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Selected Topics in Applied Functional Analysis

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Basic Facts from Functional Analysis and Banach Lattices

1.1 Spaces and Operators

1.1.1 General Notation

The symbol ':=' denotes 'equal by definition'. The sets of all natural (not including 0), integer, real, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, respectively. If $\lambda \in \mathbb{C}$, then we write $\Re\lambda$ for its real part, $\Im\lambda$ for its imaginary part, and $\bar{\lambda}$ for its complex conjugate. The symbols $[a, b]$, (a, b) denote, respectively, closed and open intervals in $\mathbb{R}$. Moreover,

$$\mathbb{R}_+ := [0, \infty),$$
$$\mathbb{N}_0 := \{0, 1, 2, \ldots\}.$$

If there is a need to emphasise that we deal with multidimensional quantities, we use boldface characters, for example $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Usually we use the Euclidean norm in $\mathbb{R}^n$, denoted by

$$|\mathbf{x}| = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

If Ω is a subset of any topological space X, then by $\overline{\Omega}$ and $Int\,\Omega$ we denote, respectively, the closure and the interior of Ω with respect to X. If (X, d) is a metric space with metric d, we denote by

$$B_{x,r} := \{y \in X;\ d(x, y) \le r\}$$

the closed ball with centre x and radius r. If X is also a linear space, then the ball with radius r, centred at the origin, is denoted by B_r.

Let f be a function defined on a set Ω and $x \in \Omega$. We use one of the following symbols to denote this function: f, $x \to f(x)$, and $f(\cdot)$. The symbol $f(x)$ is in general reserved to denote the value of f at x, however, occasionally, we abuse this convention and use it to denote the function itself.

If $\{x_n\}_{n\in\mathbb{N}}$ is a family of elements of some set, then the sequence of these elements, that is, the function $n \to x_n$, is denoted by $(x_n)_{n\in\mathbb{N}}$. However, for simplicity, we often abuse this notation and use $(x_n)_{n\in\mathbb{N}}$ also to denote $\{x_n\}_{n\in\mathbb{N}}$.

The derivative operator is usually denoted by ∂. However, as we occasionally need to distinguish different types of derivatives of the same function, we use other commonly accepted symbols for differentiation. To indicate the variable with respect to which we differentiate we write $\partial_t, \partial_x, \partial^2_{tx} \ldots$. If $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then $\partial_{\mathbf{x}} := (\partial_{x_1}, \ldots, \partial_{x_n})$ is the gradient operator.

If $\beta := (\beta_1, \ldots, \beta_n)$, $\beta_i \geq 0$ is a multi-index with $|\beta| := \beta_1 + \cdots + \beta_n = k$, then symbol $\partial^\beta_{\mathbf{x}} f$ is any derivative of f of order k. Thus, $\sum_{|\beta|=0}^k \partial^\beta f$ means the sum of all derivatives of f of order less than or equal to k.

If $\Omega \subset \mathbb{R}^n$ is an open set, then for $k \in \mathbb{N}$ the symbol $C^k(\Omega)$ denotes the set of k times continuously differentiable functions in Ω. We denote by $C(\Omega) := C^0(\Omega)$ the set of all continuous functions in Ω and

$$C^\infty(\Omega) := \bigcap_{k=0}^{\infty} C^k(\Omega).$$

Functions from $C^k(\Omega)$ need not be bounded in Ω. If they are required to be bounded together with their derivatives up to the order k, then the corresponding set is denoted by $C^k(\overline{\Omega})$.

For a continuous function f, defined on Ω, we define the *support* of f as

$$\mathrm{supp} f = \overline{\{\mathbf{x} \in \Omega;\ f(x) \neq 0\}}.$$

The set of all functions with compact support in Ω which have continuous derivatives of order smaller than or equal to k is denoted by $C^k_0(\Omega)$. As above, $C_0(\Omega) := C^0_0(\Omega)$ is the set of all continuous functions with compact support in Ω and

$$C^\infty_0(\Omega) := \bigcap_{k=0}^{\infty} C^k_0(\Omega).$$

Another important standard class of spaces are the spaces $L_p(\Omega)$, $1 \leq p \leq \infty$, of functions integrable with power p. To define them, let us establish some general notation and terminology. We begin with a *measure space* (Ω, Σ, μ), where Ω is a set, Σ is a σ-algebra of subsets of Ω, and μ is a σ-additive measure on Σ. We say that μ is *σ-finite* if Ω is a countable union of sets of finite measure.

In most applications in this book, $\Omega \subset \mathbb{R}^n$ and Σ is the σ-algebra of Lebesgue measurable sets. However, occasionally we need the family of *Borel sets* which, by definition, is the smallest σ-algebra which contains all open sets. The measure μ in the former case is called the Lebesgue measure and in the latter the Borel measure. Such measures are σ-finite.

A function $f : \Omega \to \mathbb{R}$ is said to be *measurable* (with respect to Σ, or with respect to μ) if $f^{-1}(B) \in \Sigma$ for any Borel subset B of $\mathbb{R}$. Because Σ is a

σ-algebra, f is measurable if (and only if) preimages of semi-infinite intervals are in Σ.

Remark 1.1. The difference between Lebesgue and Borel measurability is visible if one considers compositions of functions. Precisely, if f is continuous and g is measurable on $\mathbb{R}$, then $f \circ g$ is measurable but, without any additional assumptions, $g \circ f$ is not. The reason for this is that the preimage of $\{x > a\}$ through f is open and preimages of open sets through Lebesgue measurable functions are measurable. On the other hand, preimage of $\{x > a\}$ through g is only a Lebesgue measurable set and preimages of such sets through continuous are not necessarily measurable. To have measurability of $g \circ f$ one has to assume that preimages of sets of measure zero through f are of measure zero (e.g., f is Lipschitz continuous).

We identify two functions which differ from each other on a set of μ-measure zero, therefore, when speaking of a function in the context of measure spaces, we usually mean a class of equivalence of functions. For most applications the distinction between a function and a class of functions is irrelevant.

One of the most important results in applications is the Łuzin theorem.

Theorem 1.2. *If f is Lebesgue measurable and $f(x) = 0$ in the complement of a set A with $\mu(A) < \infty$, then for any $\epsilon > 0$ there exists a function $g \in C_0(\mathbb{R}^n)$ such that $\sup_{\mathbf{x}\in\mathbb{R}^n} g(\mathbf{x}) \leq \sup_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})$ and $\mu(\{\mathbf{x};\ f(\mathbf{x}) \neq g(\mathbf{x})\}) < \epsilon$.*

In other words, the theorem implies that there is a sequence of compactly supported continuous functions converging to f almost everywhere. Indeed, for any n we find a continuous function ϕ_n such that for $A_n = \{\mathbf{x};\ \phi_n(\mathbf{x}) \neq f(\mathbf{x})\}$ we have

$$\mu(A_n) \leq \frac{1}{n^2}.$$

Define

$$A = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n.$$

We see that if $\mathbf{x} \notin A$, then there is k such that for any $n \geq k$, $\mathbf{x} \notin A_k$, that is, $\phi_n(\mathbf{x}) = f(\mathbf{x})$ and hence $\phi_n(\mathbf{x}) \to f(\mathbf{x})$ whenever $\mathbf{x} \notin A$. On the other hand,

$$0 \leq \mu(A) \leq \lim_{k\to\infty} \sum_{n=k}^{\infty} \frac{1}{n^2} = 0$$

and hence $(\phi_n)_{n\in\mathbb{N}}$ converges to f almost everywhere.

The space of equivalence classes of all measurable real functions on Ω is denoted by $L_0(\Omega, d\mu)$ or simply $L_0(\Omega)$.

The integral of a measurable function f with respect to measure μ over a set Ω is written as

$$\int_{\Omega} f d\mu = \int_{\Omega} f(\mathbf{x}) d\mu_{\mathbf{x}},$$

where the second version is used if there is a need to indicate the variable of integration. If μ is the Lebesgue measure, we abbreviate $d\mu_{\mathbf{x}} = d\mathbf{x}$.

For $1 \leq p < \infty$, the spaces $L_p(\Omega)$ are defined as the subspaces of $L_0(\Omega)$ consisting of functions for which

$$\|f\|_p := \|f\|_{L_p(\Omega)} = \left(\int_{\Omega} |f(\mathbf{x})|^p d\mathbf{x} \right)^{1/p} < \infty. \tag{1.1}$$

The space $L_p(\Omega)$ with the above norm is a Banach space. It is customary to complete the scale of L_p spaces by the space $L_\infty(\Omega)$ defined to be the space of all Lebesgue measurable functions which are bounded almost everywhere in Ω, that is, bounded everywhere except possibly on a set of measure zero. The corresponding norm is defined by

$$\|f\|_\infty := \|f\|_{L_\infty(\Omega)} := \inf\{M;\ \mu(\{\mathbf{x} \in \Omega;\ |f(\mathbf{x})| > M\}) = 0\}. \tag{1.2}$$

The expression on the right-hand side of (1.2) is frequently referred to as the *essential supremum* of f over Ω and denoted $\operatorname{ess\,sup}_{\mathbf{x} \in \Omega} |f(\mathbf{x})|$.

If $\mu(\Omega) < \infty$, then for $1 \leq p \leq p' \leq \infty$ we have

$$L_{p'}(\Omega) \subset L_p(\Omega) \tag{1.3}$$

and, for $f \in L_\infty(\Omega)$,

$$\|f\|_\infty = \lim_{p \to \infty} \|f\|_p, \tag{1.4}$$

which justifies the notation. However,

$$\bigcap_{1 \leq p < \infty} L_p(\Omega) \neq L_\infty(\Omega),$$

as demonstrated by the function $f(x) = \ln x,\ x \in\ (0, 1]$. If $\mu(\Omega) = \infty$, then neither (1.3) nor (1.4) hold.

Occasionally we need functions from $L_0(\Omega)$ which are L_p only on compact subsets of $\mathbb{R}^n$. Spaces of such functions are denoted by $L_{p,loc}(\Omega)$. A function $f \in L_{1,loc}(\Omega)$ is called *locally integrable* (in Ω).

Let $\Omega \subset \mathbb{R}^n$ be an open set. It is clear that

$$C_0^\infty(\Omega) \subset L_p(\Omega)$$

for $1 \leq p \leq \infty$. If $p \in [1, \infty)$, then we have even more: $C_0^\infty(\Omega)$ is dense in $L_p(\Omega)$.

$$\overline{C_0^\infty(\Omega)} = L_p(\Omega), \tag{1.5}$$

where the closure is taken in the L_p-norm.

Example 1.3. Having in mind further applications, it is worthwhile to have some understanding of the structure of this result; see [**?**, Lemma 2.18]. Let us define the function

$$\omega(\mathbf{x}) = \begin{cases} \exp\left(\frac{1}{|\mathbf{x}|^2-1}\right) & \text{for } |\mathbf{x}| < 1, \\ 0 & \text{for } |\mathbf{x}| \geq 1. \end{cases} \tag{1.6}$$

This is a $C_0^\infty(\mathbb{R}^n)$ function with support B_1.

Using this function we construct the family

$$\omega_\epsilon(\mathbf{x}) = C_\epsilon \omega(\mathbf{x}/\epsilon),$$

where C_ϵ are constants chosen so that $\int_{\mathbb{R}^n} \omega_\epsilon(\mathbf{x})d\mathbf{x} = 1$; these are also $C_0^\infty(\mathbb{R}^n)$ functions with support B_ϵ, often referred to as *mollifiers.* Using them, we define the *regularisation* (or *mollification*) of f by taking the convolution

$$(J_\epsilon * f)(\mathbf{x}) := \int\limits_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y})\omega_\epsilon(\mathbf{y})d\mathbf{y} = \int\limits_{\mathbb{R}^n} f(\mathbf{y})\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y}. \tag{1.7}$$

Precisely speaking, if $\Omega \neq \mathbb{R}^n$, we integrate outside the domain of definition of f. Thus, in such cases below, we consider f to be extended by 0 outside Ω.

Then, we have

Theorem 1.4. *With the notation above,*

1. *Let $p \in [1, \infty)$. If $f \in L_p(\Omega)$, then*

$$\lim_{\epsilon \to 0^+} \|J_\epsilon * f - f\|_p = 0.$$

2. *If $f \in C(\Omega)$, then $J_\epsilon * f \to f$ uniformly on any $\bar{G} \Subset \Omega$.*
3. *If $\bar{\Omega}$ is compact and $f \in C(\bar{\Omega})$, then $J_\epsilon * f \to f$ uniformly on $\bar{\Omega}$.*

Proof. For 1.–3., even if $\mu(\Omega) = \infty$, then any $f \in L_p(\Omega)$ can be approximated by (essentially) bounded (simple) functions with compact supports. It is enough to consider a real nonnegative function u. For such a u, there is a monotonically increasing sequence $(s_n)_{n\in\mathbb{N}}$ of nonnegative simple functions converging point-wise to u on Ω. Since $0 \leq s_n(\mathbf{x}) \leq u(\mathbf{x})$, we have $s_n \in L_p(\Omega)$, $(u(\mathbf{x}) - s_n(\mathbf{x}))^p \leq u^p(\mathbf{x})$ and thus $s_n \to u$ in $L_p(\Omega)$ by the Dominated Convergence Theorem. Thus there exists a function s in the sequence for which $\|u - s\|_p \leq \epsilon/2$. Since $p < \infty$ and s is simple, the support of s must have finite volume. We can also assume that $s(\mathbf{x}) = 0$ outside Ω. By the Łuzin theorem, there is $\phi \in C_0(\mathbb{R}^n)$ such that $|\phi(\mathbf{x})| \leq \|s\|_\infty$ for all $\mathbf{x} \in \mathbb{R}^n$ and

$$\mu(\{\mathbf{x} \in \mathbb{R}^n;\ \phi(\mathbf{x}) \neq s(\mathbf{x})\} \leq \left(\frac{\epsilon}{4\|s\|_\infty}\right)^p.$$

Hence

$$\|s - \phi\|_p \leq \|s - \phi\|_\infty \frac{\epsilon}{4\|s\|_\infty} \leq \frac{\epsilon}{2}$$

and $\|u - \phi\|_p < \epsilon$.

Therefore, first we prove the result for continuous compactly supported functions.

Because the effective domain of integration in the second integral is $B_{\mathbf{x},\epsilon}$, $J_\epsilon * f$ is well defined whenever f is locally integrable and, similarly, if the support of f is bounded, then $\mathrm{supp}(J_\epsilon * f)$ is also bounded and it is contained in the ϵ-neighbourhood of supp$f$. The functions f_ϵ are infinitely differentiable with

$$\partial^\beta_{\mathbf{x}}(J_\epsilon * f)(\mathbf{x}) = \int\limits_{\mathbb{R}^n} f(\mathbf{y})\partial^\beta_{\mathbf{x}}\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y} \tag{1.8}$$

for any β. By Hölder inequality, if $f \in L_p(\mathbb{R}^n)$, then $J_\epsilon * f \in L_p(\mathbb{R}^n)$ with

$$\|J_\epsilon * f\|_p \leq \|f\|_p \tag{1.9}$$

for any $\epsilon > 0$. Indeed, for $p = 1$

$$\int\limits_{\mathbb{R}^n} |J_\epsilon * f(\mathbf{x})|d\mathbf{x} \leq \int\limits_{\mathbb{R}^n} |f(\mathbf{y})| \left(\int\limits_{\mathbb{R}^n} \omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{x}\right) d\mathbf{y} = \|f\|_1.$$

For $p > 1$, we have

$$\begin{aligned} |J_\epsilon * f(\mathbf{x})| &= \left|\int\limits_{\mathbb{R}^n} f(\mathbf{y})\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y}\right| \\ &\leq \left(\int\limits_{\mathbb{R}^n} \omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y}\right)^{1/q} \left(\int\limits_{\mathbb{R}^n} |f(\mathbf{y})|^p\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y}\right)^{1/p} \\ &= \left(\int\limits_{\mathbb{R}^n} |f(\mathbf{y})|^p\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y}\right)^{1/p} \end{aligned}$$

and, as above,

$$\int\limits_{\mathbb{R}^n} |J_\epsilon * f(\mathbf{x})|^p d\mathbf{x} \leq \int\limits_{\mathbb{R}^n} |f(\mathbf{y})|^p \left(\int\limits_{\mathbb{R}^n} \omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{x}\right) d\mathbf{y} = \|f\|_p^p.$$

Next (remember f is compactly supported continuous function, and thus it is uniformly continuous)

$$|(J_\epsilon * f)(\mathbf{x}) - f(\mathbf{x})| = \left| \int\limits_{\mathbb{R}^n} f(\mathbf{y})\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y} - \int\limits_{\mathbb{R}^n} f(\mathbf{x})\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y} \right|$$
$$\le \int\limits_{\mathbb{R}^n} |f(\mathbf{y}) - f(\mathbf{x})|\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y} \le \sup_{\|\mathbf{x}-\mathbf{y}\|\le\epsilon} |f(\mathbf{x}) - f(\mathbf{y})|.$$

By the compactness of support, and thus uniform continuity, of f we obtain $J_\epsilon * f \rightrightarrows f$ and, again by compactness of the support,

$$f = \lim_{\epsilon\to 0^+} f_\epsilon \qquad \text{in } L_p(\mathbb{R}^n) \tag{1.10}$$

as well as in $C(\bar{\Omega})$, where in the latter case we extend f outside Ω by a continuous function (e.g. by the Urysohn theorem).

To extend the result to an arbitrary $f \in L_p(\Omega)$, let $\phi \in C_0(\Omega)$ such that $\|f-\phi\|_p < \eta$ and $\|J_\epsilon * \phi - \phi\|_p < \eta$

$$\|J_\epsilon * f - f\|_p \le \|J_\epsilon * f - J_\epsilon * \phi)\|_p + \|J_\epsilon * \phi - \phi\|_p + \|f - \phi\|$$
$$\le 2\|f-\phi\| + \|J_\epsilon * \phi - \phi\|_p < 3\eta$$

for sufficiently small ϵ.

As an example of application, we shall consider a generalization of the du Bois-Reymond lemma. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in L_{1,loc}(\Omega)$ be such that

$$\int\limits_{\Omega} u(\mathbf{x})f(\mathbf{x})d\mathbf{x} = 0$$

for any $C_0^\infty(\Omega)$. Then $u = 0$ almost everywhere on Ω. To prove this statement, let $g \in L_\infty(\Omega)$ such that $\operatorname{supp} g$ is a compact set in Ω. We define $g_m = J_{1/m} * g$. Then $g_m \in C_0^\infty(\Omega)$ for large m. Since a compactly supported bounded function is integrable, we have $g_m \to g$ in $L_1(\Omega)$ and thus there is a subsequence (denoted by the same indices) such that $g_m \to g$ almost everywhere. Moreover, $\|g_m\|_\infty \le \|g\|_\infty$. Using compactness of the supports and dominated convergence theorem, we obtain

$$\int\limits_{\Omega} u(\mathbf{x})g(\mathbf{x})d\mathbf{x} = 0.$$

If we take any compact set $K \subset \Omega$ and define $g = \operatorname{sign} u$ on K and 0 otherwise, we find that for any K,

$$\int\limits_{K} |u(\mathbf{x})|d\mathbf{x} = 0.$$

Hence $u = 0$ almost everywhere on K and, since K was arbitrary, this holds almost everywhere on Ω.

Remark 1.5. We observe that, if f is nonnegative, then f_ϵ are also nonnegative by (1.7) and hence any non-negative $f \in L_p(\mathbb{R}^n)$ can be approximated by nonnegative, infinitely differentiable, functions with compact support.

Remark 1.6. Spaces $L_p(\Omega)$ often are defined as the completion of $C_0(\Omega)$ in the $L_p(\Omega)$ norm, thus avoiding introduction of measure theory. The theorem above shows that these two definitions are equivalent.

1.1.2 Operators

Let X, Y be real or complex Banach spaces with the norm denoted by $\|\cdot\|$ or $\|\cdot\|_X$.

An *operator* from X to Y is a linear rule $A : D(A) \to Y$, where $D(A)$ is a linear subspace of X, called the *domain* of A. The set of operators from X to Y is denoted by $L(X, Y)$. Operators taking their values in the space of scalars are called *functionals*. We use the notation $(A, D(A))$ to denote the operator A with domain $D(A)$. If $A \in L(X, X)$, then we say that A (or $(A, D(A))$) is an operator in X.

By $\mathcal{L}(X, Y)$, we denote the space of all bounded operators between X and Y; $\mathcal{L}(X, X)$ is abbreviated as $\mathcal{L}(X)$. The space $\mathcal{L}(X, Y)$ can be made a Banach space by introducing the norm of an operator X by

$$\|A\| = \sup_{\|x\| \le 1} \|Ax\| = \sup_{\|x\|=1} \|Ax\|. \tag{1.11}$$

If $(A, D(A))$ is an operator in X and $Y \subset X$, then the *part* of the operator A in Y is defined as

$$A_Y y = Ay \tag{1.12}$$

on the domain

$$D(A_Y) = \{x \in D(A) \cap Y;\ Ax \in Y\}.$$

A *restriction* of $(A, D(A))$ to $D \subset D(A)$ is denoted by $A|_D$. For $A, B \in L(X, Y)$, we write $A \subset B$ if $D(A) \subset D(B)$ and $B|_{D(A)} = A$.

Two operators $A, B \in \mathcal{L}(X)$ are said to commute if $AB = BA$. It is not easy to extend this definition to unbounded operators due to the difficulties with defining the domains of the composition. The extension is usually done to the case when one of the operators is bounded. Thus, an operator $A \in L(X)$ is said to *commute* with $B \in \mathcal{L}(X)$ if

$$BA \subset AB. \tag{1.13}$$

This means that for any $x \in D(A)$, $Bx \in D(A)$ and $BAx = ABx$.

We define the *image* of A by

$$ImA = \{y \in Y;\ y = Ax \text{ for some } x \in D(A)\}$$

and the *kernel* of A by

$$KerA = \{x \in D(A);\ Ax = 0\}.$$

We note a simple result which is frequently used throughout the book.

Proposition 1.7. *Suppose that $A, B \in L(X,Y)$ satisfy: $A \subset B$, $KerB = \{0\}$, and $ImA = Y$. Then $A = B$.*

Proof. If $D(A) \neq D(B)$, we take $x \in D(B) \setminus D(A)$ and let $y = Bx$. Because A is onto, there is $x' \in D(A)$ such that $y = Ax'$. Because $x' \in D(A) \subset D(B)$ and $A \subset B$, we have $y = Ax' = Bx'$ and $Bx' = Bx$. Because $KerB = \{0\}$, we obtain $x = x'$ which is a contradiction with $x \notin D(A)$. □

Furthermore, the *graph* of A is defined as

$$G(A) = \{(x,y) \in X \times Y;\ x \in D(A), y = Ax\}. \tag{1.14}$$

We say that the operator A is *closed* if $G(A)$ is a closed subspace of $X \times Y$. Equivalently, A is closed if and only if for any sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$, if $\lim_{n\to\infty} x_n = x$ in X and $\lim_{n\to\infty} Ax_n = y$ in Y, then $x \in D(A)$ and $y = Ax$.

An operator A in X is *closable* if the closure of its graph $\overline{G(A)}$ is itself a graph of an operator, that is, if $(0, y) \in \overline{G(A)}$ implies $y = 0$. Equivalently, A is closable if and only if for any sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$, if $\lim_{n\to\infty} x_n = 0$ in X and $\lim_{n\to\infty} Ax_n = y$ in Y, then $y = 0$. In such a case the operator whose graph is $\overline{G(A)}$ is called the *closure* of A and denoted by $\overline{A}$.

By definition, when A is closable, then

$$\begin{aligned} D(\overline{A}) = \{x \in X;\ &\text{there is } (x_n)_{n\in\mathbb{N}} \subset D(A) \text{ and } y \in X \text{ such that} \\ &\|x_n - x\| \to 0 \text{ and } \|Ax_n - y\| \to 0\}, \\ \overline{A}x = y. \end{aligned}$$

For any operator A, its domain $D(A)$ is a normed space under the *graph norm*

$$\|x\|_{D(A)} := \|x\|_X + \|Ax\|_Y. \tag{1.15}$$

The operator $A : D(A) \to Y$ is always bounded with respect to the graph norm, and A is closed if and only if $D(A)$ is a Banach space under (1.15).

The differentiation operator

One of the simplest and most often used unbounded, but closed or closable, operators is the operator of differentiation. If X is any of the spaces $C([0,1])$ or $L_p([0,1])$, then considering $f_n(x) := C_n x^n$, where $C_n = 1$ in the former case and $C_n = (np+1)^{1/p}$ in the latter, we see that in all cases $\|f_n\| = 1$. However,

$$\|f_n'\| = n\left(\frac{np+1}{np+1-p}\right)^{1/p}$$

in $L_p([0,1])$ and $\|f_n'\| = n$ in $C([0,1])$, so that the operator of differentiation is unbounded.

Let us define $Tf = f'$ as an unbounded operator on $D(T) = \{f \in X;\ Tf \in X\}$, where X is any of the above spaces. We can easily see that in $X = C([0,1])$ the operator T is closed. Indeed, let us take $(f_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} Tf_n = g$ in X. This means that $(f_n)_{n\in\mathbb{N}}$ and $(f_n')_{n\in\mathbb{N}}$ converge uniformly to, respectively, f and g, and from basic calculus f is differentiable and $f' = g$.

