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Selected Topics in Applied Functional Analysis

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Basic Facts from Functional Analysis and Banach Lattices

1.1 Spaces and Operators

1.1.1 General Notation

The symbol ':=' denotes 'equal by definition'. The sets of all natural (not including 0), integer, real, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, respectively. If $\lambda \in \mathbb{C}$, then we write $\Re\lambda$ for its real part, $\Im\lambda$ for its imaginary part, and $\bar{\lambda}$ for its complex conjugate. The symbols $[a,b]$, (a,b) denote closed and open intervals in $\mathbb{R}$. Moreover,

$$\mathbb{R}_+ := [0,\infty),$$
$$\mathbb{N}_0 := \{0,1,2,\ldots\}.$$

If there is a need to emphasise that we deal with multidimensional quantities, we use boldface characters, for example $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$. Usually we use the Euclidean norm in $\mathbb{R}^n$, denoted by,

$$|\mathbf{x}| = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

If Ω is a subset of any topological space X, then by $\overline{\Omega}$ and $Int\,\Omega$ we denote, respectively, the closure and the interior of Ω with respect to X. If (X,d) is a metric space with metric d, we denote by

$$B_{x,r} := \{y \in X;\ d(x,y) \le r\}$$

the closed ball with centre x and radius r. If X is also a linear space, then the ball with radius r centred at the origin is denoted by B_r.

Let f be a function defined on a set Ω and $x \in \Omega$. We use one of the following symbols to denote this function: f, $x \to f(x)$, and $f(\cdot)$. The symbol $f(x)$ is in general reserved to denote the value of f at x, however, occasionally we abuse this convention and use it to denote the function itself.

If $\{x_n\}_{n\in\mathbb{N}}$ is a family of elements of some set, then the sequence of these elements, that is, the function $n \to x_n$, is denoted by $(x_n)_{n\in\mathbb{N}}$. However, for simplicity, we often abuse this notation and use $(x_n)_{n\in\mathbb{N}}$ also to denote $\{x_n\}_{n\in\mathbb{N}}$.

The derivative operator is usually denoted by ∂. However, as we occasionally need to distinguish different types of derivatives of the same function, we use other commonly accepted symbols for differentiation. To indicate the variable with respect to which we differentiate we write $\partial_t, \partial_x, \partial^2_{tx}\ldots$. If $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then $\partial_{\mathbf{x}} := (\partial_{x_1}, \ldots, \partial_{x_n})$ is the gradient operator.

If $\beta := (\beta_1, \ldots, \beta_n)$, $\beta_i \geq 0$ is a multi-index with $|\beta| := \beta_1 + \cdots + \beta_n = k$, then symbol $\partial^\beta_{\mathbf{x}} f$ is any derivative of f of order k. Thus, $\sum_{|\beta|=0}^{k} \partial^\beta f$ means the sum of all derivatives of f of order less than or equal to k.

If $\Omega \subset \mathbb{R}^n$ is an open set, then for $k \in \mathbb{N}$ the symbol $C^k(\Omega)$ denotes the set of k times continuously differentiable functions in Ω. We denote by $C(\Omega) := C^0(\Omega)$ the set of all continuous functions in Ω and

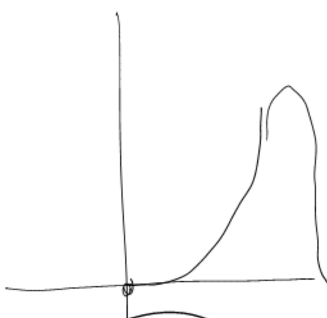
$$C^\infty(\Omega) := \bigcap_{k=0}^{\infty} C^k(\Omega).$$

Functions from $C^k(\Omega)$ need not be bounded in Ω. If they are required to be bounded together with their derivatives up to the order k, then the corresponding set is denoted by $C^k(\overline{\Omega})$.

