Remark 1.5. We observe that, if f is nonnegative, then f_ϵ are also nonnegative by (1.7) and hence any non-negative $f \in L_p(\mathbb{R}^n)$ can be approximated by nonnegative, infinitely differentiable, functions with compact support.

Remark 1.6. Spaces $L_p(\Omega)$ often are defined as a completion of $C_0(\Omega)$ in the $L_p(\Omega)$ norm, thus avoiding introduction of measure theory. The theorem above shows that these two definitions are equivalent.

1.1.2 Operators

Let X, Y be real or complex Banach spaces with the norm denoted by $\|\cdot\|$ or $\|\cdot\|_X$.

An *operator* from X to Y is a linear rule $A : D(A) \to Y$, where $D(A)$ is a linear subspace of X, called the *domain* of A. The set of operators from X to Y is denoted by $L(X, Y)$. Operators taking their values in the space of scalars are called *functionals*. We use the notation $(A, D(A))$ to denote the operator A with domain $D(A)$. If $A \in L(X, X)$, then we say that A (or $(A, D(A))$) is an operator in X.

By $\mathcal{L}(X, Y)$, we denote the space of all bounded operators between X and Y; $\mathcal{L}(X, X)$ is abbreviated as $\mathcal{L}(X)$. The space $\mathcal{L}(X, Y)$ can be made a Banach space by introducing the norm of an operator X by

$$\|A\| = \sup_{\|x\|\leq 1} \|Ax\| = \sup_{\|x\|=1} \|Ax\|. \tag{1.11}$$

If $(A, D(A))$ is an operator in X and $Y \subset X$, then the *part* of the operator A in Y is defined as

$$A_Y y = Ay \tag{1.12}$$

on the domain

$$D(A_Y) = \{x \in D(A) \cap Y;\ Ax \in Y\}.$$

A *restriction* of $(A, D(A))$ to $D \subset D(A)$ is denoted by $A|_D$. For $A, B \in L(X, Y)$, we write $A \subset B$ if $D(A) \subset D(B)$ and $B|_{D(A)} = A$.

Two operators $A, B \in \mathcal{L}(X)$ are said to commute if $AB = BA$. It is not easy to extend this definition to unbounded operators due to the difficulties with defining the domains of the composition. The extension is usually done to the case when one of the operators is bounded. Thus, an operator $A \in L(X)$ is said to *commute* with $B \in \mathcal{L}(X)$ if

$$BA \subset AB. \tag{1.13}$$

This means that for any $x \in D(A)$, $Bx \in D(A)$ and $BAx = ABx$.

We define the *image* of A by

$$ImA = \{y \in Y;\ y = Ax \text{ for some } x \in D(A)\}$$

and the *kernel* of A by

$$KerA = \{x \in D(A);\ Ax = 0\}.$$

We note a simple result which is frequently used throughout the book.

Proposition 1.7. *Suppose that $A, B \in L(X, Y)$ satisfy: $A \subset B$, $KerB = \{0\}$, and $ImA = Y$. Then $A = B$.*

Proof. If $D(A) \neq D(B)$, we take $x \in D(B) \setminus D(A)$ and let $y = Bx$. Because A is onto, there is $x' \in D(A)$ such that $y = Ax'$. Because $x' \in D(A) \subset D(B)$ and $A \subset B$, we have $y = Ax' = Bx'$ and $Bx' = Bx$. Because $KerB = \{0\}$, we obtain $x = x'$ which is a contradiction with $x \notin D(A)$. □

Furthermore, the *graph* of A is defined as

$$G(A) = \{(x, y) \in X \times Y;\ x \in D(A), y = Ax\}. \tag{1.14}$$

We say that the operator A is *closed* if $G(A)$ is a closed subspace of $X \times Y$. Equivalently, A is closed if and only if for any sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$, if $\lim_{n\to\infty} x_n = x$ in X and $\lim_{n\to\infty} Ax_n = y$ in Y, then $x \in D(A)$ and $y = Ax$.

An operator A in X is *closable* if the closure of its graph $\overline{G(A)}$ is itself a graph of an operator, that is, if $(0, y) \in \overline{G(A)}$ implies $y = 0$. Equivalently, A is closable if and only if for any sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$, if $\lim_{n\to\infty} x_n = 0$ in X and $\lim_{n\to\infty} Ax_n = y$ in Y, then $y = 0$. In such a case the operator whose graph is $\overline{G(A)}$ is called the *closure* of A and denoted by $\overline{A}$.