The picture changes, however, in L_p spaces. To simplify the notation, we take $p = 1$ and consider the sequence of functions

$$f_n(x) = \begin{cases} 0 & \text{for } 0 \le x \le \frac{1}{2}, \\ \frac{n}{2}\left(x - \frac{1}{2}\right)^2 & \text{for } \frac{1}{2} < x \le \frac{1}{2} + \frac{1}{n}, \\ x - \frac{1}{2} - \frac{1}{2n} & \text{for } \frac{1}{2} + \frac{1}{n} < x \le 1. \end{cases}$$

These are differentiable functions and it is easy to see that $(f_n)_{n\in\mathbb{N}}$ converges in $L_1([0,1])$ to the function f given by $f(x) = 0$ for $x \in [0,1/2]$ and $f(x) = x - 1/2$ for $x \in (1/2,1]$ and the derivatives converge to $g(x) = 0$ if $x \in [0,1/2]$ and to $g(x) = 1$ otherwise. The function f, however, is not differentiable and so T is not closed. On the other hand, g seems to be a good candidate for the derivative of f in some more general sense. Let us develop this idea further. First, we show that T is closable. Let $(f_n)_{n\in\mathbb{N}}$ and $(f_n')_{n\in\mathbb{N}}$ converge in X to f and g, respectively. Then, for any $\phi \in C_0^\infty((0,1))$, we have, integrating by parts,

$$\int_0^1 f_n'(x)\phi(x)dx = -\int_0^1 f_n(x)\phi'(x)dx$$

and because we can pass to the limit on both sides, we obtain

$$\int_0^1 g(x)\phi(x)dx = -\int_0^1 f(x)\phi'(x)dx. \tag{1.16}$$

Using the equivalent characterization of closability, we put $f = 0$, so that

$$\int_0^1 g(x)\phi(x)dx = 0$$

for any $\phi \in C_0^\infty((0,1))$ which yields $g(x) = 0$ almost everywhere on $[0,1]$. Hence $g = 0$ in $L_1([0,1])$ and consequently T is closable.

The domain of $\overline{T}$ in $L_1([0,1])$ is called the Sobolev space $W_1^1([0,1])$ which is discussed in more detail in Subsection **??**.

These considerations can be extended to hold in any $\Omega \subset \mathbb{R}^n$. In particular, we can use (1.16) to generalize the operation of differentiation in the following way: we say that a function $g \in L_{1,loc}(\Omega)$ is the *generalised (or distributional) derivative* of $f \in L_{1,loc}(\Omega)$ of order α, denoted by $\partial_{\mathbf{x}}^{\alpha} f$, if

$$\int_{\Omega} g(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = (-1)^{|\beta|}\int_{\Omega} f(\mathbf{x})\partial_{\mathbf{x}}^{\beta}\phi(\mathbf{x})d\mathbf{x} \tag{1.17}$$

for any $\phi \in C_0^{\infty}(\Omega)$.

This operation is well defined. This follows from the du Bois Reymond lemma.

From the considerations above it is clear that $\partial_{\mathbf{x}}^{\beta}$ is a closed operator extending the classical differentiation operator (from $C^{|\beta|}(\Omega)$). One can also prove that ∂^{β} is the closure of the classical differentiation operator.

Proposition 1.8. *If $\Omega = \mathbb{R}^n$, then ∂^{β} is the closure of the classical differentiation operator.*

Proof. We use (1.7) and (1.8). Indeed, let $f \in L_p(\mathbb{R}^n)$ and $g = \partial^{\beta} f \in L_p(\mathbb{R}^n)$. We consider $f_{\epsilon} := J_{\epsilon} * f \to f$ in L_p. By the Fubini theorem, we prove

$$\begin{aligned}\int_{\mathbb{R}^n} (J_{\epsilon} * f)(\mathbf{x})\partial^{\beta}\phi(\mathbf{x})d\mathbf{x} &= \int_{\mathbb{R}^n} \omega_{\epsilon}(y)\int_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y})\partial^{\beta}\phi(\mathbf{x})d\mathbf{x}d\mathbf{y} \\ &= (-1)^{|\beta|}\int_{\mathbb{R}^n} \omega_{\epsilon}(y)\int_{\mathbb{R}^n} g(\mathbf{x}-\mathbf{y})\phi(\mathbf{x})d\mathbf{x}d\mathbf{y} \\ &= (-1)^{|\beta|}\int_{\mathbb{R}^n} (J_{\epsilon} * g)\phi(\mathbf{x})d\mathbf{x}\end{aligned}$$

so that $\partial^{\beta} f_{\epsilon} = J_{\epsilon} * \partial^{\beta} f = J_{\epsilon} * g \to g$ as $\epsilon \to 0$ in $L_p(\mathbb{R}^n)$. This shows that action of the distributional derivative can be obtained as the closure of the classical derivation.

Otherwise the proof is more complicated (see, e.g., [**?**, Theorem 3.16]) since we do not know whether we can extend f outside Ω in such a way that the extension still will have the generalized derivative. We shall discuss it later.

Example 1.9. **A non closable operator.** Let us consider the space $X = L_2((0,1))$ and the operator $K : X \to Y$, $Y = X \times \mathbb{C}$ (with the Euclidean norm), defined by

$$Kv =< v, v(1) > \tag{1.18}$$

on the domain $D(K)$ consisting of continuous functions on $[0,1]$. We have the following lemma

Lemma 1.10. *K is not closable, but has a bounded inverse. ImK is dense in Y.*

Proof. Let $f \in C^\infty([0,1])$ be such that

$$f(x) = \begin{cases} 0 \text{ for } 0 \leq x < 1/3 \\ 1 \text{ for } 2/3 < x \leq 1. \end{cases}$$

To construct such a function, we can consider e.g. $J_\epsilon * \bar{f}$, where

$$\bar{f}(x) = \begin{cases} 1 \quad \text{ for } \quad \frac{2}{3} - \epsilon < x \leq 1, \\ 0 \text{ otherwise} \end{cases}$$

and $\epsilon < 1/3$. Let $v_n(x) = f(x^n)$ for $0 \leq x \leq 1$. Clearly, $v_n \in D(K)$ and $v_n \to 0$ in $L_2((0,1))$ as

$$\int\limits_0^1 f^2(x^n)dx = \int\limits_{3^{-1/n}}^1 f^2(x^n)dx = \frac{1}{n}\int\limits_{1/3}^1 z^{-1+1/n}f^2(z)dz.$$

However, $Kv_n =< v_n, 1 >\to< 0, 1 >\neq< 0, 0 >$.

Further, K is one-to-one with $K^{-1}(v, v(1)) = v$ and

$$\|K^{-1}(v, v(1))\|^2 = \|v\|^2 \leq \|v\|^2 + |v(1)|^2.$$

To prove that ImK is dense in Y, let $< y, \alpha >\in Y$. We know that $C_0^\infty((0,1)) \subset D(K)$ is dense in $Z = L_2((0,1))$. Let (ϕ_n) be sequence of C_0^∞-functions which approximate y in $L_2(0,1)$ and put $w_n = \phi_n + \alpha v_n$. We have $Kw_n =< w_n, \alpha >\to< y, \alpha >$.

Absolutely continuous functions

In one-dimensional spaces the concept of the generalised derivative is closely related to a classical notion of absolutely continuous function. Let $I = [a,b] \subset \mathbb{R}^1$ be a bounded interval. We say that $f : I \to \mathbb{C}$ is *absolutely continuous* if, for any $\epsilon > 0$, there is $\delta > 0$ such that for any finite collection $\{(a_i, b_i)\}_i$ of disjoint intervals in $[a,b]$ satisfying $\sum_i (b_i - a_i) < \delta$, we have $\sum_i |f(b_i) - f(a_i)| < \epsilon$. The fundamental theorem of calculus, [**?**, Theorem 8.18], states that any absolutely continuous function f is differentiable almost everywhere, its derivative f' is Lebesgue integrable on $[a,b]$, and $f(t) - f(a) = \int_a^t f'(s)ds$. It can be proved (e.g., [**?**, Theorem VIII.2]) that absolutely continuous functions on $[a,b]$ are exactly integrable functions having integrable generalised derivatives and the generalised derivative of f coincides with the classical derivative of f almost everywhere.

Let us explore this connection. We prove

Theorem 1.11. *Assume that* $u \in L_{1,loc}(\mathbb{R})$ *and its generalized derivative* Du *also satisfies* $Du \in L_{1,loc}(\mathbb{R})$. *Then there is a continuous representation* $\widetilde{u}$ *of* u *such that*

$$\widetilde{u}(x) = C + \int_0^x Du(t)dt$$

for some constant C and thus u is differentiable almost everywhere.

Proof. The proof is carried out in three steps. In Step 1, we prove that if

$$F(x) = \int_a^x f(y)dy, \tag{1.19}$$

where $f \in L_{1,loc}(\mathbb{R})$, then F is differentiable almost everywhere (it is absolutely continuous) and f is its derivative. In Step 2, we show that if an $L_{1,loc}(\mathbb{R})$ function has the generalised derivative equal to zero, then it is constant (almost everywhere). Finally, in Step 3, we show that the generalised derivative of F defined by (1.19) coincides with f, which will allow to draw the final conclusion.

Step 1. Consider

$$A_h f(x) = \frac{1}{h}\int_x^{x+h} f(y)dy.$$

Clearly it is a jointly continuous function on $\mathbb{R}_+ \times \mathbb{R}$. Further, denote

$$Hf(x) = \sup_{h>0} A_h|f|(x).$$

We restrict considerations to some bounded open interval I. Then $A_h f(x) \to f(x)$ if there is no n such that $x \in S_n = \{x;\ \limsup_{h\to 0}|A_h f(x) - f(x)| \geq 1/n\}$. Thus, we have to prove $\mu(S_n) = 0$ for any n.

Then we can assume that f is of bounded support and therefore, by the Theorem 1.4, for any ϵ there is a continuous function g with bounded support with $\int_I |f(x) - g(x)|dx \leq \epsilon$. From this it follows that

$$\begin{aligned}\epsilon &\geq \int_I |f(x) - g(x)|dx \geq \int_{\{x;\ |f(x)-g(x)|\geq 1/n)} |f(x) - g(x)|dx \\ &\geq \frac{1}{n}\mu(\{x;\ |f(x) - g(x)| \geq 1/n\}),\end{aligned}$$

that is,

$$\mu(\{x;\ |f(x) - g(x)| \geq 1/n\}) \leq n\epsilon. \tag{1.20}$$

Fix any ϵ and corresponding g. Then

$$\limsup_{h\to 0}|A_h f(x) - f(x)| \leq \sup_{h>0}|A_h(f(x) - g(x))| + \lim_{h\to 0}|A_h g(x) - g(x)| + |f(x) - g(x)|$$

The second term is zero by the continuity of g. We begin with estimating the first term. For a given ϕ consider an open set $E_\alpha = \{x \in I; H\phi(x) > \alpha\}$ (E_α is open as it is the sum of the sets $\{x \in I;\ A_h|\phi|(x) > \alpha\}$ over $h > 0$, where the latter are open by continuity of $A_h|\phi|$. For any $x \in E_\alpha$ we find r_x such that $A_{r_x}|\phi|(x) > \alpha$. Consider intervals $I_{x,r_x} = (x - r_x, x + r_x)$. Thus, E_α is covered by these intervals. From the theory of Lebesque measure, the measure of any measurable set S is supremum over measures of compact sets $K \subset S$. Thus, for any $c < \mu(E_\alpha)$ we can find compact set $K \subset E_\alpha$ with $c < \mu(K) \subset \mu(E_\alpha)$ and a finite cover of K by $I_{x_i,r_{x_i}}$, $i = 1, \ldots, i_K$. Let us modify this cover in the following way. Let I_1 be the element of maximum length $2r_1$, centred at x_1, I_2 be the largest of the remaining which are disjoint with I_1, centred at x_2, and so on, until the collection is exhausted with $j = J$. According to the construction, if some $I_{x_i,r_{x_i}}$ is not in the selected list, then there is j such that $I_{x_i,r_{x_i}} \cap I_j \neq \emptyset$. Let as take the smallest such j, that is, the largest I_j. Then $2r_{x_i}$ is at most equal to the length of I_j, $2r_j$, and thus $I_{x_i,r_{x_i}} \subset I_j^*$ where the latter is the interval with the same centre as I_j but with length $6r_j$. The collection of I_j^* also covers K and, since $A_{r_j}|\phi|(x_j) = r_j^{-1}\int_{x_j}^{x_j+r_j}|\phi(y)|dy \geq \alpha$, we obtain

$$c \leq 6\sum_{j=1}^{J} r_j \leq \frac{6}{\alpha}\sum_{j=1}^{J}\int\limits_{x_j}^{x_j+r_j}|\phi(y)|dy \leq \frac{6}{\alpha}\int\limits_{I}|\phi(y)|dy,$$

since the intervals I_j, and thus (x_j, x_j+r_j), do not overlap (and we can restrict h to be small enough for the intervals to be in I). Passing with $c \to \mu(E_\alpha)$ we get

$$\mu(E_\alpha) = \mu(\{x \in I; H\phi(x) > \alpha\}) \leq \frac{6}{\alpha}\int\limits_{I}|\phi(y)|dy.$$

Using this estimate for $\phi = f - g$ and combining it with (1.20), we see that for any $\epsilon > 0$ we have

$$\mu(S_n) \leq 6n\epsilon + n\epsilon$$

and, since ϵ is arbitrary, $\mu(S_n) = 0$ for any n. So, we have differentiability of $x \to \int_{x_0}^{x} f(y)dy$ almost everywhere.

Step 2. Next, we observe that if $f \in L_{1,loc}(\mathbb{R})$ satisfies

$$\int\limits_{\mathbb{R}} f\phi' dx = 0$$

for any $\phi \in C_0^\infty(\mathbb{R})$, then $f = const$ almost everywhere. To prove this, we observe that if $\psi \in C_0^\infty(\mathbb{R})$ satisfies $\int_{\mathbb{R}}\psi dx = 1$, then for any $\omega \in C_0^\infty(\mathbb{R})$ there is $\phi \in C_0^\infty(\mathbb{R})$ satisfying

$$\phi' = \omega - \psi\int\limits_{\mathbb{R}}\omega dx.$$

Indeed, $h = \omega - \psi \int_{\mathbb{R}} \omega dx$ is continuous compactly supported with $\int_{\mathbb{R}} h dx = 0$ and thus it has a unique compactly supported primitive.

Hence

$$\int_{\mathbb{R}} f\phi' dx = \int_{R} f(\omega - \psi \int_{\mathbb{R}} \omega dy) dx = 0$$

or

$$\int_{\mathbb{R}} (f - \int_{\mathbb{R}} f\psi dy)\omega dx = 0$$

for any $\omega \in C_0^\infty(\mathbb{R})$ and thus $f = const$ almost everywhere.

Step 3. Next, if $v(x) = \int_{x_0}^{x} f(y)dy$ for $f \in L_{1,loc}(\mathbb{R})$, then v is continuous and the generalized derivative of v, Dv, equals f. In the proof, we can put $x_0 = 0$. Then

$$\begin{aligned}
\int_{\mathbb{R}} v\phi' dx &= \int_0^\infty (\int_0^x f(y)\phi'(x)dy)dx - \int_{-\infty}^0 (\int_x^0 f(y)\phi'(x)dy)dx \\
&= \int_0^\infty f(y)(\int_y^\infty \phi'(x)dx)dy - \int_{-\infty}^0 f(y)(\int_{-\infty}^0 \phi'(x)dx)dy \\
&= -\int_{\mathbb{R}} f(y)\phi(y)dy.
\end{aligned}$$

With these results, let $u \in L_{1,loc}(\mathbb{R})$ be the distributional derivative $Du \in L_{1,loc}(\mathbb{R})$ and set $\bar{u}(x) = \int_0^x Du(t)dt$. Then $D\bar{u} = Du$ almost everywhere and hence $\bar{u} + C = u$ almost everywhere. Defining $\widetilde{u} = \bar{u} + C$, we see that $\widetilde{u}$ is continuous and has integral representation and thus it is differentiable almost everywhere.

1.2 Fundamental Theorems of Functional Analysis

The foundation of classical functional analysis are the four theorems which we formulate and discuss below.

1.2.1 Hahn–Banach Theorem

Theorem 1.12. *(Hahn–Banach) Let X be a normed space, X_0 a linear subspace of X, and x_1^* a continuous linear functional defined on X_0. Then there exists a continuous linear functional x^* defined on X such that $x^*(x) = x_1^*(x)$ for $x \in X_0$ and $\|x^*\| = \|x_1^*\|$.*

The Hahn–Banach theorem has a multitude of applications. For us, the most important one is in the theory of the dual space to X. The space $\mathcal{L}(X, \mathbb{R})$ (or $\mathcal{L}(X, \mathbb{C})$) of all continuous functionals is denoted by X^* and referred to as the *dual space*. The Hahn–Banach theorem implies that X^* is nonempty (as one can easily construct a continuous linear functional on a one-dimensional space) and, moreover, there are sufficiently many bounded functionals to separate points of x; that is, for any two points $x_1, x_2 \in X$ there is $x^* \in X^*$ such that $x^*(x_1) = 0$ and $x^*(x_2) = 1$. The Banach space $X^{**} = (X^*)^*$ is called the *second dual*. Every element $x \in X$ can be identified with an element of X^{**} by the evaluation formula

$$x(x^*) = x^*(x); \tag{1.21}$$

that is, X can be viewed as a subspace of X^{**}. To indicate that there is some symmetry between X and its dual and second dual we shall often write

$$x^*(x) = <x^*, x>_{X^* \times X},$$

where the subscript $X^* \times X$ is suppressed if no ambiguity is possible.

In general $X \neq X^{**}$. Spaces for which $X = X^{**}$ are called *reflexive*. Examples of reflexive spaces are rendered by Hilbert and L_p spaces with $1 < p < \infty$. However, the spaces L_1 and L_∞, as well as nontrivial spaces of continuous functions, fail to be reflexive.

Example 1.13. If $1 < p < \infty$, then the dual to $L_p(\Omega)$ can be identified with $L_q(\Omega)$ where $1/p + 1/q = 1$, and the duality pairing is given by

$$<f, g> = \int\limits_{\Omega} f(\mathbf{x})g(\mathbf{x})d\mathbf{x}, \quad f \in L_p(\Omega), \; g \in L_q(\Omega). \tag{1.22}$$

This shows, in particular, that $L_2(\Omega)$ is a Hilbert space and the above duality pairing gives the scalar product in the real case. If $L_2(\Omega)$ is considered over the complex field, then in order to get a scalar product, (1.22) should be modified by taking the complex adjoint of g.

Moreover, as mentioned above, the spaces $L_p(\Omega)$ with $1 < p < \infty$ are reflexive. On the other hand, if $p = 1$, then $(L_1(\Omega))^* = L_\infty(\Omega)$ with duality pairing given again by (1.22). However, the dual to L_∞ is much larger than $L_1(\Omega)$ and thus $L_1(\Omega)$ is not a reflexive space.

Another important corollary of the Hahn–Banach theorem is that for each $0 \neq x \in X$ there is an element $\bar{x}^* \in X^*$ that satisfies $\|\bar{x}^*\| = \|x\|$ and $<\bar{x}^*, x> = \|x\|$. In general, the correspondence $x \to \bar{x}^*$ is multi-valued: this is the case in L_1-spaces and spaces of continuous functions it becomes, however, single-valued if the unit ball in X is strictly convex (e.g., in Hilbert spaces or L^p-spaces with $1 < p < \infty$; see [**?**]).

1.2.2 Spanning theorem and its application

A workhorse of analysis is the spanning criterion.

Theorem 1.14. *Let X be a normed space and $\{y_j\} \subset X$. Then $z \in Y := \overline{\mathcal{L}in}\{y_j\}$ if and only if*

$$\forall_{x^* \in X^*} < x^*, y_j >= 0 \quad \text{implies} \quad < x^*, z >= 0.$$

Proof. In one direction it follows easily from linearity and continuity.

Conversely, assume $< x^*, z >= 0$ for all x^* annihilating Y and $z \neq Y$. Thus, $\inf_{y \in Y} \|z - y\| = d > 0$ (from closedness). Define $Z = \mathcal{L}in\{Y, z\}$ and define a functional y^* on Z by $< y*, \xi >=< y^*, y + az >= a$. We have

$$\|y + az\| = |a| \| \frac{y}{a} + z\| \geq |a| d$$

hence

$$| < y^*, \xi >= |a| \leq \frac{\|y + az\|}{d} = d^{-1} \|\xi\|$$

and y^* is bounded. By H.-B. theorem, we extend it to $\widetilde{y}^*$ on X with $< \widetilde{y}^*, x >= 0$ on Y and $< \widetilde{y}^*, z >= 1 \neq 0$.

Next we consider the Müntz theorem.

Theorem 1.15. *Let $(\lambda_j)_{j \in \mathbb{N}}$ be a sequence of positive numbers tending to ∞. The functions $\{t^{\lambda_j}\}_{j \in \mathbb{N}}$ span the space of all continuous functions on $[0, 1]$ that vanish at $t = 0$ if and only if*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty.$$

Proof. We prove the 'sufficient' part. Let x^* be a bounded linear functional that vanishes on all t^{λ_j}:

$$< x^*, t^{\lambda_j} >= 0, \qquad j \in \mathbb{N}.$$

For $\zeta \in \mathbb{C}$ such that $\Re\zeta > 0$, the functions $\zeta \to t^\zeta$ are analytic functions with values in $C([0, 1])$ This can be proved by showing that

$$\lim_{\mathbb{C} \ni h \to 0} \frac{t^{\zeta + h} - t^\zeta}{h} = (\ln t) t^\zeta$$

uniformly in $t \in [0, 1]$. Then

$$f(\zeta) =< x^*, t^\zeta >$$

is a scalar analytic function of ζ with $\Re\zeta > 0$. We can assume that $\|x^*\| \leq 1$. Then

$$|f(\zeta)| \leq 1$$

for $\Re\zeta > 0$ and $f(\lambda_j) = 0$ for any $j \in \mathbb{N}$.

Next, for a given N, we define a Blaschke product by

$$B_N(\zeta) = \prod_{j=1}^{N} \frac{\zeta - \lambda_j}{\zeta + \lambda_j}.$$

We see that $B_N(\zeta) = 0$ if and only if $\zeta = \lambda_j$, $|B_N(\zeta)| \to 1$ both as $\Re\zeta \to 0$ and $|\zeta| \to \infty$. Hence

$$g_N(\zeta) = \frac{f(\zeta)}{B_N(\zeta)}$$

is analytic in $\Re\zeta > 0$. Moreover, for any ϵ' there is $\delta_0 > 0$ such that for any $\delta > \delta_0$ we have $|B_N(\zeta)| \geq 1 - \epsilon'$ on $\Re\zeta = \delta$ and $|\zeta| = \delta^{-1}$. Hence for any ϵ

$$|g_N(\zeta)| \leq 1 + \epsilon$$

there and by the maximum principle the inequality extends to the interior of the domain. Taking $\epsilon \to 0$ we obtain $|g_N(\zeta)| \leq 1$ on $\Re\zeta > 0$.

Assume now there is $k > 0$ for which $f(k) \neq 0$. Then we have

$$\prod_{j=1}^{N} \left| \frac{\lambda_j + k}{\lambda_j - k} \right| \leq \frac{1}{f(k)}.$$

Note, that this estimate is uniform in N. If we write

$$\frac{\lambda_j + k}{\lambda_j - k} = 1 + \frac{2k}{\lambda_j - k}$$

then, by $\lambda_j \to \infty$ almost all terms bigger then 1. Remembering that boundedness of the product is equivalent to the boundedness of the sum

$$\sum_{j=1}^{N} \frac{1}{\lambda_j - k}$$

we see that we arrived at contradiction with the assumption. Hence, we must have $f(k) = 0$ for any $k > 0$. This means, however, that any functional that vanishes on $\{t^{\lambda_j}\}$ vanishes also on t^k for any k. But, by the Stone- Weierstrass theorem, it must vanish on any continuous function (taking value 0 at zero). Hence, by the spanning criterion, any such continuous function belongs to the closed linear span of $\{t^{\lambda_j}\}$.

Non-reflexiveness of $C([-1, 1])$

Consider the Banach space $X = C([-1, 1])$ normed with the sup norm. If X was reflexive, then we could identify X^{**} with X and thus, for every $x^* \in X^*$ there would be $x \in X$ such that

$$\|x\| = 1, \quad < x^*, x >= \|x^*\|. \tag{1.23}$$

Let us define $x^* \in X^*$ by

$$< x^*, x >= \int_{-1}^{1} \text{sign} t x(t) dt.$$

Then

$$| < x^*, x > | < 2\|x\|. \tag{1.24}$$

Indeed, restrict our attention to $\|x\| = 1$. We see then that $| < x^*, x > | < 2$. Clearly, for the integral to attain maximum possible values, the integral should be of opposite values. We can focus on the case when the integral over $(-1, 0)$ is negative and over $(0, 1)$ is positive and then for the best values, $x(t)$ must negative on $(-1, 0)$ and positive on $(0, 1)$. Then, each term is at most 1 and for this $x(t) = 1$ for $t \in (0, 1)$ and $x(t) = -1$ for $t \in (-1, 0)$. But this is impossible as g is continuous at 0. On the other hand, by choosing $x(t)$ to be -1 for $-1 < t < -\epsilon$, 1 for $\epsilon < t < 1$ and linear between $-\epsilon$ and ϵ we see that

$$< x^*, x >= 2 - \epsilon$$

with $\|x\| = 1$. Hence, $\|x^*\| = 2$. However, this is impossible by (1.24).