For a continuous function f, defined on Ω, we define the *support* of f as

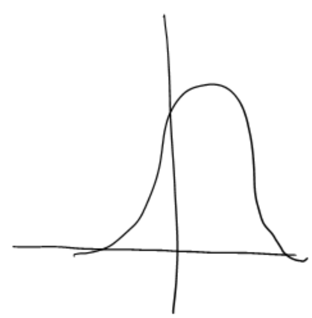
$$\text{supp} f = \overline{\{\mathbf{x} \in \Omega;\ f(x) \neq 0\}}.$$

The set of all functions with compact support in Ω which have continuous derivatives of order smaller than or equal to k is denoted by $C^k_0(\Omega)$. As above, $C_0(\Omega) := C^0_0(\Omega)$ is the set of all continuous functions with compact support in Ω and

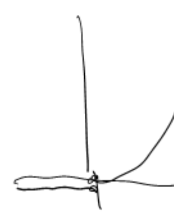


$$C^\infty_0(\Omega) := \bigcap_{k=0}^{\infty} C^k_0(\Omega).$$

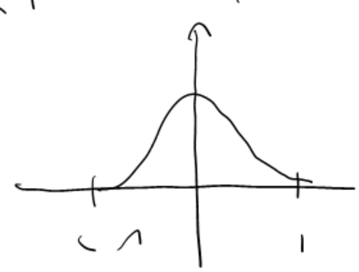
Another important standard class of spaces are the spaces $L_p(\Omega)$, $1 \leq p \leq \infty$ of functions integrable with power p. To define them, let us establish some general notation and terminology. We begin with a *measure space* (Ω, Σ, μ), where Ω is a set, Σ is a σ-algebra of subsets of Ω, and μ is a σ-additive measure on Σ. We say that μ is *σ-finite* if Ω is a countable union of sets of finite measure.



In most applications in this book, $\Omega \subset \mathbb{R}^n$ and Σ is the σ-algebra of Lebesgue measurable sets. However, occasionally we need the family of *Borel sets* which, by definition, is the smallest σ-algebra which contains all open sets. The measure μ in the former case is called the Lebesgue measure and in the latter the Borel measure. Such measures are σ-finite.



A function $f : \Omega \to \mathbb{R}$ is said to be *measurable* (with respect to Σ, or with respect to μ) if $f^{-1}(B) \in \Sigma$ for any Borel subset B of $\mathbb{R}$. Because Σ is a



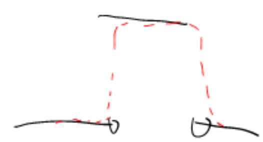
σ-algebra, f is measurable if (and only if) preimages of semi-infinite intervals are in Σ.

Remark 1.1. The difference between Lebesgue and Borel measurability is visible if one considers compositions of functions. Precisely, if f is continuous and g is measurable on $\mathbb{R}$, then $f \circ g$ is measurable but, without any additional assumptions, $g \circ f$ is not. The reason for this is that the preimage of $\{x > a\}$ through f is open and preimages of open sets through Lebesgue measurable functions are measurable. On the other hand, preimage of $\{x > a\}$ through g is only a Lebesgue measurable set and preimages of such sets through continuous are not necessarily measurable. To have measurability of $g \circ f$ one has to assume that preimages of sets of measure zero through f are of measure zero (e.g., f is Lipschitz continuous).

We identify two functions which differ from each other on a set of μ-measure zero, therefore, when speaking of a function in the context of measure spaces, we usually mean a class of equivalence of functions. For most applications the distinction between a function and a class of functions is irrelevant.

One of the most important results in applications is the Łuzin theorem.

Theorem 1.2. *If f is Lebesgue measurable and $f(x) = 0$ in the complement of a set A with $\mu(A) < \infty$, then for any $\epsilon > 0$ there exists a function $g \in C_0(\mathbb{R}^n)$ such that $\sup_{\mathbf{x}\in\mathbb{R}^n} g(\mathbf{x}) \leq \sup_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})$ and $\mu(\{\mathbf{x};\ f(\mathbf{x}) \neq g(\mathbf{x})\}) < \epsilon$.*



In other words, the theorem implies that there is a sequence of compactly supported continuous functions converging to f almost everywhere. Indeed, for any n we find a continuous function ϕ_n such that for $A_n = \{\mathbf{x};\ \phi_n(\mathbf{x}) \neq f(\mathbf{x})\}$ we have

$$\mu(A_n) \leq \frac{1}{n^2}.$$

Define

$$A = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n.$$

We see that if $\mathbf{x} \notin A$, then there is k such that for any $n \geq k$, $\mathbf{x} \notin A_k$, that is, $\phi_n(\mathbf{x}) = f(\mathbf{x})$ and hence $\phi_n(\mathbf{x}) \to f(\mathbf{x})$ whenever $\mathbf{x} \notin A$. On the other hand,

$$0 \leq \mu(A) \leq \lim_{k\to\infty} \sum_{n=k}^{\infty} \frac{1}{n^2} = 0$$

and hence $(\phi_n)_{n\in\mathbb{N}}$ converges to f almost everywhere.

