 1. If the latter condition is satisfied, then $e_\lambda(m) \to \infty$ and $e^{-\lambda}(m) \to 0$ as $m \to 1^-$.

Indeed, consider the family of functionals $\{\xi_m\}_{m \in [1-\epsilon, 1)}$ for some $\epsilon > 0$ defined by

$$\xi_m f = e_{-\lambda}(m) \int_0^m e_\lambda(s) f(s)\, ds$$

$\lim_{m\to\infty} \int_0^{1-\delta} e_\lambda(s) f(s)\, ds$

for $f \in L^1[0,1]$. We have

$$|\xi_m f| \le e_{-\lambda}(m) \int_0^m e_\lambda(s)|f(s)|\, ds \le \int_0^1 |f(s)|\, ds$$

on account of monotonicity of e_λ. Moreover, for f with support in $[0, 1-\delta]$ with any $\delta > 0$ we have $\lim_{m \to 1^-} \xi_m f = 0$ and, by Proposition 1.29, the above limit extends by density for any $f \in L^1[0,1]$.

$\xi_m \hat{f} \to 0$ from Prop 1.29

$\xi_m f$ converge for any $f \in L_1([0,1])$

$\xi_m f = \xi_m(f - \hat{f}) + \xi_m \hat{f} \to 0$

For any $\varepsilon > 0$ I find $\hat{f}$ such that $\|\hat{f} - f\| < \varepsilon$

$\|\lim \xi_m f\| \le \|f - \hat{f}\| + 0$

1.2.4 Weak compactness

In finite dimensional spaces normed spaces we have Bolzano-Weierstrass theorem stating that from any bounded sequence of elements of X_n one can select a convergent subsequence. In other words, a closed unit ball in X_n is compact.

There is no infinite dimensional normed space in which the unit ball is compact.

Weak compactness comes to the rescue. Let us begin with (separable) Hilbert spaces.

Theorem 1.30. *Each bounded sequence $(u_n)_{n\in\mathbb{N}}$ in a separable Hilbert space X has a weakly convergent subsequence.*

Proof. Let $\{v_k\}_{k\in\mathbb{N}}$ be dense in X and consider numerical sequences $((u_n, v_k))_{n\in\mathbb{N}}$ for any k. From Banach-Steinhaus theorem and

a_n^1 $k = 1$

$$|(u_n, v_k)| \leq \|u_n\|\|v_k\|$$

(a_n^1) $(a^1_{i_n})$

we see that for each k these sequences are bounded and hence each has a convergent subsequence. We use the diagonal procedure: first we select $(u_{1n})_{n\in\mathbb{N}}$ such that $(u_{1n}, v_1) \to a_1$, then from $(u_{1n})_{n\in\mathbb{N}}$ we select $(u_{2n})_{n\in\mathbb{N}}$ such that $(u_{2n}, v_2) \to a_2$ and continue by induction. Finally, we take the diagonal sequence $w_n = u_{nn}$ which has the property that $(w_n, v_k) \to a_k$. This follows from the fact that elements of $(w_n)_{n\in\mathbb{N}}$ belong to $(u_{kn}$ for $n \geq k$. Since $\{v_k\}_{k\in\mathbb{N}}$ is dense in X and $(u_n)_{n\in\mathbb{N}}$ is norm bounded, Proposition 1.29 implies $((w_n, v))_{n\in\mathbb{N}}$ converges to, say, $a(v)$ for any $v \in X$ and $v \to a(v)$ is a bounded (anti) linear functional on X. By the Riesz representation theorem, there is $w \in X$ such that $a(v) = (v, w)$ and thus $w_n \rightharpoonup w$.

$(a^2_{i_n})$ $(a^2_{i_{n_n}})$ (w_n) $w_1 = a^1_{i_1}$ $w_2 = a^2_{i_{n_2}}$

If X is not separable, then we can consider $Y = \overline{\mathcal{L}in\{u_n\}_{n\in\mathbb{N}}}$ which is separable and apply the above theorem in Y getting an element $w \in Y$ for which

$$(w_n, v) \to (w, v), \qquad v \in Y.$$

Let now $z \in X$. By orthogonal decomposition, $z = v + v^{\perp}$ by linearity and continuity (as $w \in Y$)

$$(w_n, z) = (w_n, v) \to (w, v) = (w, z)$$

(w_n) $n \geq$

and so $w_n \rightharpoonup w$ in X.

Corollary 1.31. *Closed unit ball in X is weakly sequentially compact.*

$(v_k, u_{i_{k_n}})$

Proof. We have

$$(v, w_n) \to (v, w), \quad n \to \infty$$

for any v. We can assume $w = 0$ We prove that for any k there are indices $n_1, \ldots, n_k$ such that

$$k^{-1}(w_{n_1} + \ldots + w_{n_k}) \to 0$$

$(\tilde{u}_n, v_n)$ v_n - ośrodek $\|u_n\| \leq M$