Norms of functionals

Example 1.16. The existence of an element $\bar{x}^*$ satisfying $<\bar{x}^*, x>= \|x\|$ has an important consequence for the relation between X and X^{**} in a nonreflexive case. Let B, B^*, B^{**} denote the unit balls in X, X^*, X^{**}, respectively. Because $x^* \in X^*$ is an operator over X, the definition of the operator norm gives

$$\|x^*\|_{X^*} = \sup_{x \in B} | <x^*, x> | = \sup_{x \in B} <x^*, x>, \tag{1.25}$$

and similarly, for $x \in X$ considered as an element of X^{**} according to (1.21), we have

$$\|x\|_{X^{**}} = \sup_{x^* \in B^*} | <x^*, x> | = \sup_{x^* \in B^*} <x^*, x> . \tag{1.26}$$

Thus, $\|x\|_{X^{**}} \leq \|x\|_X$. On the other hand,

$$\|x\|_X =<\bar{x}^*, x>\leq \sup_{x^* \in B^*} <x^*, x>= \|x\|_{X^{**}}$$

and

$$\|x\|_{X^{**}} = \|x\|_X. \tag{1.27}$$

Hence, in particular, the identification given by (1.21) is an isometry and X is a closed subspace of X^{**}.

First comments on weak convergence

The existence of a large number of functionals over X allows us to introduce new types of convergence. Apart from the standard *norm (or strong) convergence* where $(x_n)_{n\in\mathbb{N}} \subset X$ converges to x if

$$\lim_{n\to\infty} \|x_n - x\| = 0,$$

we define *weak convergence* by saying that $(x_n)_{n\in\mathbb{N}}$ weakly converges to x, if for any $x^* \in X^*$,

$$\lim_{n\to\infty} <x^*, x_n>=<x^*, x> .$$

In a similar manner, we say that $(x_n^*)_{n\in\mathbb{N}} \subset X^*$ converges $*$-*weakly* to x^* if, for any $x \in X$,

$$\lim_{n\to\infty} <x_n^*, x>=<x^*, x> .$$

Remark 1.17. It is worthwhile to note that we have a concept of a *weakly convergent* or *weakly Cauchy* sequence if the finite limit $\lim_{n\to\infty} <x^*, x_n>$ exists for any $x^* \in X^*$. In general, in this case we do not have a limit element. If every weakly convergent sequence converges weakly to an element of X, the Banach space is said to be *weakly sequentially complete*. It can be proved that reflexive spaces and L_1 spaces are weakly sequentially complete. On the other hand, no space containing a subspace isomorphic to the space c_0 (of sequences that converge to 0) is weakly sequentially complete (see, e.g., [**?**]).

Remark 1.18. In finite dimensional spaces weak and strong convergence is equivalent which can be seen by taking x^* being the coordinate vectors. Then weak convergence reduces to coordinate-wise convergence.

However, the weak convergence is indeed weaker than the convergence in norm. For example, consider any orthonormal basis $\{e_n\}_{n\geq 1}$ of a separable Hilbert space X. Then $\|e_n\| = 1$ but for any $f \in X$ we know that the series

$$\sum_{n=1}^{\infty} < f, e_n > e_n$$

converges in X and, equivalently,

$$\sum_{n=1}^{\infty} | < f, e_n > |^2 < \infty.$$

Thus

$$\lim_{n\to\infty} < f, e_n >= 0$$

for any $f \in X (= X^*)$ and so $(e_n)_{n\geq 0}$ weakly converges to zero.

1.2.3 Banach–Steinhaus Theorem

Another fundamental theorem of functional analysis is the Banach–Steinhaus theorem, or the Uniform Boundedness Principle. It is based on a fundamental topological results known as the Baire Category Theorem.

Theorem 1.19. *Let X be a complete metric space and let $\{X_n\}_{n\geq 1}$ be a sequence of closed subsets in X. If $Int\, X_n = \emptyset$ for any $n \geq 1$, then $Int \bigcup_{n=1}^{\infty} X_n = \emptyset$. Equivalently, taking complements, we can state that a countable intersection of open dense sets is dense.*

Remark 1.20. Baire's theorem is often used in the following equivalent form: if X is a complete metric space and $\{X_n\}_{n\geq 1}$ is a countable family of closed sets such that $\bigcup_{n=1}^{\infty} X_n = X$, then $Int X_n \neq \emptyset$ at least for one n.

Chaotic dynamical systems

We assume that X is a complete metric space, called the state space. In general, a *dynamical system* on X is just a family of states $(\mathbf{x}(t))_{t\in\mathbb{T}}$ parametrized by some parameter t (time). Two main types of dynamical systems occur in applications: those for which the time variable is discrete (like the observation times) and those for which it is continuous.

Theories for discrete and continuous dynamical systems are to some extent parallel. In what follows mainly we will be concerned with continuous dynamical systems. Also, to fix attention we shall discuss only systems defined for $t \geq 0$, that are sometimes called *semidynamical systems*. Thus by a *continuous dynamical system* we will understand a family of functions (operators) $(\mathbf{x}(t,\cdot))_{t\geq 0}$ such that for each t, $\mathbf{x}(t,\cdot) : X \to X$ is a continuous function, for each $\mathbf{x}_0$ the function $t \to \mathbf{x}(t,\mathbf{x}_0)$ is continuous with $\mathbf{x}(0,\mathbf{x}_0) = \mathbf{x}_0$. Moreover, typically it is required that the following semigroup property is satisfied (both in discrete and continuous case)

$$\mathbf{x}(t+s,\mathbf{x}_0) = \mathbf{x}(t,\mathbf{x}(s,\mathbf{x}_0)), \qquad t,s \geq 0, \tag{1.28}$$

which expresses the fact that the final state of the system can be obtained as the superposition of intermediate states.

Often discrete dynamical systems arise from iterations of a function

$$\mathbf{x}(t+1,\mathbf{x}_0) = f(\mathbf{x}(t,\mathbf{x}_0)), \qquad t \in \mathbb{N}, \tag{1.29}$$

while when t is continuous, the dynamics are usually described by a differential equation

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = A(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0 \qquad t \in \mathbb{R}_+. \tag{1.30}$$

Let (X, d) be a metric space where, to avoid non-degeneracy, we assume that $X \neq \{\mathbf{x}(t, \boldsymbol{p})\}_{t\geq 0}$ for any $\boldsymbol{p} \in X$, that is, the space does not degenerates to a single orbit). We say that the dynamical system $(\mathbf{x}(t))_{t\geq 0}$ on (X, d) is *topologically transitive* if for any two non-empty open sets $U, V \subset X$ there is $t_0 \geq 0$ such that $\mathbf{x}(t, U) \cap V \neq \emptyset$. A *periodic point* of $(\mathbf{x}(t))_{t\geq 0}$ is any point $\boldsymbol{p} \in X$ satisfying

$$\mathbf{x}(T, \boldsymbol{p}) = \boldsymbol{p},$$

for some $T > 0$. The smallest such T is called the period of $\boldsymbol{p}$. We say that the system has *sensitive dependence on initial conditions*, abbreviated as sdic, if there exists $\delta > 0$ such that for every $\boldsymbol{p} \in X$ and a neighbourhood N_p of $\boldsymbol{p}$ there exists a point $\mathbf{y} \in N_p$ and $t_0 > 0$ such that the distance between $\mathbf{x}(t_0, \boldsymbol{p})$ and $\mathbf{x}(t_0, \mathbf{y})$ is larger than δ. This property captures the idea that in chaotic systems minute errors in experimental readings eventually lead to large scale divergence, and is widely understood to be the central idea in chaos.

With this preliminaries we are able to state Devaney's definition of chaos (as applied to continuous dynamical systems).

Definition 1.21. *Let X be a metric space. A dynamical system $(\mathbf{x}(t))_{t\geq 0}$ in X is said to be chaotic in X if*

1. $(\mathbf{x}(t))_{t\geq 0}$ *is transitive,*
2. *the set of periodic points of* $(\mathbf{x}(t))_{t\geq 0}$ *is dense in* X*,*
3. $(\mathbf{x}(t))_{t\geq 0}$ *has sdic.*

To summarize, chaotic systems have three ingredients: indecomposability (property 1), unpredictability (property 3), and an element of regularity (property 2).

It is then a remarkable observation that properties 1. and 2 together imply sdic.

Theorem 1.22. *If $(\mathbf{x}(t))_{t\geq 0}$ is topologically transitive and has dense set of periodic points, then it has sdic.*

We say that X is non-degenerate, if continuous images of a compact intervals are nowhere dense in X.

Lemma 1.23. *Let X be a non-degenerate metric space. If the orbit $O(\boldsymbol{p}) = \{\mathbf{x}(t, \boldsymbol{p})\}_{t\geq 0}$ is dense in X, then also the orbit $O(\mathbf{x}(s, \boldsymbol{p})) = \{\mathbf{x}(t, \boldsymbol{p})\}_{t>s}$ is dense in X, for any $s > 0$.*

Proof. Assume that $O(\mathbf{x}(s, \boldsymbol{p}))$ is not dense in X, then there is an open ball B such that $B \cap \overline{O(\mathbf{x}(s, \boldsymbol{p}))} = \emptyset$. However, each point of the ball is a limit point of the whole orbit $O(\boldsymbol{p})$, thus we must have $\{\mathbf{x}(t, \boldsymbol{p})\}_{0\leq t\leq s} = \overline{\{\mathbf{x}(t, \boldsymbol{p})\}_{0\leq t\leq s}} \supset B$ which contradicts the assumption of nondegeneracy. ■

To fix terminology we say that a semigroup having a dense trajectory is called *hypercyclic*. We note that by continuity $\overline{O(\boldsymbol{p})} = \overline{\{\mathbf{x}(t, \boldsymbol{p})\}_{t\in\mathbb{Q}}}$, where $\mathbb{Q}$ is the set of positive rational numbers, therefore hypercyclic semigroups can exist only in separable spaces.

By X_h we denote the set of hypercyclic vectors, that is,

$$X_h = \{\boldsymbol{p} \in X;\ O(\boldsymbol{p}) \text{ is dense in } X\}$$

Note that if $(\mathbf{x}(t))_{t\geq 0}$ has one hypercyclic vector, then it has a dense set of hypercyclic vectors as each of the point on the orbit $O(\boldsymbol{p})$ is hypercyclic (by the first part of the proof above).

Theorem 1.24. *Let $(\mathbf{x}(t))_{t\geq 0}$ be a strongly continuous semigroup of continuous operators (possibly nonlinear) on a complete (separable) metric space X. The following conditions are equivalent:*

1. *X_h is dense in X,*
2. *$(\mathbf{x}(t))_{t\geq 0}$ is topologically transitive.*

Proof. Let as take the set of nonegative rational numbers and enumerate them as $\{t_1, t_2, \ldots\}$. Consider now the family $\{\mathbf{x}(t_n)\}_{n\in\mathbb{N}}$. Clearly, the orbit of $\boldsymbol{p}$ through $(\mathbf{x}(t))_{t\geq 0}$ is dense in X if and only if the set $\{\mathbf{x}(t_n)\boldsymbol{p}\}_{n\in\mathbb{N}}$ is dense.

Consider now the covering of X by the enumerated sequence of balls B_m centered at points of a countable subset of X with rational radii. Since each $\mathbf{x}(t_m)$ is continuous, the sets

$$G_m = \bigcup_{n\in\mathbb{N}} \mathbf{x}^{-1}(t_n, B_m)$$

are open. Next we claim that

$$X_h = \bigcap_{m\in\mathbb{N}} G_m.$$

In fact, let $\boldsymbol{p} \in X_h$, that is, $\boldsymbol{p}$ is hypercyclic. It means that $\mathbf{x}(t_n, \boldsymbol{p})$ visits each neigbourhood of each point of X for some n. In particular, for each m there must be n such that $\mathbf{x}(t_n, \boldsymbol{p}) \in B_m$ or $\boldsymbol{p} \in \mathbf{x}^{-1}(t_n, B_m)$ which means $\boldsymbol{p} \in \bigcap\limits_{m\in\mathbb{N}} G_m$.

Conversely, if $\boldsymbol{p} \in \bigcap\limits_{m\in\mathbb{N}} G_m$, then for each m there is n such that $\boldsymbol{p} \in \mathbf{x}^{-1}(t_n, B_m)$, that is, $\mathbf{x}(t_n, \boldsymbol{p}) \in B_m$. This means that $\{\mathbf{x}(t_n, \boldsymbol{p})\}_{n\in\mathbb{N}}$ is dense.

The next claim is condition 2. is equivalent to each set G_m being dense in X. If G_m were not dense, then for some B_r, $B_r \cap \mathbf{x}^{-1}(t_n, B_m) = \emptyset$ for any n. But then $\mathbf{x}(t_n, B_r) \cap B_m = \emptyset$ for any n. Since the continuous semigroup is topologically transitive, we know that there is $\mathbf{y} \in B_r$ such that $\mathbf{x}(t_0, \mathbf{y}) \in B_m$ for some t_0. Since B_m is open, $\mathbf{x}(t, \mathbf{y}) \in B_m$ for t from some neighbourhood of t_0 and this neighbourhood must contain rational numbers.

The converse is immediate as for given open U and V we find $B_m \subset V$ and since G_m is dense $U \cap G_m \neq \emptyset$. Thus $U \cap \mathbf{x}^{-1}(t_n, B_m) \neq \emptyset$ for some n, hence $\mathbf{x}(t_n, U) \cap B_m \neq \emptyset$.

So, if $(\mathbf{x}(t))_{t\geq 0}$ is topologically transitive, then X_h is the intersection of a countable collection of open dense sets, and by Baire Theorem in a complete space such an intersection must be still dense, thus X_h is dense.

Conversely, if X_h is dense, then each term of the intersection must be dense, thus each G_m is dense which yields the transitivity. ■

Back to the Banach–Steinhaus Theorem

To understand its importance, let us reflect for a moment on possible types of convergence of sequences of operators. Because the space $\mathcal{L}(X, Y)$ can be made a normed space by introducing the norm (1.11), the most natural concept of convergence of a sequence $(A_n)_{n\in\mathbb{N}}$ would be with respect to this norm. Such a convergence is referred to as the *uniform operator convergence*. However, for many purposes this notion is too strong and we work with the pointwise or *strong convergence*: the sequence $(A_n)_{n\in\mathbb{N}}$ is said to converge strongly if, for each $x \in X$, the sequence $(A_n x)_{n\in\mathbb{N}}$ converges in the norm of Y. In the same way we define uniform and strong boundedness of a subset of $\mathcal{L}(X, Y)$.

Note that if $Y = \mathbb{R}$ (or $\mathbb{C}$), then strong convergence coincides with $*$-weak convergence.

After these preliminaries we can formulate the Banach–Steinhaus theorem.

Theorem 1.25. *Assume that X is a Banach space and Y is a normed space. Then a subset of $\mathcal{L}(X, Y)$ is uniformly bounded if and only if it is strongly bounded.*

One of the most important consequences of the Banach–Steinhaus theorem is that a strongly converging sequence of bounded operators is always converging to a linear bounded operator. That is, if for each x there is y_x such that

$$\lim_{n\to\infty} A_n x = y_x,$$

then there is $A \in \mathcal{L}(X, Y)$ satisfying $Ax = y_x$.

Further comments on weak convergence

Example 1.26. We can use the above result to get a better understanding of the concept of weak convergence and, in particular, to clarify the relation between reflexive and weakly sequentially complete spaces. First, by considering elements of X^* as operators in $\mathcal{L}(X, \mathbb{C})$, we see that every $*$-weakly converging sequence of functionals converges to an element of X^* in $*$-weak topology. On the other hand, for a weakly converging sequence $(x_n)_{n\in\mathbb{N}} \subset X$, such an approach requires that $x_n, n \in \mathbb{N}$, be identified with elements of X^{**} and thus, by the Banach–Steinhaus theorem, a weakly converging sequence always has a limit $x \in X^{**}$. If X is reflexive, then $x \in X$ and X is weakly sequentially complete. However, for nonreflexive X we might have $x \in X^{**} \setminus X$ and then $(x_n)_{n\in\mathbb{N}}$ does not converge weakly to any element of X.

On the other hand, (1.27) implies that a weakly convergent sequence in a normed space is norm bounded. Indeed, we consider $(x_n)_{n\in\mathbb{N}}$ such that for each $x^* \in X^*$ $< x^*, x_n >$ converges. Treating x_n as elements of X^{**}, we see that the numerical sequences $< x_n, x^* >$ are bounded for each $x^* \in X^*$. X^* is a Banach space (even if X is not). Then $(\|x_n\|)_{n\geq 0}$ is bounded by the Banach-Steinhaus theorem.

We can also prove the partial reverse of this inequality: if $(x_n)_{n\in\mathbb{N}}$ is a sequence in a normed space X weakly converging to x, then

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|. \tag{1.31}$$

To prove this, there is $x^* \in X^*$ such that

$$\|x^*\| = 1, \qquad | < x^*, x > | = \|x\|.$$

Hence

$$\|x\| = | < x^*, x > | = | \lim_{n\to\infty} < x^*, x_n > | \leq \liminf_{n\to\infty} | < x^*, x_n > | \liminf_{n\to\infty} \|x_n\|.$$

However, we point out that a theorem proved by Mazur (e.g., see [**?**], p. 120) says that if $x_n \to x$ weakly, then there is a sequence of convex combinations of elements of $(x_n)_{n\in\mathbb{N}}$ that converges to x in norm. To prove this result, let us introduce the concept of the support function of a set. For a set M we define

$$S_M(x^*) = \sup_{x\in M} < x^*, x > .$$

A crucial result is

Lemma 1.27. *If X is a normed space over $\mathbb{R}$ and M is a closed convex subset of X then $z \in M$ if and only if $< x^*, z > \leq S_M(x^*)$ for any $x^* \in X^*$.*

Proof. If $z \in M$, then $< x^*, z > \leq \sup_{x\in M} < x^*, x > = S_M(x^*)$ by definition.

If $z \notin M$ then, by closedness, there is a ball $B(z, r)$ not intersecting with M. By the geometric version of the Hahn-Banach theorem, there is a linear functional z^* and a constant c such that for any $x \in M$ and $y \in B(z, r)$ we have

$$< z^*, x > \leq c \leq < z^*, y > .$$

Since $y = z + rv$, $\|v\| \leq 1$, we have

$$c \leq < z^*, z + rx > = < z^*, z > + r < z^*, x > .$$

Using the fact that $\inf_{\|x\|\leq 1} < z^*, z > = -\|z^*\|$, we obtain

$$c \leq < z^*, z + rx > = < z^*, z > - r\|z^*\|.$$

On the other hand

$$S_M(z^*) \leq c \leq < z^*, z > -r\|z^*\|$$

which yields

$$< z^*, z > \geq S_M(z^*) + r\|z^*\| > S_M(z^*)$$

and completes the proof.

With this result we can prove the Mazur theorem.

Let K be a closed convex set and $(x_n)_{n\in\mathbb{N}}$ be a sequence weakly converging to $x \in K$. Consider $S_K(x^*)$. We have

$$< x^*, x_n > \leq S_K(x^*)$$

for any $x^* \in X^*$. But this implies

$$< x^*, x > \leq S_K(x^*)$$

and the result follows by the above lemma.

The Banach-Steinhaus theorem and convergence on subsets

We note another important corollary of the Banach–Steinhaus theorem which we use in the sequel.

Corollary 1.28. *A sequence of operators $(A_n)_{n\in\mathbb{N}}$ is strongly convergent if and only if it is convergent uniformly on compact sets.*

Proof. It is enough to consider convergence to 0. If $(A_n)_{n\in\mathbb{N}}$ converges strongly, then by the Banach–Steinhaus theorem, $a = \sup_{n\in\mathbb{N}} \|A_n\| < +\infty$. Next, if $\Omega \subset X$ is compact, then for any ϵ we can find a finite set $N_\epsilon = \{x_1, \ldots, x_k\}$ such that for any $x \in \Omega$ there is $x_i \in N_\epsilon$ with $\|x - x_i\| \leq \epsilon/2a$. Because N_ϵ is finite, we can find n_0 such that for all $n > n_0$ and $i = 1, \ldots, k$ we have $\|A_n x_i\| \leq \epsilon/2$ and hence

$$\|A_n x\| = \|A_n x_i\| + a\|x - x_i\| \leq \epsilon$$

for any $x \in \Omega$. The converse statement is obvious. □

We conclude this unit by presenting a frequently used result related to the Banach–Steinhaus theorem.

Proposition 1.29. *Let X, Y be Banach spaces and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X, Y)$ be a sequence of operators satisfying $\sup_{n\in\mathbb{N}} \|A_n\| \leq M$ for some $M > 0$. If there is a dense subset $D \subset X$ such that $(A_n x)_{n\in\mathbb{N}}$ is a Cauchy sequence for any $x \in D$, then $(A_n x)_{n\in\mathbb{N}}$ converges for any $x \in X$ to some $A \in \mathcal{L}(X, Y)$.*

Proof. Let us fix $\epsilon > 0$ and $y \in X$. For this ϵ we find $x \in D$ with $\|x-y\| < \epsilon/M$ and for this x we find n_0 such that $\|A_n x - A_m x\| < \epsilon$ for all $n, m > n_0$. Thus,

$$\|A_n y - A_m y\| \leq \|A_n x - A_m x\| + \|A_n(x - y)\| + \|A_m(x - y)\| \leq 3\epsilon.$$

Hence, $(A_n y)_{n\in\mathbb{N}}$ is a Cauchy sequence for any $y \in X$ and, because Y is a Banach space, it converges and an application of the Banach–Steinhaus theorem ends the proof. □

Application–limits of integral expressions

Consider an equation describing growth of, say, cells

$$\frac{\partial N}{\partial t} + \frac{\partial (g(m)N)}{\partial m} = -\mu(m)N(t,m), \quad m \in (0,1), \tag{1.32}$$

with the boundary condition

$$g(0)N(t,0) = 0 \tag{1.33}$$

and with the initial condition

$$N(0,m) = N_0(m) \quad \text{for } m \in [0,1]. \tag{1.34}$$

Here $N(m)$ denotes cells' density with respect to their size/mass and we consider the problem in $L_1([0,1])$.

Consider the 'formal' equation for the stationary version of the equation (the resolvent equation)

$$\lambda N(m) + (g(m)N(m))' + \mu(m)N(m) = f(m) \in L_1([0,1]),$$

whose solution is given by

$$N_\lambda(m) = \frac{e^{-\lambda G(m)-Q(m)}}{g(m)} \int_0^m e^{\lambda G(s)+Q(s)} f(s)\, ds \tag{1.35}$$

where $G(m) = \int_0^m (1/g(s))\, ds$ and $Q(m) = \int_0^m (\mu(s)/g(s))\, ds$. To shorten notation we denote

$$e_{-\lambda}(m) := e^{-\lambda G(m)-Q(m)}, \qquad e_\lambda(m) := e^{\lambda G(m)+Q(m)}.$$

Our aim is to show that $g(m)N_\lambda(m) \to 0$ as $m \to 1^-$ provided $1/g$ or μ is not integrable close to 1. If the latter condition is satisfied, then $e_\lambda(m) \to \infty$ and $e^{-\lambda}(m) \to 0$ as $m \to 1^-$.

Indeed, consider the family of functionals $\{\xi_m\}_{m\in[1-\epsilon,1)}$ for some $\epsilon > 0$ defined by

$$\xi_m f = e_{-\lambda}(m) \int_0^m e_\lambda(s) f(s)\, ds$$

for $f \in L^1[0,1]$. We have

$$|\xi_m f| \le e_{-\lambda}(m) \int_0^m e_\lambda(s)|f(s)|\, ds \le \int_0^1 |f(s)|\, ds$$

on account of monotonicity of e_λ. Moreover, for f with support in $[0, 1-\delta]$ with any $\delta > 0$ we have $\lim_{m\to 1^-} \xi_m f = 0$ and, by Proposition 1.29, the above limit extends by density for any $f \in L^1[0,1]$.

1.2.4 Weak compactness

In finite dimensional spaces normed spaces we have Bolzano-Weierstrass theorem stating that from any bounded sequence of elements of X_n one can select a convergent subsequence. In other words, a closed unit ball in X_n is compact.

There is no infinite dimensional normed space in which the unit ball is compact.

Weak compactness comes to the rescue. Let us begin with (separable) Hilbert spaces.

Theorem 1.30. *Each bounded sequence $(u_n)_{n\in\mathbb{N}}$ in a separable Hilbert space X has a weakly convergent subsequence.*

Proof. Let $\{v_k\}_{k\in\mathbb{N}}$ be dense in X and consider numerical sequences $((u_n, v_k))_{n\in\mathbb{N}}$ for any k. From Banach-Steinhaus theorem and

$$|(u_n, v_k)| \leq \|u_n\|\|v_k\|$$

we see that for each k these sequences are bounded and hence each has a convergent subsequence. We use the diagonal procedure: first we select $(u_{1n})_{n\in\mathbb{N}}$ such that $(u_{1n}, v_1) \to a_1$, then from $(u_{1n})_{n\in\mathbb{N}}$ we select $(u_{2n})_{n\in\mathbb{N}}$ such that $(u_{2n}, v_2) \to a_2$ and continue by induction. Finally, we take the diagonal sequence $w_n = u_{nn}$ which has the property that $(w_n, v_k) \to a_k$. This follows from the fact that elements of $(w_n)_{n\in\mathbb{N}}$ belong to $(u_{kn}$ for $n \geq k$. Since $\{v_k\}_{k\in\mathbb{N}}$ is dense in X and $(u_n)_{n\in\mathbb{N}}$ is norm bounded, Proposition 1.29 implies $((w_n, v))_{n\in\mathbb{N}}$ converges to, say, $a(v)$ for any $v \in X$ and $v \to a(v)$ is a bounded (anti) linear functional on X. By the Riesz representation theorem, there is $w \in X$ such that $a(v) = (v, w)$ and thus $w_n \rightharpoonup w$.

If X is not separable, then we can consider $Y = \overline{\mathcal{L}in\{u_n\}_{n\in\mathbb{N}}}$ which is separable and apply the above theorem in Y getting an element $w \in Y$ for which

$$(w_n, v) \to (w, v), \qquad v \in Y.$$

Let now $z \in X$. By orthogonal decomposition, $z = v + v^\perp$ by linearity and continuity (as $w \in Y$)

$$(w_n, z) = (w_n, v) \to (w, v) = (w, z)$$

and so $w_n \rightharpoonup w$ in X.