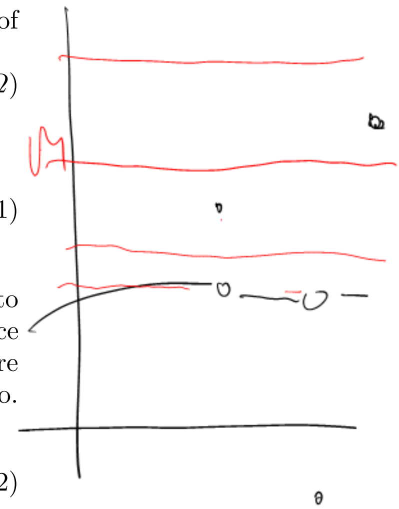
The space of equivalence classes of all measurable real functions on Ω is denoted by $L_0(\Omega, d\mu)$ or simply $L_0(\Omega)$.

The integral of a measurable function f with respect to measure μ over a set Ω is written as

$$\int_{\Omega} f d\mu = \int_{\Omega} f(\mathbf{x}) d\mu_{\mathbf{x}},$$

where the second version is used if there is a need to indicate the variable of integration. If μ is the Lebesgue measure, we abbreviate $d\mu_{\mathbf{x}} = d\mathbf{x}$.

For $1 \leq p < \infty$ the spaces $L_p(\Omega)$ are defined as subspaces of $L_0(\Omega)$ consisting of functions for which



$$\|f\|_p := \|f\|_{L_p(\Omega)} = \left(\int_{\Omega} |f(\mathbf{x})|^p d\mathbf{x} \right)^{1/p} < \infty. \tag{1.1}$$

The space $L_p(\Omega)$ with the above norm is a Banach space. It is customary to complete the scale of L_p spaces by the space $L_\infty(\Omega)$ defined to be the space of all Lebesgue measurable functions which are bounded almost everywhere in Ω, that is, bounded everywhere except possibly on a set of measure zero. The corresponding norm is defined by

$$\|f\|_\infty := \|f\|_{L_\infty(\Omega)} := \inf\{M;\ \mu(\{\mathbf{x} \in \Omega;\ |f(\mathbf{x})| > M\}) = 0\}. \tag{1.2}$$

The expression on the right-hand side of (1.2) is frequently referred to as the *essential supremum* of f over Ω and denoted $\operatorname{ess\,sup}_{\mathbf{x}\in\Omega} |f(\mathbf{x})|$.

If $\mu(\Omega) < \infty$, then for $1 \leq p \leq p' \leq \infty$ we have

$$L_{p'}(\Omega) \subset L_p(\Omega) \tag{1.3}$$

and for $f \in L_\infty(\Omega)$

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p, \tag{1.4}$$

which justifies the notation. However,

$$\bigcap_{1\leq p<\infty} L_p(\Omega) \neq L_\infty(\Omega),$$

as demonstrated by the function $f(x) = \ln x,\ x \in (0,1]$. If $\mu(\Omega) = \infty$, then neither (1.3) nor (1.4) hold.

Occasionally we need functions from $L_0(\Omega)$ which are L_p only on compact subsets of $\mathbb{R}^n$. Spaces of such functions are denoted by $L_{p,loc}(\Omega)$. A function $f \in L_{1,loc}(\Omega)$ is called *locally integrable* (in Ω).

Let $\Omega \subset \mathbb{R}^n$ be an open set. It is clear that

$$C_0^\infty(\Omega) \subset L_p(\Omega)$$

for $1 \leq p \leq \infty$. If $p \in [1,\infty)$, then we have even more: $C_0^\infty(\Omega)$ is dense in $L_p(\Omega)$.

$$\overline{C_0^\infty(\Omega)} = L_p(\Omega), \tag{1.5}$$

where the closure is taken in the L_p-norm.

Example 1.3. Having in mind further applications, it is worthwhile to have some understanding of the structure of this result; see [4, Lemma 2.18]. Let us define the function

$$\omega(\mathbf{x}) = \begin{cases} \exp\left(\frac{1}{|\mathbf{x}|^2-1}\right) & \text{for } |\mathbf{x}| < 1, \\ 0 & \text{for } |\mathbf{x}| \geq 1. \end{cases} \tag{1.6}$$

This is a $C_0^\infty(\mathbb{R}^n)$ function with support B_1.