By definition, when A is closable, then

$$\begin{aligned} D(\overline{A}) = \{x \in X;\ &\text{there is } (x_n)_{n\in\mathbb{N}} \subset D(A) \text{ and } y \in X \text{ such that} \\ &\|x_n - x\| \to 0 \text{ and } \|Ax_n - y\| \to 0\}, \\ \overline{A}x = y.& \end{aligned}$$

For any operator A, its domain $D(A)$ is a normed space under the *graph norm*

$$\|x\|_{D(A)} := \|x\|_X + \|Ax\|_Y. \tag{1.15}$$

The operator $A : D(A) \to Y$ is always bounded with respect to the graph norm, and A is closed if and only if $D(A)$ is a Banach space under (1.15).

The differentiation operator

One of the simplest and most often used unbounded, but closed or closable, operators is the operator of differentiation. If X is any of the spaces $C([0, 1])$ or $L_p([0, 1])$, then considering $f_n(x) := C_n x^n$, where $C_n = 1$ in the former case and $C_n = (np + 1)^{1/p}$ in the latter, we see that in all cases $\|f_n\| = 1$. However,

$$\|Tf_n\|_{L_p(0,1)} = \|f'_n\| = n\left(\frac{np+1}{np+1-p}\right)^{1/p}$$

$Tf = f'$

o ile f' jest zdefiniowana

in $L_p([0,1])$ and $\|f_n'\| = n$ in $C([0,1])$, so that the operator of differentiation is unbounded.

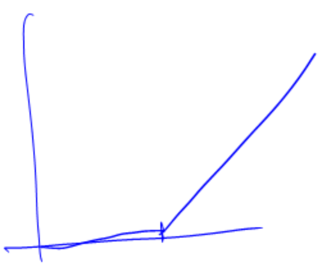
Let us define $Tf = f'$ as an unbounded operator on $D(T) = \{f \in X;\ Tf \in X\}$, where X is any of the above spaces. We can easily see that in $X = C([0,1])$ the operator T is closed. Indeed, let us take $(f_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} Tf_n = g$ in X. This means that $(f_n)_{n\in\mathbb{N}}$ and $(f_n')_{n\in\mathbb{N}}$ converge uniformly to, respectively, f and g, and from basic calculus f is differentiable and $f' = g$.

The picture changes, however, in L_p spaces. To simplify the notation, we take $p = 1$ and consider the sequence of functions

$$f_n(x) = \begin{cases} 0 & \text{for } 0 \le x \le \frac{1}{2}, \\ \frac{n}{2}\left(x - \frac{1}{2}\right)^2 & \text{for } \frac{1}{2} < x \le \frac{1}{2} + \frac{1}{n}, \\ x - \frac{1}{2} - \frac{1}{2n} & \text{for } \frac{1}{2} + \frac{1}{n} < x \le 1. \end{cases}$$

These are differentiable functions and it is easy to see that $(f_n)_{n\in\mathbb{N}}$ converges in $L_1([0,1])$ to the function f given by $f(x) = 0$ for $x \in [0,1/2]$ and $f(x) = x - 1/2$ for $x \in (1/2, 1]$ and the derivatives converge to $g(x) = 0$ if $x \in [0,1/2]$ and to $g(x) = 1$ otherwise. The function f, however, is not differentiable and so T is not closed. On the other hand, g seems to be a good candidate for the derivative of f in some more general sense. Let us develop this idea further. First, we show that T is closable. Let $(f_n)_{n\in\mathbb{N}}$ and $(f_n')_{n\in\mathbb{N}}$ converge in X to f and g, respectively. Then, for any $\phi \in C_0^\infty((0,1))$, we have, integrating by parts,