Corollary 1.31. *Closed unit ball in X is weakly sequentially compact.*

Proof. We have

$$(v, w_n) \to (v, w), \quad n \to \infty$$

for any v. We can assume $w = 0$ We prove that for any k there are indices $n_1, \ldots, n_k$ such that

$$k^{-1}(w_{n_1} + \ldots + w_{n_k}) \to 0$$

in X. Since $(w_1, w_n) \to 0$, we set $n_1 = 1$ and select n_2 such that $|(w_{n_1}, w_{n_2})| \leq 1/2$. Then we select n_3 such that $|(w_{n_1}, w_{n_3})| \leq 1/2$ and $|(w_{n_2}, w_{n_3})| \leq 1/2$ and further, n_k such that $|(w_{n_1}, w_{n_k})| \leq 1/(k-1), \ldots, |(w_{n_{k-1}}, w_{n_k})| \leq 1/(k-1)$. Since $\|w_n\| \leq C$, we obtain

$$\begin{aligned}
&\|k^{-1}(w_{n_1} + \ldots + w_{n_k})\|^2 \\
&\leq k^{-2}\left(\sum_{j=1}^{k}\|w_{n_j}\|^2 + 2\sum_{j=1}^{k-1}(w_{n_j}, w_{n_k}) + 2\sum_{j=1}^{k-2}(w_{n_j}, w_{n_{k-1}}) + \ldots\right) \\
&\leq k^{-2}(kC^2 + 2(k-1)(k-1)^{-1} + 2(k-2)(k-2)^{-1} + \ldots 2) \\
&\leq k^{-1}(C^2 + 2)
\end{aligned}$$

Note that this result shows that any closed convex set in X is weakly sequentially compact. What about other spaces?

Practically the same proof (using the fact that a closed subspace of a reflexive space is reflexive) shows that if a Banach space is reflexive, then the closed unit ball is weakly sequentially compact. The converse is also true (Eberlain).

Helly's theorem: If X is a separable Banach space and $U = X^*$, then the closed unit ball in U is weak* sequentially compact. Alaoglu removed separability.

1.2.5 The Open Mapping Theorem

The Open Mapping Theorem is fundamental for inverting linear operators. Let us recall that an operator $A : X \to Y$ is called *surjective* if $ImA = Y$ and *open* if the set $A\Omega$ is open for any open set $\Omega \subset X$.

Theorem 1.32. *Let X, Y be Banach spaces. Any surjective $A \in \mathcal{L}(X, Y)$ is an open mapping.*

One of the most often used consequences of this theorem is the Bounded Inverse Theorem.

Corollary 1.33. *If $A \in \mathcal{L}(X, Y)$ is such that $KerA = \{0\}$ and $ImA = Y$, then $A^{-1} \in \mathcal{L}(Y, X)$.*

The corollary follows as the assumptions on the kernel and the image ensure the existence of a linear operator A^{-1} defined on the whole Y. The operator A^{-1} is continuous by the Open Mapping Theorem, as the preimage of any open set in X through A^{-1}, that is, the image of this set through A, is open.

Throughout the book we are faced with invertibility of unbounded operators. An operator $(A, D(A))$ is said to be *invertible* if there is a bounded operator $A^{-1} \in \mathcal{L}(Y, X)$ such that $A^{-1}Ax = x$ for all $x \in D(A)$ and $A^{-1}y \in D(A)$ with $AA^{-1}y = y$ for any $y \in Y$. We have the following useful conditions for invertibility of A.

Proposition 1.34. *Let X, Y be Banach spaces and $A \in L(X,Y)$. The following assertions are equivalent.*

(i) A is invertible;
(ii) $ImA = Y$ and there is $m > 0$ such that $\|Ax\| \geq m\|x\|$ for all $x \in D(A)$;
(iii) A is closed, $\overline{ImA} = Y$ and there is $m > 0$ such that $\|Ax\| \geq m\|x\|$ for all $x \in D(A)$;
(iv) A is closed, $ImA = Y$, and $KerA = \{0\}$.

Proof. The equivalence of (i) and (ii) follows directly from the definition of invertibility. By Theorem 1.35, the graph of any bounded operator is closed and because the graph of the inverse is given by

$$G(A) = \{(x,y);\ (y,x) \in G(A^{-1})\},$$

we see that the graph of any invertible operator is closed and thus any such an operator is closed. Hence, (i) and (ii) imply (iii) and (iv). Assume now that (iii) holds. $G(A)$ is a closed subspace of $X \times Y$, therefore it is a Banach space itself. The inequality $\|Ax\| \geq m\|x\|$ implies that the mapping $G(A) \ni (x, Ax) \to Ax \in ImA$ is an isomorphism onto ImA and hence ImA is also closed. Thus $ImA = Y$ and (ii) follows. Finally, if (iv) holds, then Corollary 1.33 can be applied to A from $D(A)$ (with the graph norm) to Y to show that $A^{-1} \in \mathcal{L}(Y, D(A)) \subset \mathcal{L}(Y, X)$. □

Norm equivalence. An important result is that if X is a Banach space with respect to two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ and there is C such that $\|x\|_1 \leq C\|x\|_2$, then both norms are equivalent.

The Closed Graph Theorem

It is easy to see that a bounded operator defined on the whole Banach space X is closed. That the inverse also is true follows from the Closed Graph Theorem.

Theorem 1.35. *Let X, Y be Banach spaces. An operator $A \in L(X,Y)$ with $D(A) = X$ is bounded if and only if its graph is closed.*

We can rephrase this result by saying that an everywhere defined closed operator in a Banach space must be bounded.

Proof. Indeed, consider on X two norms, the original norm $\|\cdot\|$ and the graph norm

$$\|x\|_{D(A)} = \|x\| + \|Ax\|.$$

By closedness, X is a Banach space with respect to $D(A)$ and A is continuous in the norm $\|\cdot\|_{D(A)}$. Hence, the norms are equivalent and A is continuous in the norm $\|\cdot\|$.

To give a nice and useful example of an application of the Closed Graph Theorem, we discuss a frequently used notion of relatively bounded operators. Let two operators $(A, D(A))$ and $(B, D(B))$ be given. We say that B is A-*bounded* if $D(A) \subset D(B)$ and there exist constants $a, b \geq 0$ such that for any $x \in D(A)$,

$$\|Bx\| \leq a\|Ax\| + b\|x\|. \tag{1.36}$$

Note that the right-hand side defines a norm on the space $D(A)$, which is equivalent to the graph norm (1.15).

Corollary 1.36. *If A is closed and B closable, then $D(A) \subset D(B)$ implies that B is A-bounded.*

Proof. If A is a closed operator, then $D(A)$ equipped with the graph norm is a Banach space. If we assume that $D(A) \subset D(B)$ and $(B, D(B))$ is closable, then $D(A) \subset D(\overline{B})$. Because the graph norm on $D(A)$ is stronger than the norm induced from X, the operator $\overline{B}$, considered as an operator from $D(A)$ to X is everywhere defined and closed. On the other hand, $\overline{B}|_{D(A)} = B$; hence $B : D(A) \to X$ is bounded by the Closed Graph Theorem and thus B is A-bounded. □

1.3 Hilbert space methods

One of the most often used theorems of functional analysis is the Riesz representation theorem.

Theorem 1.37 (Riesz representation theorem). *If x^* is a continuous linear functional on a Hilbert space H, then there is exactly one element $y \in H$ such that*

$$< x^*, x >= (x, y). \tag{1.37}$$

1.3.1 To identify or not to identify–the Gelfand triple

Riesz theorem shows that there is a canonical isometry between a Hilbert space H and its dual H^*. It is therefore natural to identify H and H^* and is done so in most applications. There are, however, situations when it cannot be done.

Assume that H is a Hilbert space equipped with a scalar product $(\cdot, \cdot)_H$ and that $V \subset H$ is a subspace of H which is a Hilbert space in its own right, endowed with a scalar product $(\cdot, \cdot)_V$. Assume that V is densely and continuously embedded in H that is $\overline{V} = H$ and $\|x\|_H \leq c\|x\|_V$, $x \in V$, for some constant c. There is a canonical map $T : H^* \to V^*$ which is given by restriction to V of any $h^* \in H^*$:

$$< Th^*, v >_{V^* \times V} =< h^*, v >_{H^* \times H}, \quad v \in V.$$

We easily see that

$$\|Th^*\|_{V^*} \leq C\|h^*\|_{H^*}.$$

Indeed

$$\begin{aligned}\|Th^*\|_{V^*} &= \sup_{\|v\|_V \leq 1} |< Th^*, v >_{V^* \times V}| = \sup_{\|v\|_V \leq 1} |< h^*, v >_{H^* \times H}| \\ &\leq \|h^*\|_{H^*} \sup_{\|v\|_V \leq 1} \|v\|_H \leq c\|h^*\|_{H^*}.\end{aligned}$$

Further, T is injective. For, if $Th_1^* = Th_2^*$, then

$$0 =< Th_1^* - Th_2^*, v >_{V^* \times V} =< h_1^* - h_2^*, v >_{H^* \times H}$$

for all $v \in V$ and the statement follows from density of V in H. Finally, the image of TH^* is dense in V^*. Indeed, let $v \in V^{**}$ be such that $< v, Th^* >= 0$ for all $h^* \in H^*$. Then, by reflexivity,

$$0 =< v, Th^* >_{V^{**} \times V^*} =< Th^*, v >_{V^* \times V} =< h^*, v >_{H^* \times H}, \quad h^* \in H^*$$

implies $v = 0$.

Now, if we identify H^* with H by the Riesz theorem and using T as the canonical embedding from H^* into V^*, one writes

$$V \subset H \simeq H^* \subset V^*$$

and the injections are dense and continuous. In such a case we say that H is the pivot space. Note that the scalar product in H coincides with the duality pairing $< \cdot, \cdot >_{V^* \times V}$:

$$(f, g)_H =< f, g >_{V^* \times V}, \quad f \in H, g \in V.$$

Remembering now that V is a Hilbert space with scalar product $(\cdot, \cdot)_V$ we see that identifying also V with V^* would lead to an absurd – we would have $V = H = H^* = V^*$. Thus, we cannot identify simultaneously both pairs. In such situations it is common to identify the pivot space H with its dual H^* bur to leave V and V^* as separate spaces with duality pairing being an extension of the scalar product in H.

An instructive example is $H = L_2([0, 1], dx)$ (real) with scalar product

$$(u, v) = \int_0^1 u(x)v(x)dx$$

and $V = L_2([0, 1], wdx)$ with scalar product

$$(u, v) = \int_0^1 u(x)v(x)w(x)dx,$$

where w is a nonnegative unbounded measurable function. Then it is useful to identify $V^* = L_2([0,1], w^{-1}dx)$ and

$$< f, g >_{V^* \times V} = \int_0^1 f(x)g(x)dx \leq \int_0^1 f(x)\sqrt{w(x)}\frac{g(x)}{\sqrt{w(x)}}dx \leq \|f\|_V \|g\|_{V^*}.$$

1.3.2 The Radon-Nikodym theorem

Let μ and ν be finite nonnegative measures on the same σ-algebra in Ω. We say that ν is absolutely continuous with respect to μ if every set that has μ-measure 0 also has ν measure 0.

Theorem 1.38. *If ν is absolutely continuous with respect to μ then there is an integrable function g such that*

$$\nu(E) = \int_E g d\mu, \tag{1.38}$$

for any μ-measurable set $E \subset \Omega$.

Proof. Assume for simplicity that $\mu(\Omega), \nu(\Omega) < \infty$. Let $H = L_2(\Omega, d\mu + d\nu)$ on the field of reals. Schwarz inequality shows that if $f \in H$, then $f \in L_1(d\mu + d\nu)$, then the linear functional

$$< x^*, f >:= \int_\Omega f d\mu$$

is bounded on H. Indeed

$$| < x^*, f > | \leq \int_\Omega 1 \cdot f d\mu \leq \sqrt{\mu(\Omega)}\sqrt{\int_\Omega f^2 d\mu} \leq \sqrt{\mu(\Omega)}\sqrt{\int_\Omega f^2 d(\mu+\nu)} \leq \sqrt{\mu(\Omega)}\|f\|_H.$$

Thus, by the Riesz theorem, there is $y \in H$ such that

$$\int_\Omega f d\mu = \int_\Omega f y d(\mu + \nu).$$

Thus we obtain

$$\int_\Omega f(1-y)d\mu = \int_\Omega f y d\nu.$$

We claim that $0 < y \leq 1$ almost everywhere with respect to μ (and thus ν). Consider the set $F = \{x \in \Omega;\ y \leq 0\}$ and f as the characteristic function of F, $f = \chi_F$ so that

$$\int_F (1-y)d\mu = \int_F y d\nu.$$

If $\mu(F) > 0$, then the left hand side is bigger that $\mu(F) > 0$ and the right hand side is at most 0 (as it may happen that $\nu(F) = 0$ – absolute continuity works only one way). Thus, $\mu(F) = 0$ and $y > 0$ μ (and ν) almost everywhere. Let now $E = \{x \in \Omega;\ y > 0\}$ and f be the characteristic function of E so that

$$\int_E (1-y)d\mu = \int_E y d\nu.$$

Now, if $\mu(E) > 0$, then the left hand side is strictly negative whereas the right hand side is at least 0 (if $\nu(E) = 0$). Thus, $\mu(E) = 0$ and $y \leq 1$ μ (and ν) almost everywhere. We can modify y on a μ measure zero set so that $0 < y \leq 1$ everywhere so that

$$g = \frac{1-y}{y}$$

is a finite nonnegative function on Ω. Let us denote

$$E_n = \{x \in \Omega;\ y(x) \geq n^{-1}\}.$$

The sequence $(E_n)_{n\in\mathbb{N}}$ is a nested sequence with $\bigcap E_n = \emptyset$ as y is positive everywhere. Thus we ca write $\chi_{E_n} = y f_n$ for some $f_n \in H$. Indeed, $0 \leq f_n \leq \chi_{E_n}/y \leq n$ so that f is bounded and thus square integrable for each n. Therefore we can write

$$\int_\Omega \chi_{E_n} y^{-1}(1-y)d\mu = \int_\Omega \chi_{E_n} d\nu.$$

Since $\chi_{E_n} \nearrow 1$ everywhere on Ω, using the dominated convergence theorem we obtain that $g = y^{-1}(1-y)$ is integrable on Ω. Taking arbitrary measurable subset $E \subset \Omega$ and its characteristic function, we obtain

$$\nu(E) = \int_E d\nu = \int_E g d\mu.$$

1.3.3 Projection on a convex set

Corollary 1.39. *Let K be a closed convex subset of a real Hilbert space H. For any $x \in H$ there is a unique $y \in K$ such that*

$$\|x - y\| = \inf_{z\in K} \|x - z\|. \tag{1.39}$$

Moreover, $y \in K$ is a unique solution to the variational inequality

$$(x - y, z - y) \leq 0 \tag{1.40}$$

for any $z \in K$.

Proof. Let $d = \inf_{z\in K} \|x-z\|$. We can assume $x \notin K$ and so $d > 0$. Consider $f(z) = \|x - z\|$, $z \in K$ and consider a minimizing sequence $(z_n)_{n\in\mathbb{N}}$, $z_n \in K$ such that $d \le f(z_n) \le d + 1/n$. By the definition of f, $(z_n)_{n\in\mathbb{N}}$ is bounded and thus it contains a weakly convergent subsequence, say $(\zeta_n)_{n\in\mathbb{N}}$. Since K is closed and convex, by Corollary 1.31, $\zeta_n \rightharpoonup y \in K$. Further we have

$$|(h, x-y)| = \lim_{n\to\infty} |(h, x-\zeta_n)| \le \|h\| \liminf_{n\to\infty} \|x-\zeta_n\| \le \|h\| \liminf_{n\to\infty} d+\frac{1}{n} = \|h\|d$$

for any $h \in H$ and thus, taking supremum over $\|h\| \le 1$, we get $f(y) \le d$ which gives existence of a minimizer.

To prove equivalence of (1.40) and (1.39) assume first that $y \in K$ satisfies (1.39) and let $z \in K$. Then, from convexity, $v = (1-t)y + tz \in K$ for $t \in [0, 1]$ and thus

$$\|x - y\| \le \|x - ((1 - t)y + tz)\| = \|(x - y) - t(z - y)\|$$

and thus

$$\|x - y\|^2 \le \|x - y\|^2 - 2t(x - y, z - y) + t^2\|z - y\|^2.$$

Hence

$$t\|z - y\|^2 \ge 2(x - y, z - y)$$

for any $t \in (0, 1]$ and thus, passing with $t \to 0$, $(x - y, z - y) \le 0$. Conversely, assume (1.40) is satisfied and consider

$$\begin{aligned}\|x - y\|^2 - \|x - z\|^2 &= (x - y, x - y) - (x - z, x - z)\\ &= 2(x, z) - 2(x, y) + 2(y, y) - 2(y, z) + 2(y, z) - (y, y)\\ &= 2(x - y, z - y) - (y - z, y - z) \le 0\end{aligned}$$

hence

$$\|x - y\| \le \|x - z\|$$

for any $z \in K$.

For uniqueness, let y_1, y_2 satisfy

$$(x - y_1, z - y_1) \le 0, \qquad (x - y_2, z - y_2) \le 0, \qquad z \in H.$$

Choosing $z = y_2$ in the first inequality and $z = y_1$ in the second and adding them, we get $\|y_1 - y_2\|^2 \le 0$ which implies $y_1 = y_2$.

■

We call the operator assigning to any $x \in K$ the element $y \in K$ satisfying (1.39) the projection onto K and denote it by P_K.

Proposition 1.40. *Let K be a nonempty closed and convex set. Then P_K is non expansive mapping.*

Proof. Let $y_i = P_K x_i$, $i = 1, 2$. We have

$$(x_1 - y_1, z - y_1) \leq 0, \qquad (x_2 - y_2, z - y_2) \leq 0, \qquad z \in H$$

so choosing, as before, $z = y_2$ in the first and $z = y_1$ in the second inequality and adding them together we obtain

$$\|y_1 - y_2\|^2 \leq (x_1 - x_2, y_1 - y_2),$$

hence $\|P_K x_1 - P_K x_2\| \leq \|x_1 - x_2\|$.

1.3.4 Theorems of Stampacchia and Lax-Milgram

1.3.5 Motivation

Consider the Dirichlet problem for the Laplace equation in $\Omega \subset \mathbb{R}^n$

$$-\Delta u = f \qquad \text{in} \quad \Omega, \tag{1.41}$$

$$u|_{\partial\Omega} = 0. \tag{1.42}$$

Assume that there is a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$. If we multiply (1.41) by a test function $\phi \in C_0^\infty(\Omega)$ and integrate by parts, then we obtain the problem

$$\int\limits_\Omega \nabla u \cdot \nabla \phi dx = \int\limits_\Omega f \phi dx. \tag{1.43}$$

Conversely, if u satisfies (1.43), then it is a distributional solution to (1.41).

Moreover, if we consider the minimization problem for

$$J(u) = \frac{1}{2} \int\limits_\Omega |\nabla u|^2 dx - \int\limits_\Omega f u dx$$

over $K = \{u \in C^2(\Omega);\ u|_{\partial\Omega} = 0\}$ and if u is a solution to this problem then for any $\epsilon \in \mathbb{R}$ and $\phi \in C_0^\infty(\Omega)$ we have

$$J(u + \epsilon\phi) \geq J(u),$$

then we also obtain (1.43). The question is how to obtain the solution.

In a similar way, we consider the obstacle problem, to minimize J over $K = \{u \in C^2(\Omega);\ u|_{\partial\Omega} = 0,\ u \geq g\}$ over some continuous function g satisfying $g|_{\partial\Omega} < 0$. Note that K is convex. Again, if $u \in K$ is a solution then for any $\epsilon > 0$ and $\phi \in K$ we obtain that $u + \epsilon(\phi - u) = (1 - \epsilon)u + \epsilon\phi$ is in K and therefore

$$J(u + \epsilon(\phi - u)) \geq J(u).$$

Here, we obtain only

$$\int_{\Omega} \nabla u \cdot \nabla(\phi - u)dx \geq \int_{\Omega} f(\phi - u)dx. \tag{1.44}$$

for any $\phi \in K$. For twice differentiable u we obtain

$$\int_{\Omega} \Delta u(\phi - u)dx \geq \int_{\Omega} f(\phi - u)dx$$

and choosing $\phi = u + \psi$, $0 \leq \psi \in C_0^\infty(\Omega)$ we get

$$-\Delta u \geq f$$

almost everywhere on Ω. As u is continuous, the set $N = \{x \in \Omega;\ u(x) > g(x)\}$ is open. Thus, taking $\psi \in C_0^\infty(N)$, we see that for sufficiently small $\epsilon > 0$, $u \pm \epsilon\phi \in K$. Then, on N

$$-\Delta u = f$$

Summarizing, for regular solutions the minimizer satisfies

$$\begin{aligned} -\Delta u &\geq f \\ u &\geq g \\ (\Delta u + f)(u - g) &= 0 \end{aligned}$$

on Ω.

Hilbert space theory

We begin with the following definition.

Definition 1.41. *Let H be a Hilbert space. A bilinear form $a : H \times H \to \mathbb{R}$ is said to be*

(i) continuous of there is a constant C such that

$$|a(x, y)| \leq C\|x\|\|y\|, \qquad x, y \in H;$$

coercive if there is a constant $\alpha > 0$ such that

$$a(x, x) \geq \alpha\|x\|^2.$$

Note that in the complex case, coercivity means $|a(x, x)| \geq \alpha\|x\|^2$.

Theorem 1.42. *Assume that $a(\cdot, \cdot)$ is a continuous coercive bilinear form on a Hilbert space H. Let K be a nonempty closed and convex subset of H. Then, given any $\phi \in H^*$, there exists a unique element $x \in K$ such that for any $y \in K$*

$$a(x, y - x) \geq < \phi, y - x >_{H^* \times H} \tag{1.45}$$

Moreover, if a is symmetric, then x is characterized by the property

$$x \in K \quad \text{and} \quad \frac{1}{2}a(x,x) - < \phi, x >_{H^* \times H} = \min_{y \in K} \frac{1}{2}a(y,y) - < \phi, y >_{H^* \times H} . \tag{1.46}$$

Proof. First we note that from Riesz theorem, there is $f \in H$ such that $< \phi, y >_{H^* \times H} = (f, y)$ for all $y \in H$. Now, if we fix $x \in H$, then $y \to a(x, y)$ is a continuous linear functional on H. Thus, again by the Riesz theorem, there is an operator $A : H \to H$ satisfying $a(x, y) = (Ax, y)$. Clearly, A is linear and satisfies

$$\|Ax\| \leq C\|x\|, \tag{1.47}$$

$$(Ax, x) \geq \alpha\|x\|^2. \tag{1.48}$$

Indeed,

$$\|Ax\| = \sup_{\|y\|=1} |(Ax, y)| \leq C\|x\| \sup_{\|y\|=1} \|y\|,$$

and (1.48) is obvious.

Problem (1.45) amounts to finding $x \in K$ satisfying, for all $y \in K$,

$$(Ax, y - x) \geq (f, y - x). \tag{1.49}$$

Let us fix a constant ρ to be determined later. Then, multiplying both sides of (1.49) by ρ and moving to one side, we find that (1.49) is equivalent to

$$(\rho f - \rho Ax + x - x, y - x) \leq 0. \tag{1.50}$$

Here we recognize the equivalent formulation of the projection problem (1.40), that is, we can write

$$x = P_K(\rho f - \rho Ax + x) \tag{1.51}$$

This is a fixed point problem for x in K. Denote $Sy = P_K(\rho f - \rho Ay + y)$ Clearly $S : K \to K$ as it is a projection onto K and K, being closed, is a complete metric space in the metric induced from H. Since P_K is nonexpansive, we have

$$\|Sy_1 - Sy_2\| \leq \|(y_1 - y_2) - \rho(Ay_1 - Ay_2)\|$$

and thus

$$\begin{aligned}\|Sy_1 - Sy_2\|^2 &= \|y_1 - y_2\|^2 - 2\rho(Ay_1 - Ay_2, y_1 - y_2) + \rho^2\|Ay_1 - Ay_2\|^2 \\ &\leq \|y_1 - y_2\|^2(1 - 2\rho\alpha + \rho^2C^2)\end{aligned}$$

We can choose ρ in such a way that $k^2 = 1 - 2\rho\alpha + \rho^2C^2 < 1$ we see that S has a unique fixed point in K.