Using this function we construct the family

$$\omega_\epsilon(\mathbf{x}) = C_\epsilon \omega(\mathbf{x}/\epsilon),$$

where C_ϵ are constants chosen so that $\int_{\mathbb{R}^n} \omega_\epsilon(\mathbf{x})d\mathbf{x} = 1$; these are also $C_0^\infty(\mathbb{R}^n)$ functions with support B_ϵ, often referred to as *mollifiers*. Using them, we define the *regularisation* (or *mollification*) of f by taking the convolution

$$(J_\epsilon * f)(\mathbf{x}) := \int\limits_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y})\omega_\epsilon(\mathbf{y})d\mathbf{y} = \int\limits_{\mathbb{R}^n} f(\mathbf{y})\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y}. \tag{1.7}$$

Precisely speaking, if $\Omega \neq \mathbb{R}^n$, we integrate outside the domain of definition of f. Thus, in such cases below, we consider f to be extended by 0 outside Ω.

Then, we have

Theorem 1.4. *With the notation above,*

1. *Let $p \in [1, \infty)$. If $f \in L_p(\Omega)$, then*

$$\lim_{\epsilon \to 0^+} \|J_\epsilon * f - f\|_p = 0.$$

2. *If $f \in C(\Omega)$, then $J_\epsilon * f \to f$ uniformly on any $\bar{G} \subset \Omega$.*
3. *If $\bar{\Omega}$ is compact and $f \in C(\bar{\Omega})$, then $J_\epsilon * f \to f$ uniformly on $\bar{\Omega}$.*

Proof. For 1.–3., even if $\mu(\Omega) = \infty$, then any $f \in L_p(\Omega)$ can be approximated by (essentially) bounded (simple) functions with compact supports. It is enough to consider a real nonnegative function u. For such a u, there is a monotonically increasing sequence $(s_n)_{n\in\mathbb{N}}$ of nonnegative simple functions converging point-wise to u on Ω. Since $0 \leq s_n(\mathbf{x}) \leq u(\mathbf{x})$, we have $s_n \in L_p(\Omega)$, $(u(\mathbf{x}) - s_n(\mathbf{x}))^p \leq u^p(\mathbf{x})$ and thus $s_n \to u$ in $L_p(\Omega)$ by the Dominated Convergence Theorem. Thus there exists a function s in the sequence for which $\|u - s\|_p \leq \epsilon/2$. Since $p < \infty$ and s is simple, the support of s must have finite volume. We can also assume that $s(\mathbf{x}) = 0$ outside Ω. By the Łuzin theorem, there is $\phi \in C_0(\mathbb{R}^n)$ such that $|\phi(\mathbf{x})| \leq \|s\|_\infty$ for all $\mathbf{x} \in \mathbb{R}^n$ and

$$\mu(\{\mathbf{x} \in \mathbb{R}^n;\ \phi(\mathbf{x}) \neq s(\mathbf{x})\} \leq \left(\frac{\epsilon}{4\|s\|_\infty}\right)^p.$$

Hence

$$\|s-\phi\|_p \le \|s-\phi\|_\infty \frac{\epsilon}{4\|s\|_\infty} \le \frac{\epsilon}{2}$$

and $\|u-\phi\|_p < \epsilon$.

Therefore, first we prove the result for continuous compactly supported functions.

Because the effective domain of integration in the second integral is $B_{\mathbf{x},\epsilon}$, $J_\epsilon * f$ is well defined whenever f is locally integrable and, similarly, if the support of f is bounded, then $\mathrm{supp} J_\epsilon * f$ is also bounded and it is contained in the ϵ-neighbourhood of $\mathrm{supp} f$. The functions f_ϵ are infinitely differentiable with

$$\partial^\beta_{\mathbf{x}}(J_\epsilon * f)(\mathbf{x}) = \int\limits_{\mathbb{R}^n} f(\mathbf{y})\partial^\beta_{\mathbf{x}}\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y} \tag{1.8}$$

for any β. By Hölder inequality, if $f \in L_p(\mathbb{R}^n)$, then $J_\epsilon * f \in L_p(\mathbb{R}^n)$ with

$$\|J_\epsilon * f\|_p \le \|f\|_p \tag{1.9}$$

for any $\epsilon > 0$. Indeed, for $p = 1$

$$\int\limits_{\mathbb{R}^n} |J_\epsilon * f(\mathbf{x})|d\mathbf{x} \le \int\limits_{\mathbb{R}^n} |f(\mathbf{y})| \left(\int\limits_{\mathbb{R}^n} \omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{x}\right) d\mathbf{y} = \|f\|_1.$$