$f_n \to f$ w L_1

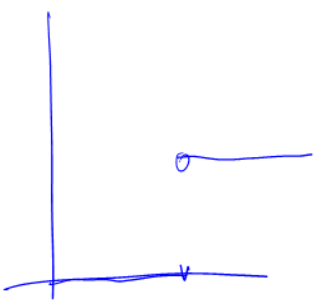


$$\int_0^1 f_n'(x)\phi(x)dx = -\int_0^1 f_n(x)\phi'(x)dx$$

and because we can pass to the limit on both sides, we obtain

$\int f_n \varphi \to \int f \varphi$

$$\int_0^1 g(x)\phi(x)dx = -\int_0^1 f(x)\phi'(x)dx. \tag{1.16}$$



Using the equivalent characterization of closability, we put $f = 0$, so that

$\left|\int (f_n - f)\varphi\right| \le \int |f_n - f|\cdot|\varphi| \le \max|\varphi|\, \|f_n - f\|_{L_1}$

$$\int_0^1 g(x)\phi(x)dx = 0$$

for any $\phi \in C_0^\infty((0,1))$ which yields $g(x) = 0$ almost everywhere on $[0,1]$. Hence $g = 0$ in $L_1([0,1])$ and consequently T is closable.

The domain of $\overline{T}$ in $L_1([0,1])$ is called the Sobolev space $W_1^1([0,1])$ which is discussed in more detail in Subsection 2.3.1.

These considerations can be extended to hold in any $\Omega \subset \mathbb{R}^n$. In particular, we can use (1.16) to generalize the operation of differentiation in the following

$|x|' = ?$

$$-\int_{-\infty}^{\infty} |x| \cdot \varphi'(x)\,dx = +\int_{-\infty}^{0} x\,\varphi'(x)\,dx - \int_{0}^{\infty} x\,\varphi'(x)\,dx$$

$C_0^\infty(\mathbb{R})$

$$= = \int_{-\infty}^{0} 1\cdot\varphi - x\,\cancel{\varphi(x)}\Big|_{-R}^{0}$$

$+\int_0^\infty 1\,\varphi\,dx = \int_{-\infty}^{\infty} \operatorname{sign} x \cdot \varphi\,dx$

way: we say that a function $g \in L_{1,loc}(\Omega)$ is the *generalised (or distributional) derivative* of $f \in L_{1,loc}(\Omega)$ of order α, denoted by $\partial_{\mathbf{x}}^{\alpha} f$, if

$$\int_{\Omega} g(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = (-1)^{|\beta|} \int_{\Omega} f(\mathbf{x})\partial_{\mathbf{x}}^{\beta}\phi(\mathbf{x})d\mathbf{x} \tag{1.17}$$

for any $\phi \in C_0^\infty(\Omega)$.

This operation is well defined. This follows from the du Bois Reymond lemma.

From the considerations above it is clear that $\partial_{\mathbf{x}}^{\beta}$ is a closed operator extending the classical differentiation operator (from $C^{|\beta|}(\Omega)$). One can also prove that $\partial_{\mathbf{x}}^{\beta}$ is the closure of the classical differentiation operator.

f, $g = D^\alpha f$

Proposition 1.8. *If $\Omega = \mathbb{R}^n$, then $\partial_{\mathbf{x}}^{\beta}$ is the closure of the classical differentiation operator.*

du $\partial^\alpha f$

ניתן..

Proof. We use (1.7) and (1.8). Indeed, let $f \in L_p(\mathbb{R}^n)$ and $g = D^\alpha f \in L_p(\mathbb{R}^n)$. We consider $f_\epsilon = J_\epsilon * f \to f$ in L_p. By the Fubini theorem, we prove

$$\begin{aligned}
\int_{\mathbb{R}^n} (J_\epsilon * f)(\mathbf{x}) D^\alpha \phi(\mathbf{x}) d\mathbf{x} &= \int_{\mathbb{R}^n} \omega_\epsilon(y) \int_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y}) D^\alpha \phi(\mathbf{x}) d\mathbf{x} d\mathbf{y} \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \omega_\epsilon(y) \int_{\mathbb{R}^n} g(\mathbf{x}-\mathbf{y}) \phi(\mathbf{x}) d\mathbf{x} d\mathbf{y} \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^n} (J_\epsilon * g)\phi(\mathbf{x}) d\mathbf{x}
\end{aligned}$$

so that $D^\alpha f_\epsilon = J_\epsilon * D^\alpha f = J_\epsilon * g \to g$ as $\epsilon \to 0$ in L_p. This shows that action of the distributional derivative can be obtained as the closure of the classical derivation.