Assume now that a is symmetric. Then $(x, y)_1 = a(x, y)$ defines a new scalar product which defines an equivalent norm $\|x\|_1 = \sqrt{a(x,x)}$ on H. Indeed, by continuity and coerciveness

$$\|x\|_1 = \sqrt{a(x,x)} \leq \sqrt{C}\|x\|$$

and

$$\|x\| = \sqrt{a(x,x)} \geq \sqrt{\alpha}\|x\|.$$

Using again Riesz theorem, we find $g \in H$ such that

$$< \phi, y >_{H^* \times H} = a(g, y)$$

and then (1.45) amounts to finding $x \in K$ such that

$$a(g - x, y - x) \leq 0$$

for all $y \in K$ but this is nothing else but finding projection x onto K with respect to the new scalar product. Thus, there is a unique $x \in K$

$$\sqrt{a(g-x, g-x)} = \min_{y \in K} \sqrt{a(g-x, g-x)}.$$

However, expanding, this is the same as finding minimum of the function

$$y \to a(g-y, g-y) = a(g,g) + a(y,y) - 2a(g,y) = a(y,y) - 2 < \phi, y >_{H^* \times H} + a(g,g).$$

Taking into account that $a(g, g)$ is a constant, we see that x is the unique minimizer of

$$y \to \frac{1}{2}a(y,y) - < \phi, y >_{H^* \times H} .$$

Corollary 1.43. *Assume that $a(\cdot, \cdot)$ is a continuous coercive bilinear form on a Hilbert space H. Then, given any $\phi \in H^*$, there exists a unique element $x \in H$ such that for any $y \in H$*

$$a(x, y) =< \phi, y >_{H^* \times H} \tag{1.52}$$

Moreover, if a is symmetric, then x is characterized by the property

$$x \in H \quad \text{and} \quad \frac{1}{2}a(x,x) - < \phi, x >_{H^* \times H} = \min_{y \in H} \frac{1}{2}a(y,y) - < \phi, y >_{H^* \times H} . \tag{1.53}$$

Proof. We use the Stampacchia theorem with $K = H$. Then there is a unique element $x \in H$ satisfying

$$a(x, y - x) \geq < \phi, y - x >_{H^* \times H} .$$

Using linearity, this must hold also for

$$a(x, ty - x) \geq < \phi, ty - x >_{H^* \times H} .$$

for any $t \in R, v \in H$. Factoring out t, we find

$$ta(x, y - xt^{-1}) \geq t < \phi, y - xt^{-1} >_{H^* \times H} .$$

and passing with $t \to \pm\infty$, we obtain

$$a(x, y) \geq < \phi, y >_{H^* \times H}, \qquad a(x, y) \leq < \phi, y >_{H^* \times H} .$$

Remark 1.44. Elementary proof of the Lax–Milgram theorem. As we noted earlier

$$a(x, y) = < \phi, y >_{H^* \times H}$$

can be written as the equation

$$(Ax, y) = (f, y)$$

for any $y \in H$, where $A : H \to H$, $\|Ax\| \leq C\|x\|$ and $(Ax, x) \geq \alpha\|x\|^2$. From the latter, $Ax = 0$ implies $x = 0$, hence A is injective. Further, if $y = Ax$, $x = A^{-1}y$ and

$$\|x\|^2 = \|A^{-1}y\|\|x\| \leq \alpha^{-1}(y, x) \leq \alpha^{-1}\|y\|\|x\|$$

so A^{-1} is bounded. This shows that the range of A, $R(A)$, is closed. Indeed, if $(y_n)_{n\in\mathbb{N}}$, $y_n \in R(A)$, $y_n \to y$, then $(y_n)_{n\in\mathbb{N}}$ is Cauchy, but then $(x_n)_{n\in\mathbb{N}}$, $x_n = A^{-1}$ is also Cauchy and thus converges to some $x \in A$. But then, from continuity of A, $Ax = y$. On the other hand, $R(A)$ is dense. For, if for some $v \in H$ we have $0 = (Ax, v)$ for any $x \in H$, we can take $v = x$ and

$$0 = (Av, v) \geq \alpha\|v\|^2$$

so $v = 0$ and so $R(A)$ is dense.

1.3.6 Dirchlet problem

Let us recall the variational formulation of the Dirichlet problem: find $u \in ?$ such that

$$\int\limits_{\Omega} \nabla u \cdot \nabla \phi dx = \int\limits_{\Omega} f \phi dx. \tag{1.54}$$

for all $C_0^\infty(\Omega)$. We also recall the associated minimization problem for

$$J(u) = \frac{1}{2}\int\limits_{\Omega} |\nabla u|^2 dx - \int\limits_{\Omega} f u dx \tag{1.55}$$

over some closed subspace $K = \{u \in ?\}$.

Let us consider the space $H = L_2(\Omega)$, $\Omega \subset \mathbb{R}^n$ bounded, with the scalar product

$$(u,v)_0 = \int_\Omega u(x)v(x)dx.$$

We know that $\overline{C_0^\infty(\Omega)}^H = H$. The relation (1.54) suggests that we should consider another scalar product, initially on $C_0^\infty(\Omega)$, given by

$$(u,v)_{0,1} = \int_\Omega \nabla u(x)\nabla v(x)dx.$$

Note that due to the fact that u, v have compact supports, this is a well defined scalar product as

$$0 = (u,u)_{0,1} = \int_\Omega |\nabla u(x)|^2 dx$$

implies $u_{x_i} = 0$ for all x_i, $i = 1, \ldots, n$ hence $u = const$ and thus $u \equiv 0$. Note that this is not a scalar product on a space $C^\infty(\bar{\Omega})$.

A fundamental role in the theory is played by the Zaremba - Poincarè-Friedrichs lemma.

Lemma 1.45. *There is a constant d such that for any $u \in C_0^\infty(\Omega)$*

$$\|u\|_0 \le d\|u\|_{0,1}. \tag{1.56}$$

Proof. Let R be a box $[a_1,b_1] \times \ldots \times [a_n,b_n]$ such that $\bar{\Omega} \subset R$ and extend u by zero to R. Since u vanishes at the boundary of R, for any $\mathbf{x} = (x_1, \ldots, x_n)$ we have

$$u(\mathbf{x}) = \int_{a_i}^{x_i} u_{x_i}(x_1, \ldots, t, \ldots)dt$$

and, by Schwarz inequality,

$$u^2(\mathbf{x}) = \left(\int_{a_i}^{x_i} u_{x_i}(x_1, \ldots, t, \ldots, x_n)dt\right)^2 \le \left(\int_{a_i}^{x_i} 1dt\right)\left(\int_{a_i}^{x_i} u_{x_i}^2(x_1, \ldots, t, \ldots, x_n)dt\right)$$

$$\le (b_i - a_i)\int_{a_i}^{b_i} u_{x_i}^2(x_1, \ldots, t, \ldots, x_n)dt$$

for any $\mathbf{x} \in R$. Integrating over R we obtain

$$\int_R u^2(\mathbf{x})d\mathbf{x} \le (b_i - a_i)^2 \int_R u_{x_i}^2(\mathbf{x})d\mathbf{x}.$$

This can be re-written

$$\int_{\Omega} u^2(\mathbf{x})d\mathbf{x} \leq (b_i - a_i)^2 \int_{\Omega} u^2_{x_i}(\mathbf{x})d\mathbf{x} \leq c \int_{\Omega} |\nabla u(\mathbf{x})|^2 d\mathbf{x}$$

We see that the lemma remains valid if Ω is bounded just in one direction. Let us define $\overset{\circ}{W}{}^1_2(\Omega)$ as the completion of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{0,1}$. We have

Theorem 1.46. *The space $\overset{\circ}{W}{}^1_2(\Omega)$ is a separable Hilbert space which can be identified with a subspace continuously and densely embedded in $L_2(\Omega)$. Every $v \in \overset{\circ}{W}{}^1_2(\Omega)$ has generalized derivatives $D_{x_i}v \in L_2(\Omega)$. Furthermore, the distributional integration by parts formula*

$$\int_{\Omega} D_{x_i} v u d\mathbf{x} = -\int_{\Omega} v D_{x_i} u d\mathbf{x} \tag{1.57}$$

is valid for any $u, v \in \overset{\circ}{W}{}^1_2(\Omega)$.

Proof. . The completion in the scalar product gives a Hilbert space. By Lemma 1.45, every equivalence class of the completion in the norm $\|\cdot\|_{0,1}$ is also an equivalence class in $\|\cdot\|_0$ and thus can be identified with the element of $\overline{C_0^\infty(\Omega)}^{\|\cdot\|_0}$ and thus with an element $v \in L_2(\Omega)$. This identification is one-to-one. Density follows from $C_0^\infty(\Omega) \subset \overset{\circ}{W}{}^1_2(\Omega) \subset L_2(\Omega)$ and continuity of injection from Lemma 1.45.

If $(v_n)_{n\in\mathbb{N}}$ of $C_0^\infty(\Omega)$ functions converges to $v \in \overset{\circ}{W}{}^1_2(\Omega)$ in $\|\cdot\|_{0,1}$, then $v_n \to v$ in $L_2(\Omega)$ and $D_{x_i}v_n \to v^i$ in $L_2(\Omega)$ for some functions $v^i \in L_2(\Omega)$. Taking arbitrary $\phi \in C_0^\infty(\Omega)$, we obtain

$$\int_{\Omega} D_{x_i} v_n \phi d\mathbf{x} = -\int_{\Omega} v_n D_{x_i} \phi d\mathbf{x}$$

and we can pass to the limit

$$\int_{\Omega} v^i \phi d\mathbf{x} = -\int_{\Omega} v D_{x_i} \phi d\mathbf{x}$$

showing that $v^i = D_{x_i}v$ in generalized sense. Furthermore, we can pass to the limit in $\|\cdot\|_{0,1}$ with $\phi \to u \in \overset{\circ}{W}{}^1_2(\Omega)$ and, by the above, $D_{x_i}\phi \to D_{x_i}u$ in $L_2(\Omega)$, giving (1.57). This also shows that $\overset{\circ}{W}{}^1_2(\Omega)$ can be identified with a closed subspace of $(L_2(\Omega))^n$ (the graph of gradient) and thus it is a separable space.

Consider now on $\overset{\circ}{W}{}_2^1(\Omega)$ the bilinear form

$$a(u,v) = \int_\Omega \nabla u \nabla v d\mathbf{x}.$$

Clearly, by Schwarz inequality

$$|a(u,v)| \leq \|u\|_{0,1}\|v\|_{0,1}$$

and

$$a(u,u) = \int_\Omega \nabla u \nabla u d\mathbf{x} = \|u\|_{0,1}^2$$

and thus a is a continuous and coercive bilinear form on $\overset{\circ}{W}{}_2^1(\Omega)$. Thus, if we take $f \in (\overset{\circ}{W}{}_2^1(\Omega))^* \supset L_2(\Omega)$ then there is a unique $u \in \overset{\circ}{W}{}_2^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \nabla v d\mathbf{x} = < f, v >_{(\overset{\circ}{W}{}_2^1(\Omega))^* \times \overset{\circ}{W}{}_2^1(\Omega)}$$

for any $v \in \overset{\circ}{W}{}_2^1(\Omega)$ or, equivalently, minimizing the functional

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 d\mathbf{x} - < f, v >_{(\overset{\circ}{W}{}_2^1(\Omega))^* \times \overset{\circ}{W}{}_2^1(\Omega)}$$

over $K = \overset{\circ}{W}{}_2^1(\Omega)$.

The question is what this solution represents. Clearly, taking $v \in C_0^\infty(\Omega)$ we obtain

$$-\Delta u = f$$

in the sense of distribution. However, to get a deeper understanding of the meaning of the solution, we have investigate the structure of $\overset{\circ}{W}{}_2^1(\Omega)$.

1.3.7 Sobolev spaces

Let Ω be a nonempty open subset of $\mathbb{R}^n$, $n \geq 1$ and let $m \in \mathbb{N}$. The Sobolev space $W_2^m(\Omega)$ consists of all $u \in L_2(\Omega)$ for which all generalized derivatives $D^\alpha u$ exist and belong to L_2. $W_2^m(\Omega)$ is equipped with the scalar product

$$(u,v)_m = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u D^\alpha v d\mathbf{x}. \tag{1.58}$$

In particular,

$$(u,v)_1 = \int_\Omega uv + \nabla u \nabla v d\mathbf{x}.$$

We obtain

Proposition 1.47. *The space $W_2^m(\Omega)$ is a separable Hilbert space.*

Proof. The proof follows since the generalized differentiation is a closed operator in $L_2(\Omega)$.

We note that $\overset{\circ}{W}{}_2^1(\Omega)$ is a closed subspace of $W_2^1(\Omega)$ as the norms $\|\cdot\|_{0,1}$ and $\|\cdot\|_1$ coincide there.

We shall focus on the case $m = 1$. A workhorse of the theory is the Friedrichs lemma.

Lemma 1.48. *Let $u \in W_2^1(\Omega)$. Then there exists a sequence $(u_k)_{k\in\mathbb{N}}$ from $C_0^\infty(\mathbb{R}^n)$ such that*

$$u_k|_\Omega \to u \quad \text{in} \quad L_2(\Omega) \tag{1.59}$$

and for any $\Omega' \Subset \Omega$

$$\nabla u_k|_{\Omega'} \to \nabla u \quad \text{in} \quad L_2(\Omega') \tag{1.60}$$

If $\Omega = \mathbb{R}^n$, then both convergences occur in $\mathbb{R}^n$.

Proof. Set

$$u^e(x) = \begin{cases} u(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \notin \Omega \end{cases}$$

and define $v_\epsilon = u^e * \omega_\epsilon$. We know $v_\epsilon \in C^\infty(\mathbb{R}^n)$ and $v_\epsilon \to u$ in $L_2(\Omega)$. Let us take $\Omega' \Subset \Omega$ and fix a function $\alpha \in C_0^\infty(\Omega)$ which equals 1 on a neighbourhood of Ω'. Then, for sufficiently small ϵ, we have

$$\omega_\epsilon * (\alpha u)^\epsilon = \omega_\epsilon * u^\epsilon$$

on Ω'. Then, by Proposition 1.8,

$$\partial_{x_j}(\omega_\epsilon * (\alpha u)^\epsilon) = \omega_\epsilon * \left(\alpha \partial_{x_j} u + \partial_{x_j} \alpha u\right)^e$$

hence

$$\partial_{x_j}(\omega_\epsilon * (\alpha u)^\epsilon) \to \left(\alpha \partial_{x_j} u + \partial_{x_j} \alpha u\right)^e$$

in $L_2(\Omega)$ and, in particular,

$$\partial_{x_j}(\omega_\epsilon * (\alpha u)^\epsilon) \to p_j u$$

in $L_2(\Omega')$. But on Ω' we can discard α to get

$$\partial_{x_j}(\omega_\epsilon * u^\epsilon) \to p_j u.$$

If v_k do not have compact support (e.g. when Ω is not bounded), then we multiply v_k by a sequence of smooth cut-off functions $\zeta_k = \zeta(x/k)$ where $\zeta(x) = 1$ for $|x| \le 1$ and $\zeta(x) = 0$ for $|x| \ge 2$.

As an immediate application we show

Proposition 1.49. *(i) Let $u, v \in W_2^1(\Omega) \cap L_\infty(\Omega)$. Then $uv \in W_2^1(\Omega) \cap L_\infty(\Omega)$ with*

$$\partial_{x_j}(uv) = \partial_j uv + u\partial_{x_j} v, \qquad i = 1, \ldots, n \tag{1.61}$$

(ii) Let Ω, Ω_1 be two open sets in $\mathbb{R}^n$ and let $H : \Omega_1 \to \Omega$ be a $C^1(\bar{\Omega})$ diffeomorphism. If $u \in W_2^1(\Omega)$ then $u \circ H \in W_2^1(\Omega')$ and

$$\int\limits_{\Omega_1} (u \circ H)\partial_{y_j} \phi d\mathbf{y} = -\int\limits_{\Omega_1} \sum_{i=1}^{n} (\partial_{x_i} u \circ H))\partial_{y_j} H_i \phi d\mathbf{y} \tag{1.62}$$

Proof. Using Friedrichs lemma, we find sequences $(u_k)_{k\in\mathbb{N}}$, $(v_k)_{k\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ such that

$$u_k \to u, \qquad v_k \to v$$

in $L_2(\Omega)$ and for any $\Omega' \Subset \Omega$ we have

$$\nabla u_k \to \nabla u, \qquad \nabla v_k \to \nabla v$$

in $L_2(\Omega')$. Moreover, from the construction of the mollifiers we get

$$\|u_k\|_{L_\infty(\Omega)} \leq \|u\|_{L_\infty(\Omega)} \qquad \|v_k\|_{L_\infty(\Omega)} \leq \|v\|_{L_\infty(\Omega)}.$$

On the other hand

$$\int\limits_{\Omega} u_k v_k \partial_{x_j} \phi d\mathbf{x} = -\int\limits_{\Omega} (\partial_j u_k v_k + u_k \partial_j v_k)\phi d\mathbf{x}$$

for any $\phi \in C_0^\infty(\Omega)$. Thanks to the compact support of ϕ, the integration actually occurs over compact subsets of Ω and we can use L_2 convergence of $\nabla u_k, \nabla v_k$. Thus

$$\int\limits_{\Omega} uv\partial_{x_j} \phi d\mathbf{x} = -\int\limits_{\Omega} (\partial_{x_j} uv + u\partial_{x_j} v)\phi d\mathbf{x}$$

and the fact that $uv \in W_2^1(\Omega)$ follows from $\partial_{x_j} u, \partial_{x_j} v \in W_2^1(\Omega)$ and $u, v \in L_\infty(\Omega)$. The proof of the second statement follows similarly. We select sequence $(u_k)_{k\in\mathbb{N}}$ as above; then clearly $u_k \circ H \to u \circ H$ in $L_2(\Omega_1)$ and

$$(\partial_{x_i} u_k \circ H)\partial_{y_j} H_i \to (\partial_{x_i} u \circ H)\partial_{y_j} H_i$$

in $L_2(\Omega_1')$ for any $\Omega_1' \Subset \Omega$. For any $\psi \in C_0^\infty(\Omega_1)$ we get

$$\int\limits_{\Omega_1} (u_k \circ H)\partial_{y_j} \phi d\mathbf{y} = -\int\limits_{\Omega_1} \sum_{i=1}^{k} (\partial_{x_i} u_k \circ H)\partial_{y_j} H_i \phi d\mathbf{y}$$

and in the limit we obtain (1.62).

Sometimes it will be necessary to indicate the domain of the definition of a Sobolev space. Then we use the $\|\cdot\|_{0,\Omega}$ to denote the norm in $L_2(\Omega)$ and analogous convention is used for the Sobolev space norms.

Proposition 1.50. *The following properties are equivalent:*

(i) $u \in W_2^1(\Omega)$,
(ii) there is C *such that for any* $\phi \in C_0^\infty(\Omega)$ *and* $i = 1, \ldots, n$

$$\left| \int_\Omega u \partial_i \phi d\mathbf{x} \right| \leq C\|\phi\|_0, \tag{1.63}$$

(iii) there is a constant C *such that for any* $\Omega' \Subset \Omega$ *and all* $\mathbf{h} \in \mathbb{R}^n$ *with* $|\mathbf{h}| \leq \operatorname{dist}(\Omega', \partial\Omega)$ *we have*

$$\|\tau_h u - u\|_{0,\Omega'} \leq C|\mathbf{h}|, \tag{1.64}$$

where $(\tau_h u)(\mathbf{x}) = u(\mathbf{x} + \mathbf{h})$*. In particular, if* $\Omega = \mathbb{R}^n$*, then*

$$\|\tau_h u - u\|_0 \leq |\mathbf{h}| \|\nabla u\|_0. \tag{1.65}$$

Proof. $(i) \Rightarrow (ii)$ follows from the definition.
$(ii) \Rightarrow (i)$. Eqn. (1.63) shows that

$$\phi \to \int_\Omega u \partial_i \phi d\mathbf{x},$$

extends to a bounded functional on $L_2(\Omega)$ and thus there is $v_i \in L_2(\Omega)$ such that

$$\int_\Omega u \partial_i \phi d\mathbf{x} = -\int_\Omega v_i \phi d\mathbf{x},$$

for any $\phi \in C_0^\infty(\Omega)$.
$(i) \Rightarrow (iii)$. Let us take $u \in C_0^\infty(\mathbb{R}^n)$. For $\mathbf{x}, \mathbf{h} \in \mathbb{R}^n$ and $t \in \mathbb{R}$ we define

$$v(t) = u(\mathbf{x} + t\mathbf{h}).$$

Then $v'(t) = \mathbf{h}\nabla u(\mathbf{x} + t\mathbf{h})$ and

$$u(\mathbf{x} + \mathbf{h}) - u(\mathbf{x}) = v(1) - v(0) = \int_0^1 \mathbf{h}\nabla u(\mathbf{x} + t\mathbf{h}) dt.$$

Hence

$$|\tau_h u(\mathbf{x}) - u(\mathbf{x})|^2 \leq |\mathbf{h}|^2 \int_0^1 |\nabla u(\mathbf{x} + t\mathbf{h})|^2 dt$$

so that

$$\int_{\Omega'} |\tau_h u(\mathbf{x}) - u(\mathbf{x})|^2 d\mathbf{x} \leq |\mathbf{h}|^2 \int_0^1 \left(\int_{\Omega'} |\nabla u(x+th)|^2 d\mathbf{x} \right) dt$$
$$= |\mathbf{h}|^2 \int_0^1 \left(\int_{\Omega'+t\mathbf{h}} |\nabla u(\mathbf{y})|^2 d\mathbf{x} \right) dt.$$

If $|\mathbf{h}| < \mathrm{dist}(\Omega', \partial\Omega)$, then there is Ω'' such that $\Omega' + t\mathbf{h} \subset \Omega'' \Subset \Omega$ for all $t \in [0,1]$ and thus

$$\int_{\Omega'} |\tau_h u(\mathbf{x}) - u(\mathbf{x})|^2 d\mathbf{x} \leq |\mathbf{h}|^2 \int_{\Omega''} |\nabla u(\mathbf{y})|^2 d\mathbf{x}$$

which gives (1.64) for $u \in C_0^\infty(\mathbb{R}^n)$. Let $u \in W_2^1(\Omega)$. Then, by the Friedrichs lemma, we find $(u_k)_{k\in\mathbb{N}}$, $u_k \in C_0^\infty(\mathbb{R}^n)$ such that $u_k \to u$ in $L_2(\Omega)$ and $\nabla u_k \to \nabla u$ in $L_2(\Omega')$ for any $\Omega' \Subset \Omega$. Noting that $\tau_h u_k \to \tau_h u$ in $L_2(\Omega')$ we can pass to the limit above, obtaining,

$$\|\tau_h u - u\|_{0,\Omega'} \leq |\mathbf{h}| \sqrt{\int_{\Omega''} |\nabla u(\mathbf{y})|^2 d\mathbf{x}} \leq C|\mathbf{h}|.$$

If $\Omega = \mathbb{R}^n$, then in all calculations above we can replace Ω', Ω'' by $\mathbb{R}^n$.

(iii) $\Rightarrow$ *(ii)*. If (1.64) holds then, taking $\Omega' \Subset \Omega$, $\phi \in C_0^\infty(\Omega)$ with $\mathrm{supp}\phi \subset \Omega'$ and $|\mathbf{h}| < \mathrm{dist}(\Omega', \partial\Omega)$, we obtain

$$\left| \int_{\Omega} (\tau_h u - u)\phi d\mathbf{x} \right| \leq C|\mathbf{h}|\|\phi\|_0.$$

On the other hand

$$\int_{\Omega} (\tau_h u - u)(\mathbf{x})\phi(\mathbf{x}) d\mathbf{x} = \int_{\Omega} u(\mathbf{y})(\tau_{-h}\phi - \phi)(\mathbf{y}) d\mathbf{y},$$

so

$$\int_{\Omega} u \frac{(\tau_{-h}\phi - \phi)}{|\mathbf{h}|} d\mathbf{y} \leq C\|\phi\|_0.$$

Choosing $\mathbf{h} = t\mathbf{e}_i, i = 1, \ldots, n$ and passing to the limit with $t \to 0$, we obtain (1.63).

1.3.8 Localization and flattening of the boundary

Assume that Ω is an open, bounded set with boundary $\partial\Omega$ which is an $n-1$ dimensional C^m manifold; further assume that that Ω lies locally at one side of the boundary. Denote $Q = \{\mathbf{y} \in \mathbb{R}^n;\ |y_i| < 1, i = 1, \ldots, n\}$, $Q_0 = \{\mathbf{y} \in Q;\ y_n = 0\}$ and $Q_+ = \{\mathbf{y} \in Q;\ x_n > 0\}$. Then we have a finite local atlas on $\partial\Omega$, that is, a finite collection $\{B_j, H^j j\}_{1\leq j\leq N}$ where B_j are open sets covering $\partial\Omega$, $H^j : Q \to B_j$ are C^m diffeomorphisms with positive Jacobians which are bijections of Q, Q_0 and Q_+ onto $B_j, B_j \cap \partial\Omega$ and $B_j \cap \Omega$, respectively.

Given the local atlas $\{B_j, H^j\}_{1\leq j\leq N}$, we construct a finite open subcover $\{G_j\}_{1\leq j\leq N}$ in such a way that $G_j \Subset B_j$ and $\partial\Omega \subset \bigcup_{j=1}^N G_j$. In fact, we can take $G_j = B_j^k$ where $B_j^k = \{\mathbf{x} \in B_j;\ \mathrm{dist}(\mathbf{x}, \partial\Omega > 1/k\}$ for some k. Indeed, suppose it is impossible, then for any k there is $\mathbf{x}_k \in \partial\Omega$ such that $\mathbf{x}_k \notin \bigcup_{j=1}^N B_j^k$. From compactness of $\partial\Omega$ we obtain an accumulation point $\mathbf{x} \in \partial\Omega$. Hence $\mathbf{x} \in B_j$ for some j and thus $x \in B_j^k$ for sufficiently large k. This contradicts the construction that x is an accumulation point of points which are outside $\bigcup_{j=1}^N B_j^k$. Defining $G_0 = \Omega \setminus \bigcup_{j=1}^N \bar{G}_j$ we further get an open set G_0 with $\bar{G}_0 \subset \Omega$. Thus

$$\bar{\Omega} \subset \Omega \cup \bigcup_{j=1}^N G_j, \qquad \Omega \subset \bigcup_{j=0}^N \bar{G}_j.$$

Now, we choose $\alpha_j \in C_0^\infty(\mathbb{R}^n)$ satisfying $0 \leq \alpha \leq 1$, $\mathrm{supp}\alpha_j \subset B_j$ and $\alpha_j = 1$ on $\bar{G}_j$. Further, $\alpha \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$\mathrm{supp}\alpha \subset \Omega \cup \bigcup_{j=1}^N G_j, \quad 0 \leq \alpha \leq 1, \quad \alpha = 1 \text{ on } \bar{\Omega}.$$

Then define

$$\beta_j(\mathbf{x}) = \frac{\alpha(\mathbf{x})\alpha_j(\mathbf{x})}{\sum\limits_{k=0}^N \alpha_k(\mathbf{x})}$$

for $\mathbf{x} \in \bigcup_{j=0}^N \bar{G}_j$ and $\beta_j(\mathbf{x}) = 0$ for $\mathbf{x} \in \mathbb{R}^n \setminus \bigcup_{j=0}^N \bar{G}_j$. We note that each β_j is well defined. Indeed, at least one $\alpha_j(\mathbf{x})$ is equal 1 on $\bigcup_{j=0}^N \bar{G}_j$ so that the denominator is at least 1 there. On the other hand, α vanishes outside a compact set contained in $\bigcup_{j=0}^N G_j$. Hence, $\beta_j \in C_0^\infty(\mathbb{R}^n)$, $\mathrm{supp}\beta_j \subset B_j$, $\beta_j \geq 0$ and

$$\sum_{j=0}^N \beta_j(\mathbf{x}) = 1$$

for $\mathbf{x} \in \bar{\Omega}$.