For $p > 1$, we have

$$\begin{aligned} |J_\epsilon * f(\mathbf{x})| &= \left|\int\limits_{\mathbb{R}^n} f(\mathbf{y})\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y}\right| \\ &\le \left(\int\limits_{\mathbb{R}^n} \omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y}\right)^{1/q} \left(\int\limits_{\mathbb{R}^n} |f(\mathbf{y})|^p\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y}\right)^{1/p} \\ &= \left(\int\limits_{\mathbb{R}^n} |f(\mathbf{y})|^p\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y}\right)^{1/p} \end{aligned}$$

and, as above,

$$\int\limits_{\mathbb{R}^n} |J_\epsilon * f(\mathbf{x})|^p d\mathbf{x} \le \int\limits_{\mathbb{R}^n} |f(\mathbf{y})|^p \left(\int\limits_{\mathbb{R}^n} \omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{x}\right) d\mathbf{y} = \|f\|_p^p.$$

Next (remember f is compactly supported continuous function, and thus it is uniformly continuous)

$$|(J_\epsilon * f)(\mathbf{x}) - f(\mathbf{x})| = \left| \int\limits_{\mathbb{R}^n} f(\mathbf{y})\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y} - \int\limits_{\mathbb{R}^n} f(\mathbf{x})\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y} \right|$$
$$\leq \int\limits_{\mathbb{R}^n} |f(\mathbf{y}) - f(\mathbf{x})|\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y} \leq \sup_{\|\mathbf{x}-\mathbf{y}\| \leq \epsilon} |f(\mathbf{x}) - f(\mathbf{y})|.$$

By the compactness of support, and thus uniform continuity, of f we obtain $J_\epsilon * f \rightrightarrows f$ and, again by compactness of the support,

$$f = \lim_{\epsilon \to 0^+} f_\epsilon \qquad \text{in } L_p(\mathbb{R}^n) \tag{1.10}$$

as well as in $C(\bar{\Omega})$, where in the latter case we extend f outside Ω by a continuous function (e.g. by the Urysohn theorem).

To extend the result to an arbitrary $f \in L_p(\Omega)$, let $\phi \in C_0(\Omega)$ such that $\|f - \phi\|_p < \eta$ and $\|J_\epsilon * \phi - \phi\|_p < \eta$

$$\|J_\epsilon * f - f\|_p \leq \|J_\epsilon * f - J_\epsilon * \phi)\|_p + \|J_\epsilon * \phi - \phi\|_p + \|f - \phi\|$$
$$\leq 2\|f - \phi\| + \|J_\epsilon * \phi - \phi\|_p < \eta$$

for sufficiently small ϵ.

As an example of application, we shall consider a generalization of the duBois-Reymond lemma. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in L_{1,loc}(\Omega)$ be such that

$$\int\limits_{\Omega} u(\mathbf{x})f(\mathbf{x})d\mathbf{x} = 0$$

for any $C_0^\infty(\Omega)$. Then $u = 0$ almost everywhere on Ω. To prove this statement, let $g \in L_\infty(\Omega)$ such that supp$g$ is a compact set in Ω. We define $g_m = J_{1/m} * g$. Then $g_m \in C_0^\infty(\Omega)$ for large m. Since a compactly supported bounded function is integrable, we have $g_m \to g$ in $L_1(\Omega)$ and thus there is a subsequence (denoted by the same indices) such that $g_m \to g$ almost everywhere. Moreover, $\|g_m\|_\infty \leq \|g\|_\infty$. Using compactness of the supports and dominated convergence theorem, we obtain

$$\int\limits_{\Omega} u(\mathbf{x})g(\mathbf{x})d\mathbf{x} = 0.$$

If we take any compact set $K \subset \Omega$ and define $g = \text{sign}u$ on K and 0 otherwise, we find that for any K,

$$\int\limits_{K} |u(\mathbf{x})|d\mathbf{x} = 0.$$

Hence $u = 0$ almost everywhere on K and, since K was arbitrary, this holds almost everywhere on Ω.