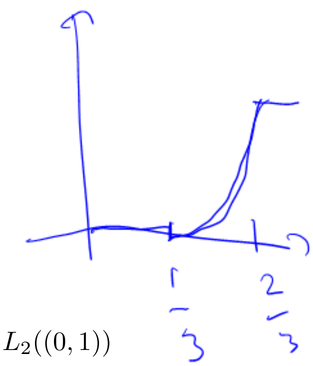
Otherwise the proof is more complicated (see, e.g., [4, Theorem 3.16]) since we do not know whether we can extend f outside Ω in such a way that the extension still will have the generalized derivative. We shall discuss it later.

Example 1.9. **A non closable operator.** Let us consider the space $X = L_p((0,1))$ and the operator $K : X \to Y$, $Y = X \times \mathbb{C}$ (with the Euclidean norm), defined by

$$Kv =< v, v(1) > \tag{1.18}$$

on the domain $D(K)$ consisting of continuous functions on $[0,1]$. We have the following lemma

Lemma 1.10. *K is not closeable, but has a bounded inverse. ImK is dense in Y.*



Proof. Let $f \in C^\infty([0,1])$ be such that

$$f(x) = \begin{cases} 0 \text{ for } 0 \le x < 1/3 \\ 1 \text{ for } 2/3 < x \le 1. \end{cases}$$

To construct such a function, we can consider e.g. $J_{1/3} * \bar{f}$ where

$$\bar{f}(x) = \begin{cases} 1 \quad \text{ for } \quad 2/3 < x \le 1, \\ 0 \text{ otherwise} \end{cases}$$

Let $v_n(x) = f(x^n)$ for $0 \le x \le 1$. Clearly, $v_n \in D(K)$ and $v_n \to 0$ in $L_2((0,1))$ as

$$\int\limits_0^1 f^2(x^n)dx = \int\limits_{3^{-1/n}}^1 f^2(x^n)dx = \frac{1}{n}\int\limits_{1/3}^1 z^{-1+1/n} f^2(z)dz.$$

However, $Kv_n = < v_n, 1 > \to < 0, 1 > \neq < 0, 0 >$.

Further, K is one-to-one with $K^{-1}(v, v(1)) = v$ and

$$\|K^{-1}(v, v(1))\|^2 = \|v\|^2 \le \|v\|^2 + |v(1)|^2.$$

To prove that ImK is dense in Y, let $< y, \alpha > \in Y$. We know that $C_0^\infty((0,1)) \subset D(K)$ is dense in $Z = L_2((0,1))$. Let (ϕ_n) be sequence of C_0^∞-functions which approximate y in $L_2(0,1)$ and put $w_n = \phi_n + \alpha v_n$. We have $Kw_n = < w_n, \alpha > \to < y, \alpha >$.

Absolutely continuous functions

In one-dimensional spaces the concept of the generalised derivative is closely related to a classical notion of absolutely continuous function. Let $I = [a,b] \subset \mathbb{R}^1$ be a bounded interval. We say that $f : I \to \mathbb{C}$ is *absolutely continuous* if, for any $\epsilon > 0$, there is $\delta > 0$ such that for any finite collection $\{(a_i, b_i)\}_i$ of disjoint intervals in $[a,b]$ satisfying $\sum_i (b_i - a_i) < \delta$, we have $\sum_i |f(b_i) - f(a_i)| < \epsilon$. The fundamental theorem of calculus, [150, Theorem 8.18], states that any absolutely continuous function f is differentiable almost everywhere, its derivative f' is Lebesgue integrable on $[a,b]$, and $f(t) - f(a) = \int_a^t f'(s)ds$. It can be proved (e.g., [61, Theorem VIII.2]) that absolutely continuous functions on $[a,b]$ are exactly integrable functions having integrable generalised derivatives and the generalised derivative of f coincides with the classical derivative of f almost everywhere.

Let us explore this connection. We prove

Theorem 1.11. *Assume that $u \in L_{1,loc}(\mathbb{R})$ and its generalized derivative Du also satisfies $Du \in L_{1,loc}(\mathbb{R})$. Then there is a continuous representation $\widetilde{u}$ of u such that*

$$\widetilde{u}(x) = C + \int\limits_0^x Du(t)dt$$

for some constant C and thus u is differentiable almost everywhere.