We call the collection $\{\beta_j\}_{j=0}^N$ a partition of unity subordinated to the open cover $\{G_j\}_{j=0}^N$ of Ω and $\{\beta_j\}_{j=1}^N$ a partition of unity subordinated to the open cover $\{G_j\}_{j=1}^N$ of Ω of $\partial\Omega$.

Suppose now we have $u \in W_2^1(\Omega)$. Then $u = \sum_{j=0}^{N} \beta_j u$ on Ω and, by Proposition 1.49 (i), $\beta_j u \in W_2^1(\Omega \cap G_j)$, $j = 1, \ldots, N$. Using Proposition 1.49 (ii) we see that for each $j = 1, \ldots, N$ we $(\beta_j u) \circ H_j \in W_2^1(Q_+)$ with support in Q. Define $\Lambda : W_2^1(\Omega) \to \overset{\circ}{W}{}_2^1(Q) \times [W_2^1(\Omega)]^N$ by

$$\Lambda u = (\beta_0 u, \beta_1 u \circ H^1, \ldots, \beta_N u \circ H^N).$$

Note that we can write $\beta_0 u \in \overset{\circ}{W}{}_2^1(Q)$ as $\beta_0 u$ has compact support in Ω and thus, by Friedrichs lemma, it can be approximated by $C_0^\infty(Q)$ functions. The mapping Λ is a linear injection as if $u(x) \neq 0$, then at least one entry of Λ must be nonzero as βs sum up to 1. Also, using Proposition 1.49, we can show that the norm on $\Lambda W_2^1(\Omega)$ is equivalent to the norm on $W_2^1(\Omega)$ and thus Λ is an isomorphism of $W_2^1(\Omega)$ onto its closed image.

1.3.9 Extension operator

We observed that one of the main obstacles in proving that $W_2^1(\Omega)$ can be obtained by closure of restrictions of $C_0^\infty(\mathbb{R}^n)$ functions to Ω is that we have no control over the regularization at points close to the boundary of Ω. A remedy could be if we are able to show that any function $W_2^1(\Omega)$ can be extended to a function from $W_2^1(\Omega)$.

Indeed, we have

Theorem 1.51. *Suppose that Ω is bounded with a C^1 boundary $\partial\Omega$. Then there exists a linear extension operator*

$$E : W_2^1(\Omega) \to W_2^1(\mathbb{R}^n)$$

such that for any $u \in W_2^1(\Omega)$

1. *$Eu|_\Omega = u$;*
2. *$\|Eu\|_{0,\mathbb{R}^n} \leq C\|u\|_{0,\Omega}$;*
3. *$\|Eu\|_{1,\mathbb{R}^n} \leq C\|u\|_{1,\Omega}$;*

Proof. We begin by showing that we can construct an extension operator from $W_2^1(Q_+)$ to $W_2^1(Q)$. Let $u \in W_2^1(Q_+)$ and define extension by reflection

$$u^*(\mathbf{x}', x_n) = \begin{cases} u(\mathbf{x}', x_n) & \text{for } x_n > 0, \\ u(\mathbf{x}', -x_n) & \text{for } x_n < 0 \end{cases}$$

where $\mathbf{x}' = (x_1, \ldots, x_{n-1})$. In the same way, we define the odd reflection

$$u^\bullet(\mathbf{x}', x_n) = \begin{cases} u(\mathbf{x}', x_n) & \text{for } x_n > 0, \\ -u(\mathbf{x}', -x_n) & \text{for } x_n < 0 \end{cases}$$

Further, we define a cut-off function close to $x_n = 0$, that is, we take a $C^\infty(\mathbb{R})$ function η which satisfies $\eta(t) = 1$ for $t \geq 1$ and $\eta(t) = 0$ for $t \leq 1/2$ and define $\eta_k(x_n) = \eta(kx_n)$. Let us take $\phi \in C_0^\infty(Q)$ and consider, for $1 \leq i \leq n-1$,

$$\int_Q u^* \partial_{x_i} \phi d\mathbf{x} = \int_{Q_+} u \partial_{x_i} \psi d\mathbf{x}$$

where $\psi(\mathbf{x}', x_n) = \phi(\mathbf{x}', x_n) + \phi(\mathbf{x}', -x_n)$. Typically, ψ is not zero at Q_0 and cannot be used as a test function. However, $\eta_k(x_n)\psi(\mathbf{x}) \in C_0^\infty(Q_+)$ and we can write

$$\int_{Q_+} u \partial_{x_i}(\eta_k \psi) d\mathbf{x} = \int_{Q_+} (\partial_{x_i} u) \eta_k \psi d\mathbf{x}.$$

However, $\partial_{x_i}(\eta_k \psi) = \eta_k \partial_{x_i} \psi$ as η does not depend on x_i, $i = 1, \ldots, n-1$ and hence

$$\int_{Q_+} \eta_k u \partial_{x_i} \psi d\mathbf{x} = -\int_{Q_+} (\partial_{x_i} u) \eta_k \psi d\mathbf{x}.$$

We can pass to the limit by dominated convergence getting

$$\int_{Q_+} u \partial_{x_i} \psi d\mathbf{x} = -\int_{Q_+} (\partial_{x_i} u) \psi d\mathbf{x},$$

so that, returning to Q

$$\int_Q u^* \partial_{x_i} \phi d\mathbf{x} = -\int_{Q_+} (\partial_{x_i} u) \psi d\mathbf{x} = -\int_Q (\partial_{x_i} u)^* \phi d\mathbf{x}.$$

Now let us consider differentiability with respect to x_n. Again, taking $\phi \in C_0^\infty(Q)$

$$\int_Q u^* \partial_{x_n} \phi d\mathbf{x} = \int_{Q_+} u \partial_{x_n} \chi d\mathbf{x}$$

where $\chi(\mathbf{x}', x_n) = \phi(\mathbf{x}', x_n) - \phi(\mathbf{x}', -x_n)$. If we again use η_k, then

$$\partial_{x_n}(\eta_k \chi) = \eta_k \partial_{x_n} \chi + \chi \partial_{x_n} \eta_k$$

where $\partial_{x_n} \eta_k(x_n) = k\eta'(kx_n)$. Then

$$\begin{aligned} k\left|\int_{Q_+} u(\mathbf{x}) \eta'(kx_n) \chi(\mathbf{x}) dx\right| &\le kCM \int_{Q_0} \left(\int_0^{1/k} |u(\mathbf{x})| x_n dx_n \right) d\mathbf{x}' \\ &\le CM \int_{Q_+} |u(\mathbf{x})| d\mathbf{x} \to 0 \end{aligned}$$

as $k \to \infty$, where $C = \sup_{t\in[0,1]} |\eta'(t)|$ and M is obtained from the estimate

$$|\chi(\mathbf{x}', x_n) \leq M|x_n|$$

on Q. Thus

$$\int\limits_{Q_+} u\partial_{x_n}\eta_k\chi d\mathbf{x} = \int\limits_{Q_+} u(\eta_k\partial_{x_n}\chi + \chi\partial_{x_n}\eta_k)d\mathbf{x} \to \int\limits_{Q_+} u\eta_k\partial_{x_n}\chi$$

and thus we obtain in the limit

$$\int\limits_{Q_+} u\partial_{x_n}\chi d\mathbf{x} = -\int\limits_{Q_+} (\partial_{x_n})u\chi d\mathbf{x}.$$

Returning to Q, we obtain

$$\int\limits_{Q} u^*\partial_{x_n}\phi d\mathbf{x} = \int\limits_{Q_+} u\partial_{x_n}\chi d\mathbf{x} = \int\limits_{Q} (\partial_{x_n}u)^\bullet\phi d\mathbf{x}.$$

We also obtain estimates

$$\|u^*\|_{0,Q} \leq 2\|u\|_{0,Q_+} \qquad \|u^*\|_{1,Q} \leq 2\|u\|_{1,Q_+}.$$

Now we can pass to the general result. Let $u \in W_2^1(\Omega)$, Ω bounded with C^1 boundary. Let $\{B_j, H^j\}_{j=1}^N$ be the atlas on the boundary and $\{G_j\}_{j=1}^N$ be the finite subcover constructed in the previous section, that is $G_0 \subset \bar{G}_0 \subset \Omega$, $\bar{G}_j \subset B_j$ with $\partial\Omega \subset \bigcup G_j$ and let $\{\beta\}_{j=1}^N$ be a subordinate partition of unity. Then we take

$$u = \sum_{j=0}^N \beta_j u = \sum_{j=0}^N u_j$$

with $u_0 \in \overset{\circ}{W}{}_2^1(\Omega)$ and $u_j \in W_2^1(\Omega \cap B_j)$. Clearly, $\|u_0\|_{1,\Omega} \leq C_0\|u\|_{1,\Omega}$ and $\|u_j\|_{1,\Omega\cap B_j} \leq C_j\|u\|_{1,\Omega}$, $j = 1, \ldots, n$. The function u_0 can be extended to $\hat{u}_0 \in W_2^1(\mathbb{R}^n)$ by zero in a continuous way. Then $v_j := u_j \circ H^j \in W_2^1(Q_+)$ and we can extend by reflection to $v_j^* \in W_2^1(Q)$. We note that v_j^* has support in Q since the support of u_j only can touch $\partial(B_j \cap \Omega$ at the points of $\partial\Omega$. Again,

$$\|v_j^*\|_{1,Q} \leq 2\|v_j\|_{1,Q_+} \leq C_j''\|u_j\|_{1,\Omega\cap B_j} \leq C_j'\|u\|_{1,\Omega}.$$

Next, we define $w_j = v_j^* \circ (H^j)^{-1} \in W_2^1(B_j)$, again with $\|w_j\|_{1,B_j} \leq C_j''\|u\|_{1,\Omega}$. Moreover, we have $w_j(\mathbf{x}) = u_j(\mathbf{x})$ whenever $\mathbf{x} \in B_j \cap \bar{\Omega}$ as

$$v_j^*((H^j)^{-1}(\mathbf{x})) = v_j((H^j)^{-1}(\mathbf{x})) = u_j(H^j((H^j)^{-1}(\mathbf{x}))) = u_j(\mathbf{x})$$

for such $\mathbf{x}$. We also notice that for each $j = 1, \ldots, N$, support of w_j is contained in B_j and thus can extend w_j by zero to $\mathbb{R}^n$ continuously in $W_2^1(\mathbb{R}^n)$

and denote this extension by $\hat{u}_j$. We note that $\hat{u}_j(\mathbf{x}) = u_j(\mathbf{x})$ for $\mathbf{x} \in \bar{\Omega}$. Indeed, if $\mathbf{x} \in \bar{\Omega}$, for a given j either $\mathbf{x} \in B_j \cap \bar{\Omega}$ and then $\hat{u}_j(\mathbf{x}) = w_j(\mathbf{x}) = u_j(\mathbf{x})$ or $\mathbf{x} \notin B_j \cap \bar{\Omega}$ in which case $\hat{u}_j(\mathbf{x}) = 0$ but then also $u_j(\mathbf{x}) = 0$ by definition. The same argument applies to $j = 0$. Now we define the operator

$$Eu = \hat{u}_0 + \sum_{j=1}^{n} \hat{u}_j$$

and we clearly have

$$Eu(\mathbf{x}) = \hat{u}_0(\mathbf{x}) + \sum_{j=1}^{n} \hat{u}_j(\mathbf{x}) = u_0(\mathbf{x}) + \sum_{j=1}^{n} u_j(\mathbf{x}) = u(\mathbf{x}).$$

Linearity and continuity follows from continuity and linearity of each operation and the fact that the sum is finite.

Remark 1.52. Similar argument allows to prove that there is an extension from $W_2^m(\Omega)$ to $W_2^m(\mathbb{R}^n)$ (as well as for $W_p^m(\Omega)$, $1 \leq p \leq \infty$) but this requires the boundary to be a C^m-manifold (so that the flattening preserves the differentiability). However, the extension across the hyperplane $x_n = 0$ is done according to the following reflection

$$u^*(\mathbf{x}', x_n) = \begin{cases} u(\mathbf{x}', x_n) & \text{for } x_n > 0 \\ \lambda_1 u(\mathbf{x}', -x_n) + \lambda_2 u\left(\mathbf{x}', -\frac{x_n}{2}\right) + \ldots + \lambda_m u\left(\mathbf{x}', -\frac{x_n}{m}\right) & \text{for } x_n < 0, \end{cases}$$

where $\lambda_1, \ldots, \lambda_m$ is the solution of the system

$$\begin{aligned} \lambda_1 + \lambda_2 + \ldots + \lambda_m &= 1, \\ -(\lambda_1 + \lambda_2/2 + \ldots + \lambda_m/m) &= 1, \\ &\cdots \\ (-1)^m(\lambda_1 + \lambda_2/2^{m-1} + \ldots + \lambda_m/m^{m-1}) &= 1 \end{aligned}$$

These conditions ensure that the derivatives in the x_n direction are continuous across $x_n = 0$.

An immediate consequence of the extension theorem is

Theorem 1.53. *Let Ω be a bounded set with a C^1 boundary $\partial\Omega$ and $u \in W_2^1(\Omega)$. Then there exits $(u_n)_{n\in\mathbb{N}}$, $u_n \in C_0^\infty(\mathbb{R}^n)$ such that*

$$\lim_{n\to\infty} u_n|_\Omega = u, \qquad \text{in } W_2^1(\Omega).$$

In other words, the set of restriction to Ω of functions from $C_0^\infty(\Omega)$ is dense in $W_2^1(\Omega)$.

Proof. If Ω is bounded then, using Theorem 1.51, we can extend u to a function $Eu \in W_2^1(\mathbb{R}^n)$ with bounded support. The existence of a $C_0^\infty(\mathbb{R}^n)$ sequence converging to u follows from the Friedrichs lemma. If Ω is unbounded (but not equal to $\mathbb{R}^n$), then first we approximate u by a sequence $(\chi_n u)_{n\in\mathbb{N}}$ where χ_n are cut-off functions. Next we construct an extension of $\chi_n u$ to $\mathbb{R}^n$. This is possible as it involves only the part of $\partial\Omega$ intersecting the ball $B(0, 2n+1)$ and χ_n is equal to zero where the sphere intersects $\partial\Omega$. For this extension we pick up an approximating function from $C_0^\infty(\mathbb{R}^n)$.

1.4 Basic applications of the density theorem

1.4.1 Sobolev embedding

In Subsection 1.1.2 we have seen that in one dimension it is possible to identify a $W_2^1(\mathbb{R})$ function. Unfortunately, this is not true in higher dimensions.

Example 1.54. We can consider in $D = \{(x,y) \in \mathbb{R}^2;\ x^2+y^2 < 1\}$

$$u(x,y) = \left|\frac{1}{2}\ln(x^2+y^2)\right|^{1/3} = (-\ln r)^{1/3}.$$

The function u is not continuous (even not bounded) at $(x,y) = (0,0)$. It is in $L_2(D)$ and for derivatives we have

$$u_x = -\frac{1}{3}(-\ln r)^{-2/3}\frac{x}{r^2}, \qquad u_y = -\frac{1}{3}(-\ln r)^{-2/3}\frac{y}{r^2}$$

and, since

$$\int\limits_D (u_x^2 + u_y^2)dxdy = \frac{2}{9}\int\limits_0^1 \frac{dr}{r(-\ln r)^{4/3}} = \frac{2}{9}\int\limits_1^\infty u^{-4/3}du < \infty$$

we see that $u \in W_2^1(D)$.

However, there is still a link between Sobolev spaces and classical calculus provided we take sufficiently high order of derivatives (or index p in L_p spaces). The link is provided by the Sobolev lemma.

Let Ω be an open and bounded subset of $\mathbb{R}^n$. We say that Ω satisfies the cone condition if there are numbers $\rho > 0$ and $\gamma > 0$ such that each $\mathbf{x} \in \Omega$ is a vertex of a cone $K(\mathbf{x})$ of radius ρ and volume $\gamma\rho^n$. Precisely speaking, if σ_n is the $n-1$ dimensional measure of the unit sphere in $\mathbb{R}^n$, then the volume of a ball of radius ρ is $\sigma_n\rho^n/n$ and then the (solid) angle of the cone is $\gamma n/\omega_n$.

Lemma 1.55. *If Ω satisfies the cone condition, then there exists a constant C such that for any $u \in C^m(\bar{\Omega})$ with $2m > n$ we have*

$$\sup_{\mathbf{x}\in\Omega} |u(\mathbf{x})| \leq C\|u\|_m \tag{1.66}$$

Proof. Let us introduce a cut-off function $\phi \in C_0^\infty(\mathbb{R})$ which satisfies $\phi(t) = 1$ for $|t| \leq 1/2$ and $\phi(t) = 0$ for $|t| \geq 1$. Define $\tau(t) = \phi(t/\rho)$ and note that there are constants A_k, $k = 1, 2, \ldots$ such that

$$\left|\frac{d^k\tau(t)}{dt^k}\right| \leq \frac{A_k}{\rho^k}. \tag{1.67}$$

Let us take $u \in C^m(\bar{\Omega})$ and assume $2m > n$. For $\mathbf{x} \in \bar{\Omega}$ and the cone $K(\mathbf{x})$ we integrate along the ray $\{\mathbf{x} + r\boldsymbol{\omega};\ 0 \leq r \leq \rho, |\boldsymbol{\omega}| = 1$

$$u(\mathbf{x}) = -\int\limits_0^\rho D_r(\tau(r)u(\mathbf{x} + r\boldsymbol{\omega}))dr.$$

Integrating over the surface Γ of the cone we get

$$\int\limits_\Gamma \int\limits_0^\rho D_r(\tau(r)u(\mathbf{x} + \boldsymbol{\omega}))drd\boldsymbol{\omega} = -u(\mathbf{x}) \int\limits_C d\boldsymbol{\omega} = -u(\mathbf{x})\frac{\gamma n}{\omega_n}.$$

Next we integrate $m - 1$ times by parts, getting

$$u(\mathbf{x}) = \frac{(-1)^m\omega_n}{\gamma n(m-1)!} \int\limits_C \int\limits_0^\rho D_r^m(\tau(r)u(\mathbf{x} + r\boldsymbol{\omega}))r^{m-1}drd\boldsymbol{\omega}.$$

and changing to Cartesian coordinates and applying Cauchy-Schwarz inequality we obtain

$$\begin{aligned} |u(\mathbf{x})|^2 &\leq \left(\frac{\omega_n}{\gamma n(m-1)!} \int\limits_{K(\mathbf{x})} |D_r^m(\tau u)|r^{m-n}d\mathbf{y}\right)^2 \\ &\leq \left(\frac{\omega_n}{\gamma n(m-1)!}\right)^2 \int\limits_{K(\mathbf{x})} |D_r^m(\tau u)|^2 d\mathbf{y}d\mathbf{y} \int\limits_{K(\mathbf{x})} r^{2(m-n)}d\mathbf{y}. \end{aligned}$$

The last term can be evaluated as

$$\int\limits_{K(\mathbf{x})} r^{2(m-n)}d\mathbf{y} = \int\limits_C \int\limits_0^\rho r^{2m-n-1}drd\boldsymbol{\omega} = \frac{\gamma n\rho^{2m-n}}{\omega_n(2m-n)}$$

so that

$$|u(\mathbf{x})|^2 \leq C(m,n)\rho^{2m-n} \int\limits_{K(\mathbf{x})} |D_r^m(\tau u)|^2 d\mathbf{y}. \tag{1.68}$$

Let us estimate the derivative. From (1.67) we obtain by the chain rule and the Leibniz formula

$$|D_r^m(\tau u)| = \left|\sum_{k=0}^{m}\binom{n}{k}D_r^{m-k}\tau D_r^k u\right| \le \sum_{k=0}^{m}\binom{n}{k}\frac{A_{m-k}}{\rho^{m-k}}\left|D_r^k u\right|,$$

hence

$$|D_r^m(\tau u)|^2 \le C'\sum_{k=0}^{m}\frac{1}{\rho^{2(m-k)}}\left|D_r^k u\right|^2$$

for some constant C'. With this estimate we can re-write (1.68) as

$$|u(\mathbf{x})|^2 \le C(m,n)C'\sum_{k=0}^{m}\rho^{2k-n}\int\limits_{K(\mathbf{x})}|D_r^m(u)|^2 d\mathbf{y}. \tag{1.69}$$

Since by the chain rule

$$|D_r^m u|^2 \le C''\sum_{|\alpha|\le k}|D^\alpha u|^2$$

by extending the integral to Ω we obtain

$$\sup_{\mathbf{x}\in\Omega}|u(\mathbf{x})| \le C\|u\|_m$$

which is (1.66).

Theorem 1.56. *Assume that Ω is a bounded open set with C^m boundary and let $m > k + n/2$ where m and k are integers. Then the embedding*

$$W_2^m(\Omega) \subset C^k(\bar{\Omega})$$

is continuous.

Proof. Under the assumptions, the problem can be reduced to the set $G_0 \Subset \Omega$ consisting of internal point, separated from the boundary by a fixed positive distance, and points in the boundary strip, covered by sets $\bar{\Omega}\cap B_j$ which are transformed onto $Q_+ \cup Q_0$. Any point in G_0 satisfies the cone conditions. Points on $Q_0 \cup Q_+$ also satisfy the condition so, if $u \in W_2^m(\Omega)$, then extending the boundary components of Λu to Q we obtain functions in $W_2^1(\Omega)$ and $W_2^1(Q)$ with compact supports in respective domains. By Friedrichs lemma, restrictions to Ω and Q of $C^\infty(\mathbb{R}^n)$ functions are dense in, respectively, $W_2^m(\Omega)$ and $W_2^m(Q)$ and therefore the estimate (1.66) can be extended by density to $W_2^m(\Omega)$ showing that the canonical injection into $C(\bar{\Omega})$ is continuous. To obtain the result for higher derivatives we substitute higher derivatives of u for u in (1.66). Thus, all components of Λu are they are C^k functions. Transferring them back, we see that $u \in C^k(\bar{\Omega})$, by regularity of the local atlas and $m > k$, we obtain the thesis.

1.4.2 Compact embedding and Rellich–Kondraschov theorem

Lemma 1.57. *let $Q = \{\mathbf{x};\ a_j \le x_j \le b_j\}$ be a cube in $\mathbb{R}^n$ with edges of length $d > 0$. If $u \in C^1(\bar{Q})$, then*

$$\|u\|_{0,Q}^2 \le d^{-n} \left(\int\limits_Q u d\mathbf{x} \right)^2 + \frac{nd^2}{2} \sum_{j=1}^n \|\partial_{x_j} u\|_{0,Q}^2 \tag{1.70}$$

Proof. For any $\mathbf{x}, \mathbf{y} \in Q$ we can write

$$u(\mathbf{x}) - u(\mathbf{y}) = \sum_{j=1}^n \int\limits_{y_j}^{x_j} \partial_{x_j} u(y_1, \ldots, y_{j-1}, s, x_{j+1}, \ldots, x_n) ds.$$

Squaring this identity and using Cauchy-Schwarz inequality we obtain

$$u^2(\mathbf{x}) + u^2(\mathbf{y}) - 2u(\mathbf{x})u(\mathbf{y}) \le nd \sum_{j=1}^n \int\limits_{a_j}^{b_j} (\partial_j u)^2 (y_1, \ldots, y_{j-1}, s, x_{j+1}, \ldots, x_n) ds.$$

Integrating the above inequality with respect to all variables, we obtain

$$2d^n \|u\|_{0,Q}^2 \le 2 \left(\int\limits_Q u d\mathbf{x} \right)^2 + nd^{n+2} \sum_{j=1}^n \|\partial_j u\|_{0,Q}^2$$

as required.

Theorem 1.58. *Let Ω be open and bounded. If the sequence $(u_k)_{k\in\mathbb{N}}$ of elements of $\overset{\circ}{W}{}_2^1(\Omega)$ is bounded, then there is a subsequence which converges in in $L_2(\Omega)$. In other words, the injection $\overset{\circ}{W}{}_2^1(\Omega) \subset L_2(\Omega)$ is compact.*

Proof. By density, we may assume $u_k \in C_0^\infty$. Let $M = \sup_k\{\|u_k\|_1\}$. We enclose Ω in a cube Q; we may assume the edges of Q to be of unit length. Further, we extend each u_k by zero to $Q \setminus \Omega$.

We decompose Q into N^n cubes of edges of length $1/N$. Since clearly $(u_k)_{k\in\mathbb{N}}$ is bounded in $L_2(Q)$ it contains a weakly convergent subsequence (which we denote again by $(u_k)_{k\in\mathbb{N}}$). For any ϵ' there is n_0 such that

$$\left| \int\limits_{Q_j} (u_k - u_l) d\mathbf{x} \right| < \epsilon', \qquad k, l \ge n_0 \tag{1.71}$$

for each $j = 1, \ldots, N^n$. Now, we apply (2.36) on each Q_j and sum over all j getting

$$\|u_k - u_l\|_{0,Q}^2 \leq N^n \epsilon' + \frac{n}{2N^2} 2M^2.$$

Now, we see that for a fixed ϵ we can find N large that $nM^2/N^2 < e$ and, having fixed N, for $\epsilon' = \epsilon/2N^n$ we can find n_0 such that (1.71) holds. Thus $(u_k)_{k\in\mathbb{N}}$ is Cauchy in $L_2(\Omega)$.

Corollary 1.59. *If Ω is a bounded open subset of $\mathbb{R}^n$, then the embedding $\overset{\circ}{W}{}_2^m(\Omega) \subset \overset{\circ}{W}{}_2^{m-1}(\Omega)$ is compact.*

Proof. Applying the previous theorem to the sequences of derivatives, we see that the derivatives form bounded sequences in $W_2^1(\Omega)$ and thus contain subsequences converging in $L_2(\Omega)$. Selecting common subsequence we get convergence in $W_2^1(\Omega)$ etc, (by closedness of derivatives).

Theorem 1.60. *If $\partial\Omega$ is a C^m boundary of a bounded open set Ω. Then the embedding $W_2^m(\Omega) \subset W_2^{m-1}(\Omega)$ is compact.*

Proof. The result follows by extension to $\overset{\circ}{W}{}_2^m(\Omega')$ where Ω' is a bounded set containing Ω.

1.4.3 Trace theorems

We know that if $u \in W_2^m(\Omega)$ with $m > n/2$ then u can be represented by a continuous function and thus can be assigned a value at the boundary of Ω (or, in fact, at any point). The requirement on m is, however, too restrictive — we have solved the Dirichlet problem, which requires a boundary value of the solution, in $\overset{\circ}{W}{}_2^1(\Omega)$. In this space, unless $n = 1$, the solution need not be continuous. It turns out that it is possible to give a meaning to the operation of taking the boundary value of a function even if it is not continuous.

We begin with the simplest (nontrivial) case when $\Omega = \mathbb{R}_+^n := \{\mathbf{x};\ \mathbf{x} = (\mathbf{x}', x_n), 0 < x_n\}$.

Theorem 1.61. *The trace operator $\gamma_0 : C^1(\overline{\mathbb{R}_+^n}) \cap W_2^1(\mathbb{R}_+^n) \to C^0(\mathbb{R}^{n-1})$ defined by*

$$(\gamma_0\phi)(\mathbf{x}') = \phi(\mathbf{x}', 0), \qquad \phi \in C^1(\overline{\mathbb{R}_+^n}) \cap W_2^1(\mathbb{R}_+^n), \mathbf{x}' \in \mathbb{R}^{n-1},$$

has a unique extension to a continuous linear operator $\gamma_0 : W_2^1(\mathbb{R}_+^n) \to L_2(\mathbb{R}^{n-1})$ whose range in dense in $L_2(\mathbb{R}^{n-1})$. The extension satisfies

$$\gamma_0(\beta u) = \gamma_0(\beta)\gamma_0(u), \qquad \beta \in C^1(\overline{\mathbb{R}_+^n}) \cap L_\infty(\mathbb{R}_+^n), u \in W_2^1(\mathbb{R}_+^n).$$

Proof. Let $\phi \in C^1(\overline{\mathbb{R}_+^n}) \cap W_2^1(\mathbb{R}_+^n)$. Then, from continuity, for any $\mathbf{x}'$, $\partial_{x_n}|\phi(\mathbf{x}', x_n)|^2 \in L_2(\mathbb{R}_+)$ we can write

$$|\phi(\mathbf{x}', r)|^2 - |\phi(\mathbf{x}', 0)|^2 = \int\limits_0^r \partial_{x_n}|u(\mathbf{x}', x_n)|^2 dx_n$$

and thus $|\phi(\mathbf{x}', r)|^2$ has a limit which must equal 0. Hence

$$|\phi(\mathbf{x}', 0)|^2 = -\int_0^\infty \partial_{x_n} |\phi(\mathbf{x}', x_n)|^2 dx_n.$$

Integrating over $\mathbb{R}^{n-1}$ we obtain

$$\begin{aligned}\|\phi(\mathbf{x}', 0)\|^2_{0,\mathbb{R}^{n-1}} &\leq 2 \int_{\mathbb{R}^n_+} \partial_{x_n} \phi(\mathbf{x}) \phi(\mathbf{x}) d\mathbf{x} \\ &\leq 2\|\partial_{x_n}\phi\|_{0,\mathbb{R}^n_+} \|\phi\|_{0,\mathbb{R}^n_+} \leq \|\partial_{x_n}\phi\|^2_{0,\mathbb{R}^n_+} + \|\phi\|^2_{0,\mathbb{R}^n_+}.\end{aligned}$$

Hence, by density, the operation of taking value at $x_n = 0$ extends to $W^1_2(\mathbb{R}^n_+)$.

If $\phi \in C_0^\infty(\mathbb{R}^{n-1})$ and τ is a truncation function $\tau(t) = 1$ for $|t| \leq 1$ and $\tau(t) = 0$ for $|t| \geq 0$ then $\phi(\mathbf{x}) = \psi(\mathbf{x}')\tau(x_n) \in C^1(\overline{\mathbb{R}^n_+}) \cap W^1_2(\mathbb{R}^n_+)$ and $\gamma_0(\phi) = \psi$ so that the range of the trace operator contains $C_0^\infty(\mathbb{R}^{n-1})$ and thus is dense. The last identity follows from continuity of the trace operator and of the operator of multiplication by bounded differentiable functions in $W^1_2(\mathbb{R}^n_+)$.

Theorem 1.62. *Let $u \in W^1_2(\mathbb{R}^n_+)$. Then $u \in \overset{\circ}{W}{}^1_2(\mathbb{R}^n_+)$ if an only if $\gamma_0(u) = 0$,*

Proof. If $u \in \overset{\circ}{W}{}^1_2(\mathbb{R}^n_+)$, then u is the limit of a sequence $(\phi_k)_{k\in\mathbb{N}}$ from $C_0^\infty(\mathbb{R}^n_+)$ in $W^1_2(\mathbb{R}^n_+)$. Since $\gamma_0(\phi_k) = 0$ for any k, we obtain $\gamma_0(u) = 0$.

Conversely, let $u \in W^1_2(\mathbb{R}^n_+)$ with $\gamma_0 u = 0$. By using the truncating functions, we may assume that u has compact support in $\overline{\mathbb{R}^n_+}$.

Next we use the truncating functions $\eta_k \in C^\infty(\mathbb{R})$, as in Theorem 1.51, by taking function η which satisfies $\eta(t) = 1$ for $t \geq 1$ and $\eta(t) = 0$ for $t \leq 1/2$ and define $\eta_k(x_n) = \eta(kx_n)$. To simplify notation, we assume that $0 \leq \eta' \leq 3$ for $t \in [1/2, 1]$ so that $0 \leq \eta_k'(x_n) \leq 3k$. Then the extension by 0 to $\mathbb{R}^n_-$ of $\mathbf{x} \to \eta_k(x_n)u(\mathbf{x}', x_n)$ is in $W^1_2(\mathbb{R}^n)$ and can be approximated by $C_0^\infty(\mathbb{R}^n_+)$ functions in $W^1_2(\mathbb{R}^n_+)$. Hence, we have to prove that $\eta_k u \to u$ in $W^1_2(\mathbb{R}^n_+)$.

As in the proof of Theorem 1.51 we can prove $\eta_k u \to u$ in $L_2(\mathbb{R}^n_+)$ and for each $i = 1, \ldots, n-1$, $\partial_{x_i}(\eta_k u) = \eta_k \partial_{x_i} u \to \partial_{x_i} u$ in $L_2(\mathbb{R}^n_+)$ as $k \to \infty$.

Since

$$\partial_{x_n}(\eta_k u) = u\partial_{x_n}\eta_k + \eta_k \partial_{x_n} u$$

we see that we have to prove that $u\partial_{x_n}\eta_k \to 0$ in $L_2(\mathbb{R}^n_+)$ as $k \to \infty$. For this, first we prove that if $\gamma_0(u) = 0$, then

$$u(\mathbf{x}', s) = \int_0^s \partial_{x_n} u(\mathbf{x}', t) dt \tag{1.72}$$

almost everywhere on $\mathbb{R}^n_+$. Indeed, let u_r be a bounded support C^1 function approximating u in $W^1_2(\mathbb{R}^n_+)$. Then $\int_0^s \partial_{x_n} u_r(\mathbf{x}', t)dt \to \int_0^s \partial_{x_n} u(\mathbf{x}', t)dt$ in

$L_2(\mathbb{R}^n_+)$. This follows from $\partial_{x_n} u_r \to \partial_{x_n} u$ in $L_2(\mathbb{R}^n_+)$ and, taking Q to be the box enclosing support of all u_r, u, with edges of length at most d

$$\int_Q \left| \int_0^s \partial_{x_n} u_r(\mathbf{x}', t)dt - \int_0^s \partial_{x_n} u(\mathbf{x}', t)dt \right|^2 d\mathbf{x}$$
$$\leq d^2 \int_Q \left| \partial_{x_n} u_r(\mathbf{x}', t) - \partial_{x_n} u(\mathbf{x}', t) \right|^2 d\mathbf{x}$$

Then we have, for any s, $0 \leq s \leq d$

$$\int_Q \left| \int_0^s \partial_{x_n} u_r(\mathbf{x}', t)dt - u_r(\mathbf{x}', s) \right|^2 d\mathbf{x} = \int_Q |u_r(\mathbf{x}', 0)|^2 d\mathbf{x} = d \int_{\mathbb{R}^{n-1}} |u_r(\mathbf{x}', 0)|^2 d\mathbf{x}'$$

and, since the right hand side goes to zero as $r \to \infty$, we obtain (1.72). Then, by Cauchy-Schwarz inequality

$$|u(\mathbf{x}', s)|^2 \leq s \int_0^s |\partial_{x_n} u(x', t)|^2 dt$$

and therefore

$$\int_0^\infty |\eta_k'(s) u(\mathbf{x}', s)|^2 ds \leq 9k^2 \int_0^{2/k} s \int_0^s |\partial_{x_n} u(\mathbf{x}', t)|^2 dt ds$$
$$18k \int_0^{2/k} \int_0^s |\partial_{x_n} u(\mathbf{x}', t)|^2 dt ds = 18k \int_0^{2/k} \int_t^{2/k} |\partial_{x_n} u(\mathbf{x}', t)|^2 ds dt$$
$$\leq 36 \int_0^{2/k} |\partial_{x_n} u(\mathbf{x}', t)|^2 dt.$$

Integration over $\mathbb{R}^{n-1}$ gives

$$\|\eta_k' u\|^2_{0,\mathbb{R}^n_+} \leq 36 \int_{\mathbb{R}^{n-1} \times 2/k} |\partial_{x_n} u|^2 d\mathbf{x}$$

which tends to 0.

The consideration above can be extended to the case where Ω is an open bounded region in $\mathbb{R}^n$ lying locally on one side of its C^1 boundary. Using the partition of unity, we define

$$\gamma_0(u) := \sum_{j=1}^{N} (\gamma_0((\beta_j u) \circ H^j)) \circ (H^j)^{-1}$$

It is clear that if $u \in C^1(\bar{\Omega})$, then $\gamma_0 u$ is the restriction of u to $\partial\Omega$. Thus, we have the following result

Theorem 1.63. *Let Ω be a bounded open subset of $\mathbb{R}^n$ which lies on one side of its boundary $\partial\Omega$ which is assumed to be a C^1 manifold. Then there exists a unique continuous and linear operator $\gamma_0 : W_2^1(\Omega) \to L_2(\partial\Omega)$ such that for each $u \in C^1(\bar{\Omega})$, γ_0 is the restriction of u to $\partial\Omega$. The kernel of γ_0 is equal to $\overset{\circ}{W}{}_2^1(\Omega)$ and its range is dense in $L_2(\partial\Omega)$.*

1.4.4 Regularity of variational solutions to the Dirichlet problem

From Subsection 1.3.6 we know that there is a unique variational solution $u \in \overset{\circ}{W}{}_2^1(\Omega)$ of the problem

$$\int_\Omega \nabla u \nabla v d\mathbf{x} = < f, v >_{(\overset{\circ}{W}{}_2^1(\Omega))^* \times \overset{\circ}{W}{}_2^1(\Omega)}, \qquad v \in \overset{\circ}{W}{}_2^1(\Omega).$$

Moreover, now we can say that $\gamma_0 u = 0$ on $\partial\Omega$ (provided $\partial\Omega$ is C^1).

We have the following theorem

Theorem 1.64. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with C^2 boundary (or $\Omega = \mathbb{R}^n_+$). Let $f \in L_2(\Omega)$ and let $u \in \overset{\circ}{W}{}_2^1(\Omega)$ satisfy*

$$\int_\Omega \nabla u \nabla v d\mathbf{x} = (f, v), \qquad v \in \overset{\circ}{W}{}_2^1(\Omega). \tag{1.73}$$

Then $u \in W_2^2(\Omega)$ and $\|u\|_{2,\Omega} \leq C\|f\|_{0,\Omega}$ where C is a constant depending only on Ω. Furthermore, if Ω is of class C^{m+2} and $f \in W_2^m(\Omega)$, then

$$u \in W_2^{m+2}(\Omega) \quad \text{and} \quad \|u\|_{m+2,\Omega} \leq C\|f\|_{m,\Omega}.$$

In particular, if $m \geq n/2$, then $u \in C^2(\bar{\Omega})$ is a classical solution.

Moreover, if Ω is bounded, then the solution operator $G : L_2(\Omega) \to \overset{\circ}{W}{}_2^1(\Omega)$ is self-adjoint and compact.

Proof. The proof naturally splits into two cases: interior estimates and boundary estimates. Let Ω be bounded with at least C^1 boundary and consider the partition of unity $\{\beta_j\}_{j=0}^N$ subordinated to the covering $\{G_j\}_{j=0}^N$. For the interior estimates let us consider $u_0 = \beta_0 u$ and let $v \in \overset{\circ}{W}{}_2^1(\Omega)$. Then we can write

$$\begin{aligned}
\int_\Omega \nabla(\beta_0 u)\nabla v d\mathbf{x} &= \int_\Omega \beta_0 \nabla u \nabla v d\mathbf{x} + \int_\Omega u \nabla \beta_0 \nabla v d\mathbf{x} \\
&= \int_\Omega \nabla u \nabla(\beta_0 v) d\mathbf{x} - \int_\Omega v \nabla u \nabla \beta_0 d\mathbf{x} + \int_\Omega u \nabla v \nabla \beta_0 d\mathbf{x} \\
&= \int_\Omega \nabla u \nabla(\beta_0 v) d\mathbf{x} - \int_\Omega v \nabla u \nabla \beta_0 d\mathbf{x} - \int_\Omega \nabla(u \nabla \beta_0) v d\mathbf{x} \\
&= \int_\Omega \nabla u \nabla(\beta_0 v) d\mathbf{x} - 2\int_\Omega v \nabla u \nabla \beta_0 d\mathbf{x} - \int_\Omega uv \Delta \beta_0 d\mathbf{x} \\
&= \int_\Omega (f\beta_0 - \Delta\beta_0 u - 2\nabla u \nabla \beta_0) v d\mathbf{x} = \int_\Omega F v d\mathbf{x}, \qquad v \in \overset{\circ}{W}{}_2^1(\Omega),
\end{aligned}$$

where $F \in L_2(\Omega)$ and we used $v \in \overset{\circ}{W}{}_2^1(\Omega)$ to get

$$\int_\Omega u \nabla v \nabla \beta_0 d\mathbf{x} = -\int_\Omega \nabla(u \nabla \beta_0) v d\mathbf{x}.$$

Hence, the function $w = \beta_0 u$ is the variational solution to the above problem in $\mathbb{R}^n$. Let us define $D_h u = |\mathbf{h}|^{-1}(\tau_h u - u)$ and take $v = D_{-h}(D_h w)$. It is possible since w has compact support in Ω and thus $v \in \overset{\circ}{W}{}_2^1(\Omega)$ for sufficiently small $\mathbf{h}$. Thus we obtain

$$\int_\Omega |\nabla D_h w|^2 d\mathbf{x} = \int_\Omega F D_{-h}(D_h w) d\mathbf{x},$$

that is,

$$\|D_h w\|_{1,\Omega}^2 \le \|F\|_{0,\Omega} \|D_{-h}(D_h w)\|_{0,\Omega}. \tag{1.74}$$

On the other hand, from Friedrichs lemma, for any $v \in W_2^1(\Omega)$ with compact support

$$\|D_{-h} v\|_{0,\Omega}^2 \le \|\nabla v\|_{0,\Omega}. \tag{1.75}$$

Applying this to $v = D_h u$, we obtain

$$\|D_h w\|_{1,\Omega}^2 \le \|F\|_{0,\Omega} \|\nabla D_h w\|_{0,\Omega} \le \|F\|_{0,\Omega} \|D_h w\|_{1,\Omega},$$

that is,

$$\|D_h w\|_{1,\Omega} \le \|F\|_{0,\Omega}.$$

In particular, we obtain

$$\|D_h \partial_{x_i} w\|_{0,\Omega} \le \|F\|_{0,\Omega}, \quad i = 1, \ldots, n,$$

which yields $\partial_{x_i} w \in W_2^1(\Omega)$, that is, $w \in W_2^2(\Omega)$.

In the next step, we shall move to estimates close to the boundary. Let us fix some some set B_j and corresponding function β_j, $1 \leq j \leq N$ from the partition of unity and drop the index j. Then we have a C^2 diffeomorphism $H : Q \to B$ the inverse of which we denote $J = H^{-1}$ so that $H(Q_+) = \Omega \cap B$ and $H(Q_0) = \partial\Omega \cap B$. We denote $\mathbf{x} = H(\mathbf{y})$, $\mathbf{y} \in Q$ and $\mathbf{y} = J(\mathbf{x})$. As before, we see that $w = \beta u$ is a variational solution to

$$\int\limits_{\Omega\cap B} \nabla w \nabla v d\mathbf{x} = \int\limits_{\Omega\cap B} (f\beta - u\Delta\beta - 2\nabla u \nabla\beta) v d\mathbf{x} = \int\limits_{\Omega\cap B} g v d\mathbf{x}, \qquad v \in \overset{\circ}{W}{}_2^1(\Omega) \tag{1.76}$$

where the Green's formula

$$\int\limits_{\Omega\cap B} u \nabla v \nabla \beta_0 d\mathbf{x} = -\int\limits_{\Omega\cap B} \nabla(u\nabla\beta_0) v d\mathbf{x}.$$

can be justified by noting that the integration is actually carried out over the domain $G \Subset B$ and we can use a function χv, where χ is equal to 1 on G and has support in B, instead of v. Function $\chi v \in \overset{\circ}{W}{}_2^1(\Omega \cap B)$ (as v can be approximated by ϕ compactly supported in Ω and χv can be approximated by $\chi\phi$ compactly supported in $\Omega \cap B$).

Now we transfer (1.76) to Q_+. We have $z(\mathbf{y}) = w(H(\mathbf{y}))$ for $\mathbf{y} \in Q_+$ or $w(\mathbf{x}) = z(J(\mathbf{x}))$ for $\mathbf{x} \in \Omega \cap B$. Let $\psi \in \overset{\circ}{W}{}_2^1(Q_+)$ and $\phi(\mathbf{x}) = \psi(J(\mathbf{x}))$. Then $\phi \in \overset{\circ}{W}{}_2^1(\Omega \cap B)$ and we have

$$\partial_{x_j} w = \sum_{k=1}^{n} \partial_{y_k} z \partial_{x_j} J_k, \qquad \partial_{x_j}\phi = \sum_{l=1}^{n} \partial_{y_l}\psi \partial_{x_j} J_l$$

and hence

$$\int\limits_{\Omega\cap B} \nabla w \nabla \phi d\mathbf{x} = \int\limits_{Q_+} \sum_{k,j,l=1}^{n} \partial_{x_j} J_k \partial_{x_j} J_l \partial_{y_k} z \partial_{y_l}\psi |\det\mathcal{J}_H| d\mathbf{y} = \int\limits_{Q_+} \sum_{k,l=1}^{n} a_{k,l} \partial_{y_k} z \partial_{y_l} \psi d\mathbf{y}$$

where $\mathcal{J}$ is the Jacobi matrix of H. We note that we can write

$$a_{k,l} = |\det\mathcal{J}_H| \mathcal{J}_J \mathcal{J}_J^T$$

and thus we have

$$\sum_{k,l=1}^{n} a_{k,l}\xi_k\xi_l = |\det\mathcal{J}_H| (\mathcal{J}_J^T \boldsymbol{\xi}, \mathcal{J}_J^T \boldsymbol{\xi}) \geq \alpha|\boldsymbol{\xi}|^2 \tag{1.77}$$

for all $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ since both Jacobi matrices $\mathcal{J}_H, \mathcal{J}_J$ are nonsingular. Also

$$\int_{\Omega\cap B} g\phi d\mathbf{x} = \int_{Q_+} (g\circ H)\psi|\det\mathcal{J}_H|d\mathbf{y} = \int_{Q_+} G\psi d\mathbf{y}$$

where $G \in L_2(Q_+)$ so that $z \in \overset{\circ}{W}{}^1_2(Q)$ is a solution to the (elliptic) variational problem

$$\int_{Q_+} \sum_{k,l=1}^{n} a_{k,l}\partial_{y_k} z\partial_{y_l}\psi d\mathbf{y} = \int_{Q_+} G\psi d\mathbf{y}, \qquad \psi \in \overset{\circ}{W}{}^1_2(Q_+). \tag{1.78}$$

Next the process is split into two cases. First we shall consider the method of finite differences, as in the G_0 case but only in the directions parallel to the boundary. Thus, we take $\psi = D_{-h}(D_h z)$ for $|\mathbf{h}|$ small enough to still have $\psi \in \overset{\circ}{W}{}^1_2(Q_+)$. Then, as above

$$\int_{Q_+} D_h\left(\sum_{k,l=1}^{n} a_{k,l}\partial_{y_k} z\right)\partial_{y_l}(D_h z)d\mathbf{y} = \int_{Q_+} GD_{-h}(D_h z)d\mathbf{y}.$$

Since $D_h x \in \overset{\circ}{W}{}^1_2(Q_+)$, we can use Friedrichs lemma to estimate

$$\int_{Q_+} GD_{-h}(D_h z)d\mathbf{y} \le \|G\|_{0,Q_+}\|D_{-h}(D_h z)\|_{0,Q_+} \le \|G\|_{0,Q_+}\|\nabla(D_h z)\|_{0,Q_+}.$$

Then, using $\tau_h(fg) - fg = \tau_h f(\tau_h g - g) + (\tau_h f - f)g$, we find

$$D_h\left(\sum_{k,l=1}^{n} a_{k,l}\partial_{y_k} z\right)(\mathbf{y}) = a_{k,l}(\mathbf{y}+\mathbf{h})\partial_{y_k} D_h z(\mathbf{y}) + (D_h a_{k,l})(\mathbf{y})\partial_{y_k}(\mathbf{y})$$

and thus we can write, be the reverse Cauchy-Schwarz inequality

$$\begin{aligned}
&\int_{Q_+} D_h\left(\sum_{k,l=1}^{n} a_{k,l}\partial_{y_k} z\right)\partial_{y_l}(D_h z)d\mathbf{y} \\
&= \int_{Q_+} \sum_{k,l=1}^{n} (\tau_h a_{k,l})\partial_{y_k}(D_h z)\partial_{y_l}(D_h z)d\mathbf{y} + \int_{Q_+} \sum_{k,l=1}^{n} (D_h a_{k,l})\partial_{y_k} z\partial_{y_l}(D_h z)d\mathbf{y} \\
&\ge \alpha\|\nabla(D_h z)\|^2_{0,Q_+} - C\|\nabla z\|_{0,Q_+}\|\nabla(D_h z)\|_{0,Q_+}
\end{aligned}$$

where C depends on the C^1 norm of $a_{k,l}$ (and thus C^2 norm of the local atlas). Thus

$$\begin{aligned}
\|\nabla(D_h z)\|^2_{0,Q_+} &\le \alpha^{-1}\left(\|G\|_{0,Q_+}\|\nabla(D_h z)\|_{0,\Omega} + C\|z\|_{1,\Omega}\|\nabla(D_h z)\|_{0,Q_+}\right) \\
&\le C'\|G\|_{0,Q_+}\|\nabla(D_h z)\|_{0,Q_+},
\end{aligned} \tag{1.79}$$

where we have used the $W_2^1(\Omega)$ estimates for solutions to (1.78): for $\psi = z \in \overset{\circ}{W}{}_2^1(Q_+)$

$$\alpha\|\nabla z\|^2 \leq \int\limits_{Q_+} \sum_{k,l=1}^{n} a_{k,l}\partial_{y_k} z \partial_{y_l} z d\mathbf{y} = \int\limits_{Q_+} G z d\mathbf{y} \leq \|G\|_{0,Q_+}\|\nabla z\|_{0,Q_+}.$$

Note that in the last inequality we used the Poincarè inequality as $z \in \overset{\circ}{W}{}_2^1(Q_+)$ and the constant in this inequality can be taken 1.

Thus we have

$$\|\nabla(D_h z)\|_{0,Q_+} \leq C'\|G\|_{0,Q_+}, \tag{1.80}$$

for any $\mathbf{h}$ which is parallel to Q_0. Let $j = 1, \dots, n$, $\mathbf{h} = |\mathbf{h}|\mathbf{e}_k$, $k = 1, \dots, n-1$ and $\phi \in C_0^\infty(Q_+)$. Then we can write

$$\int\limits_{Q_+} D_h \partial_{y_j} z \phi d\mathbf{y} = -\int\limits_{Q_+} \partial_{y_j} z D_{-h}\phi d\mathbf{y}$$

and, by (1.80),

$$\left|\int\limits_{Q_+} \partial_{y_j} z D_{-h}\phi d\mathbf{y}\right| = \left|\int\limits_{Q_+} D_h \partial_{y_j} z \phi d\mathbf{y}\right| \leq C'\|G\|_{0,Q_+}\|\phi\|_{0,Q_+}$$

which, passing to the limit as $|h| \to 0$ gives for any $(j,k) \neq (n,n)$

$$\left|\int\limits_{Q_+} \partial_{y_j} z \partial_{y_k}\phi d\mathbf{y}\right| \leq C'\|G\|_{0,Q_+}\|\phi\|_{0,Q_+}. \tag{1.81}$$

To conclude, we have to show also the above estimate for $k = n$. First we observe that $a_{nn} \geq \alpha$ on Q_+. This follows from (1.77) by taking $\boldsymbol{\xi} = (1, 0, \dots, 0)$. Thus, we can replace in (1.78) ψ by ψ/a_{nn}. Then we rewrite (1.78) as

$$\begin{aligned}\int\limits_{Q_+} a_{n,n}\partial_{y_k} z \partial_{y_l}(a_{n,n}^{-1}\psi) d\mathbf{y} &= \int\limits_{Q_+} a_{n,n} G(a_{n,n}^{-1}\psi) d\mathbf{y} \\ &\quad - \int\limits_{Q_+} \sum_{(k,l)\neq(n,n)} a_{k,l}\partial_{y_k} z \partial_{y_l}(a_{n,n}^{-1}\psi) d\mathbf{y},\end{aligned}$$

and differentiating on the left hand side

$$\int\limits_{Q_+} \partial_{y_k} z \partial_{y_l} \psi d\mathbf{y} = \int\limits_{Q_+} a_{n,n}^{-1} \psi \partial_{y_n} a_{n,n} \partial_{y_k} z d\mathbf{y} + \int\limits_{Q_+} a_{n,n} G \cdot (a_{n,n}^{-1}\psi) d\mathbf{y}$$
$$- \int\limits_{Q_+} \sum_{(k,l)\neq(n,n)} (a_{n,n}^{-1}\psi)\partial_{y_l} a_{k,l} \partial_{y_k} z d\mathbf{y}$$
$$\int\limits_{Q_+} \sum_{(k,l)\neq(n,n)} \partial_{y_k} z \partial_{y_l} (a_{n,n}^{-1} a_{k,l}\psi) d\mathbf{y},$$

Applying now (1.83), we get

$$\left|\int\limits_{Q_+} \partial_{y_k} z \partial_{y_l} \psi d\mathbf{y}\right| \leq C(\|G\|_{0,Q_+} + \|z\|_{1,Q_+})\|\psi\|_{0,Q}. \tag{1.82}$$

This shows that

$$\left|\int\limits_{Q_+} \partial_{y_j} z \partial_{y_k} \phi d\mathbf{y}\right| \leq C'\|G\|_{0,Q_+}\|\phi\|_{0,Q_+}. \tag{1.83}$$

for any $j,k = 1,\ldots n$ and thus, by Proposition 1.50, each first derivative of z belongs to $W_2^1(Q_+)$ and thus $z \in W_2^2(Q_+)$. Using the first part of the proof and transferring the solution back to Ω shows that $u \in W_2^2(\Omega)$.

Let us consider higher derivatives. As before, we split u according to the partition of unity and separately argue argue in $G_0 \Subset \Omega$ and in Q_+. Let us begin with $u \in W_2^2(\Omega)\cap \overset{\circ}{W}{}_2^1(\Omega)$ and consider $w = \beta_0 u$. Let $f \in W_2^1(\Omega)$ and consider any derivative ∂u, $i = 1,\ldots,n$. We know that $\partial u \in W_2^1(\Omega)$. Then we can use $\phi \in C_0^\infty$ and take $\partial\phi$ as the test function in (1.73) so that , integrating by parts

$$-\int\limits_{\Omega} \partial f \phi d\mathbf{x} = \int\limits_{\Omega} f \partial\phi d\mathbf{x} = \int\limits_{\Omega} \nabla u \nabla \partial\phi d\mathbf{x} = -\int\limits_{\Omega} \nabla \partial u \nabla \phi d\mathbf{x}$$

so that ∂u is a variational solution with square integrable right hand side and thus $\partial u \in W_2^2(\Omega)$ and $u \in W_2^3(\Omega)$. Then we can proceed by induction.

Let us consider $z \in W_2^2(Q_+)\cap \overset{\circ}{W}{}_2^1$ and let ∂u be any derivative in direction tangential to Q_0. We claim that $\partial z \in \overset{\circ}{W}{}_2^1$. First, we note that $D_h z \in \overset{\circ}{W}{}_2^1$ if $\mathbf{h}$ is parallel to Q_0 for sufficiently small $|\mathbf{h}|$. By (1.80), $D_h z$ is bounded in $W_0^1(Q)$ and thus we have a subsequence $\mathbf{h}_n$ such that $D_{h_n} z \rightharpoonup g \in \overset{\circ}{W}{}_2^1(Q)$. Clearly, $D_{h_n} z$ converges weakly in $L_2(Q_+)$ and thus for any $\phi \in C_0^\infty(Q_+)$

$$\int\limits_{Q_+} (D_{h_n} z)\phi d\mathbf{y} = \int\limits_{Q_+} z D_{-h_n} \phi d\mathbf{y}$$

and thus passing to the limit

$$\int_{Q_+} g\phi d\mathbf{y} = -\int_{Q_+} z\partial\phi d\mathbf{y}$$

and thus $\partial z \in \overset{\circ}{W}{}_2^1(Q_+)$. Then, as before

$$\int_{\Omega} \partial G\psi d\mathbf{y} = \int_{\Omega} \sum_{k,l=1}^{n} \partial_{y_k}(\partial z)\partial_{y_l}\psi d\mathbf{y} \tag{1.84}$$

for any $\phi \in \overset{\circ}{W}{}_2^1(Q_+)$. We argue by induction in m. Let $f \in W_2^{m+1}(Q_+)$. From induction assumption, we have $u \in W^{m+2}(Q_+)$. Also ∂u in any tangential derivative is in $\overset{\circ}{W}{}_2^1(Q_+)$ and satisfies (1.84). By induction assumption to ∂u and ∂G we see that $\partial u \in W_2^{m+2}(Q_+)$. Finally we can write

$$\partial^2_{x_n x_n} u = \frac{1}{a_{n,n}}\left(-G - \int_{\Omega} \sum_{(k,l)\neq(n,n)} \partial_{y_k}(\partial z)\partial_{y_l}\psi d\mathbf{y}\right)$$

so that the claim follows.

2

An Overview of Semigroup Theory

In this chapter we are concerned with methods of finding solutions of the Cauchy problem.

Definition 2.1. *Given a Banach space and a linear operator $\mathcal{A}$ with domain $D(\mathcal{A})$ and range $Im\mathcal{A}$ contained in X and also given an element $u_0 \in X$, find a function $u(t) = u(t, u_0)$ such that*

1. *$u(t)$ is continuous on $[0, \infty)$ and continuously differentiable on $(0, \infty)$,*
2. *for each $t > 0$, $u(t) \in D(\mathcal{A})$ and*

$$u'(t) = \mathcal{A}u(t), \quad t > 0, \tag{2.1}$$

3.

$$\lim_{t \to 0^+} u(t) = u_0 \tag{2.2}$$

in the norm of X.

A function satisfying all conditions above is called the classical (or strict) solution of (2.1), (2.2).

2.1 What the semigroup theory is about

In the theory of differential equations, one of the first differential equations encountered is

$$u'(t) = \alpha u(t), \qquad \alpha \in \mathbb{C} \tag{2.3}$$

with initial condition $u(0) = u_0$. It is not difficult to verify that $u(t) = e^{t\alpha}u_0$ is a solution of Eq. (2.3).

As early as in 1887, G.P. Peano showed that the system of linear ordinary differential equations with constant coefficients

$$\begin{aligned} u_1' &= \alpha_{11}u_1 + \cdots + \alpha_{1n}u_n, \\ &\vdots \\ u_n' &= \alpha_{n1}u_1 + \cdots + \alpha_{nn}u_n, \end{aligned} \tag{2.4}$$

can be written in a matrix form as

$$u'(t) = Au(t), \tag{2.5}$$

where A is an $n \times n$ matrix $\{\alpha_{ij}\}_{1 \le i,j \le n}$ and u is an n-vector whose components are unknown functions, and can be solved using the explicit formula

$$u(t) = e^{tA}u_0, \tag{2.6}$$

where the matrix exponential e^{tA} is defined by

$$e^{tA} = I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \cdots . \tag{2.7}$$

Taking a norm on $\mathbb{C}^n$ and the corresponding matrix-norm on $M_n(\mathbb{C})$, the space of all complex $n \times n$ matrices, one shows that the partial sums of the series (2.7) form a Cauchy sequence and converge. Moreover, the map $t \to e^{tA}$ is continuous and satisfies the properties, [**?**, Proposition I.2.3]:

$$\begin{aligned} e^{(t+s)A} &= e^{tA}e^{sA} \qquad \text{for all } t, s \ge 0 \\ e^{0A} &= I. \end{aligned} \tag{2.8}$$

Thus the one-parameter family $\{e^{tA}\}_{t\ge 0}$ is a homomorphism of the additive semigroup $[0, \infty)$ into a multiplicative semigroup of matrices M_n and forms what is termed a semigroup of matrices.

The representation (2.7) can be used to obtain a solution of the abstract Cauchy problem (2.1-2.2) where $\mathcal{A} : X \to X$ is a bounded linear operator, as in this case the series in (2.7) is still convergent with respect to the norm in the space of linear operators $\mathcal{L}(X)$.

In general, however, the operators coming from applications, such as, for example, differential operators, are not bounded on the whole space X and (2.7) cannot be used to obtain a solution of the abstract Cauchy problem (**??**). This is due to the fact that the domain of the operator $\mathcal{A}$ in such cases is a proper subspace of X and because (2.7) involves iterates of A, their common domain could shrink to the trivial subspace $\{0\}$. For the same reason, another common representation of the exponential function

$$e^{t\mathcal{A}} = \lim_{n\to\infty} \left(1 + \frac{t}{n}\mathcal{A}\right)^n, \tag{2.9}$$

cannot be used. For a large class of unbounded operators a variation of the latter, however, makes the representation (2.6) meaningful with $e^{t\mathcal{A}}$ calculated according to the formula

$$e^{t\mathcal{A}}x = \lim_{n\to\infty} \left(I - \frac{t}{n}\mathcal{A}\right)^{-n} x = \lim_{n\to\infty} \left[\frac{n}{t}\left(\frac{n}{t} - \mathcal{A}\right)^{-1}\right]^n x. \tag{2.10}$$

The aim of the semigroup theory is to find conditions under which such a generalization of the exponential function satisfying (2.8) is possible.

2.2 Rudiments

2.2.1 Definitions and Basic Properties

If the solution to (2.1), (2.2) is unique, then we can introduce the family of operators $(G(t))_{t\geq 0}$ such that $u(t, u_0) = G(t)u_0$. Ideally, $G(t)$ should be defined on the whole space for each $t > 0$, and the function $t \to G(t)u_0$ should be continuous for each $u_0 \in X$, leading to well-posedness of (2.1), (2.2). Moreover, uniqueness and linearity of $\mathcal{A}$ imply that $G(t)$ are linear operators. A fine-tuning of these requirements leads to the following definition.

Definition 2.2. *A family $(G(t))_{t\geq 0}$ of bounded linear operators on X is called a C_0-semigroup, or a strongly continuous semigroup, if*

(i) $G(0) = I$;
(ii) $G(t+s) = G(t)G(s)$ *for all* $t, s \geq 0$;
(iii) $\lim_{t\to 0^+} G(t)x = x$ *for any* $x \in X$.

A linear operator A is called the (infinitesimal) generator of $(G(t))_{t\geq 0}$ if

$$Ax = \lim_{h\to 0^+} \frac{G(h)x - x}{h}, \tag{2.11}$$

with $D(A)$ defined as the set of all $x \in X$ for which this limit exists. If we need to use differen generators, then typically the semigroup generated by A will be denoted by $(G_A(t))_{t\geq 0}$, otherwise simply by $(G(t))_{t\geq 0}$.

Proposition 2.3. *Let $(G(t))_{t\geq 0}$ be a C_0-semigroup. Then there are constants $\omega \geq 0$, $M \geq 1$ such that*

$$\|G(t)\| \leq Me^{\omega t}, \qquad t \geq 0. \tag{2.12}$$

Proof. First we observe that $\|G(t)\|$ is bounded on some interval. Indeed, if not, there is $(t_n)_{n\in\mathbb{N}}$, $t_n \to 0$, $\|G(t_n)\| \geq n$, that is $(G(t_n))$ is unbounded. But, by the Banach-Steinhaus theorem there is an $x \in X$ and a subsequence $(t_{n_k})_{n_k\in\mathbb{N}}$ such that $(G(t_{n_k})x)$ is unbounded, contrary to (iii). So, $\|G(t)\| \leq M$ for $0 \leq t \leq \eta$ for some η and $M \geq 1$ as $G(0) = I$. For any $t \geq 0$ we take $t = n\eta + \delta$, $0 \leq \delta < \eta$ and, by the semigroup property,

$$\|G(t)\| = \|G(\delta)(G(\eta))^n\| \leq MM^n = Me^{(t-\delta)\ln M/\eta} \leq Me^{\omega t}$$

where $\omega = \eta^{-1} \ln M \geq 0$.

As a corollary, we have

Corollary 2.4. *Let $(G(t))_{t\geq 0}$ be a C_0-semigroup. Then for every $x \in X$, $t \to G(t)x \in C(\mathbb{R}_+ \cup \{0\}, X)$.*

Proof. We have for $t, h \geq 0$

$$\|G(t+h)x - G(t)x\| \leq \|G(t)\|\|G(h)x - x\| \leq Me^{\omega t}\|G(h)x - x\|$$

and for $t \geq h \geq 0$

$$\|G(t-h)x - G(t)x\| \leq \|G(t-h)\|\|G(h)x - x\| \leq Me^{\omega t}\|G(h)x - x\|$$

and the statement follows from condition (iii).

Remark 2.5. As we have seen above, for semigroups, the existence of a one-sided limit at some $t_0 > 0$ yields the existence of the limit.

Let $(G(t))_{t\geq 0}$ be a semigroup generated by the operator A. The following properties of $(G(t))_{t\geq 0}$ are frequently used.

Lemma 2.6. *Let $(G(t))_{t\geq 0}$ be a C_0-semigroup generated by A.*

(a) For $x \in X$

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} G(s)x ds = G(t)x. \tag{2.13}$$

(b) For $x \in X$, $\int_0^t G(s)xds \in D(A)$ and

$$A \int_0^t G(s)x ds = G(t)x - x. \tag{2.14}$$

(c) For $x \in D(A)$, $G(t)x \in D(A)$ and

$$\frac{d}{dt} G(t)x = AG(t)x = G(t)Ax. \tag{2.15}$$

(d) For $x \in D(A)$,

$$G(t)x - G(s)x = \int_s^t G(\tau)Axd\tau = \int_s^t AG(\tau)xd\tau. \tag{2.16}$$

Proof. (a) follows from continuity of the semigroup. To prove (b) we consider $x \in X$ and $h > 0$. Then

$$\begin{aligned}
\frac{G(h) - I}{h} \int_0^t G(s)xds &= \frac{1}{h} \int_0^t (G(s+h)x - G(s)x)ds \\
&= \frac{1}{h} \int_t^{t+h} G(s)xds - \frac{1}{h} \int_0^h G(s)x)ds
\end{aligned}$$

and the right hand side tends to $G(t)x - x$ by (a) which proves that $\int_0^t G(s)xds \in D(A)$ and (2.14). To prove (c), let $x \in D(A)$ and $h > 0$. As above

$$\frac{G(h) - I}{h} G(t)x = G(t) \left(\frac{G(h) - I}{h} \right) x \to T(t)x$$

as $h \to 0$. Thus, $G(t)x \in D(A)$ and $AG(t)x = G(t)Ax$ for $x \in D(A)$. The limit above also shows that

$$\frac{d^+}{dt} G(t)x = AG(t)x = G(t)Ax,$$

that is, the right derivative of $G(t)x$ is $AG(t)$. Take now $t > 0$ and $h \leq t$. Then

$$\lim_{h \to 0} \left(\frac{G(t-h)x - G(t)x}{-h} - AG(t)x \right)$$
$$\lim_{h \to 0} G(t-h) \left(\frac{G(h)x - x}{h} - Ax \right) + \lim_{h \to 0} (G(t-h)Ax - G(t)Ax)$$

and we see that both limits are 0 by uniform boundedness of $(G(t))_{t \geq 0}$, strong continuity and $x \in D(A)$.

Part (d) is obtained by integrating (2.15).

Why C_0-semigroups?

Proposition 2.7. *If $(G(t))_{t \geq 0}$ is uniformly bounded, then its generator is bounded.*

Proof. Since $\rho^{-1} \int_0^\rho G(s)ds \to I$ in the uniform operator norm, then there is $\rho > 0$ such that $\|\rho^{-1} \int_0^\rho G(s)ds - I\| < 1$ and thus $\rho^{-1} \int_0^\rho G(s)ds$ and hence $\int_0^\rho G(s)ds$ are invertible.

$$\frac{G(h) - I}{h} \int_0^\rho G(s)ds = \frac{1}{h} \int_0^\rho (G(s+h) - G(s))ds$$
$$= \frac{1}{h} \int_h^{\rho+h} G(s)xds - \frac{1}{h} \int_0^\rho G(s)xds = \frac{1}{h} \int_\rho^{\rho+h} G(s)xds - \frac{1}{h} \int_0^h G(s)xds$$

Thus

$$\frac{G(h)-I}{h} = \left(\frac{1}{h}\int\limits_{\rho}^{\rho+h} G(s)xds - \frac{1}{h}\int\limits_{0}^{h} G(s)xds\right)\left(\int\limits_{0}^{\rho} G(s)ds\right)^{-1}.$$

Letting $h \to 0$, we see that $(G(h)-I)/h \to (G(\rho)-I)(\int_0^\rho G(s)ds)^{-1}$ in the uniform norm and thus the generator is bounded.

From (2.15) and condition (iii) of Definition 2.2 we see that if A is the generator of $(G(t))_{t\geq 0}$, then for $x \in D(A)$ the function $t \to G(t)x$ is a classical solution of the following Cauchy problem,

$$\partial_t u(t) = A(u(t)), \quad t > 0, \tag{2.17}$$

$$\lim_{t\to 0^+} u(t) = x. \tag{2.18}$$

We note that ideally the generator A should coincide with $\mathcal{A}$ but in reality very often it is not so.

Remark 2.8. We noted above that for $x \in D(A)$ the function $u(t) = G(t)x$ is a classical solution to (2.17), (2.18). For $x \in X \setminus D(A)$, however, the function $u(t) = G(t)x$ is continuous but, in general, not differentiable, nor $D(A)$-valued, and, therefore, not a classical solution. Nevertheless, from (2.14), it follows that the integral $v(t) = \int_0^t u(s)ds \in D(A)$ and therefore it is a strict solution of the integrated version of (2.17), (2.18):

$$\begin{aligned}&\partial_t v = Av + x, \quad t > 0\\ &v(0) = 0,\end{aligned} \tag{2.19}$$

or equivalently,

$$u(t) = A\int\limits_0^t u(s)ds + x. \tag{2.20}$$

We say that a function u satisfying (2.19) (or, equivalently, (2.20)) is a *mild solution* or *integral solution* of (2.17), (2.18).

Corollary 2.9. *If $(G(t))_{t\geq 0}$ is a C_0-semigroup generated by A, then A is a closed densely defined linear operator.*

Proof. For $x \in X$ we set $x_t = t^{-1}\int\limits_0^t G(s)xds$. By (b), $x_t \in D(A)$ and by (a), $x_t \to x$ as $t \to 0$. To prove closedness, let $D(A) \ni x_n \to x \in X$ and let $Ax_n \to y \in X$. From (d) we have

$$G(t)x_n - x_n = \int\limits_0^t G(s)Ax_n ds.$$

By local boundedness of $(G(t))_{t\geq 0}$ we have that $G(s)Ax_n \to G(s)y$ uniformly on bounded intervals, hence, by letting $n \to \infty$,

$$G(t)x - x = \int_0^t G(s)yds.$$

Thus, using (a), $x \in D(A)$ and $Ax = y$.

Thus, if we have a semigroup, we can identify the Cauchy problem of which it is a solution. Usually, however, we are interested in the reverse question, that is, in finding the semigroup for a given equation.

A first step in this direction is

Theorem 2.10. *Let $(G_A(t))_{t\geq 0}$ and $(G_B(t))_{t\geq 0}$ be C_0 semigroups generated by, respectively, A and B. If $A = B$, then $G_A(t) = G_B(t)$.*

Proof. Let $x \in D(A) = D(B)$. Consider the function

$$s \to G_A(t-s)G_B(s)x, \quad 0 \leq s \leq t,$$

continuous on $[0,t]$. Writing, for appropriate s, h

$$\begin{aligned}&\frac{G_A(t-(s+h))G_B(s+h)x - G_A(t-s)G_B(s)x}{h}\\ &= \frac{G_A(t-(s+h))G_B(s+h)x - G_A(t-(s+h))G_B(s)x}{h}\\ &\quad + \frac{G_A(t-(s+h))G_B(s)x - G_A(t-s)G_B(s)x}{h}\end{aligned}$$

we see that by local boundedness both terms converge and, by (c), we obtain

$$\begin{aligned}\frac{d}{ds}G_A(t-s)G_B(s)x &= -AG_A(t-s)G_B(s)x + G_A(t-s)BG_B(s)x\\ &= -G_A(t-s)AG_B(s)x + G_A(t-s)BG_B(s)x = 0.\end{aligned}$$

Thus $G_A(t-s)G_B(s)x$ is constant and, in particular, evaluating at $s = 0$ and $s = t$ we get $G_A(t)x = G_B(t)x$ for any t and $x \in D(A)$. From density, we obtain the equality on X.

The final answer is given by the Hille–Yoshida theorem (or, more properly, the Feller–Miyadera–Hille–Phillips–Yosida theorem). Before, however, we need to discuss the concept of the resolvent.

Let A be any operator in X. The *resolvent set* of A is defined as

$$\rho(A) = \{\lambda \in \mathbb{C};\ \lambda I - A : D(A) \to X \text{ is invertible}\}. \tag{2.21}$$

We call $(\lambda I - A)^{-1}$ the resolvent of A and denote it by $R(\lambda, A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$. The complement of $\rho(A)$ in $\mathbb{C}$ is called the *spectrum* of A and denoted by $\sigma(A)$.

The resolvent of any operator A satisfies the *resolvent identity*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A), \qquad \lambda, \mu \in \rho(A), \tag{2.22}$$

from which it follows, in particular, that $R(\lambda, A)$ and $R(\mu, A)$ commute. Writing

$$R(\mu, A) = R(\lambda, A)(I - (\mu - \lambda) R(\mu, A))$$

we see by the Neuman expansion that $R(\lambda, A)$ can be written as the power series

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^{n+1} \tag{2.23}$$

for $|\mu - \lambda| < \|R(\mu, A)\|^{-1}$ so that $\rho(A)$ is open and $\lambda \to R(\lambda, A)$ is an analytic function in $\rho(A)$. The iterates of the resolvent and its derivatives are related by

$$\frac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1}. \tag{2.24}$$

2.2.2 The Hille–Yosida Theorem

We begin with the simplest case of contractive semigroups. A C_0 semigroup $(G_A(t))_{t \geq 0}$ is called contractive if

$$\|G_A(t)\| \leq 1$$

Theorem 2.11. *A is the generator of a contractive semigroup $(G_A(t))_{t \geq 0}$ if and only if*

(a) A is closed and densely defined,
(b) $(0, \infty) \subset \rho(A)$ and for all $\lambda > 0$,

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}. \tag{2.25}$$

Proof. (Necessity) If A is the generator of a C_0 semigroup $(G_A(t))_{t \geq 0}$, then it is densely defined and closed. Let us define

$$R(\lambda) x = \int_0^{\infty} e^{-\lambda t} G(t) x dt \tag{2.26}$$

is valid for all $x \in X$. Since $(G_A(t))_{t \geq 0}$ is contractive, the integral exists for $\lambda > 0$ as an improper Riemann integral and defines a bounded linear operator $R(\lambda)x$ (by the Banach-Steinhaus theorem). $R(\lambda)$ satisfeis

$$\|R(\lambda) x\| \leq \frac{1}{\lambda} \|x\|.$$

Furthermore, $h > 0$,

$$\frac{G_A(t) - I}{h} R(\lambda)x = \frac{1}{h}\int_0^\infty e^{-\lambda t}(G_A(t+h)x - G_A(t)x)dt$$
$$= \frac{1}{h}\left(\int_h^\infty e^{-\lambda(t-h)} G_A(t)xdt - \int_0^\infty e^{-\lambda t} G_A(t)xdt\right)$$
$$= \frac{e^{\lambda h} - 1}{h}\int_h^\infty e^{-\lambda t} G_A(t)xdt - \frac{1}{h}\int_0^h e^{-\lambda t} G_A(t)xdt.$$

By strong continuity of G_A, the right hand side converges to $\lambda R(\lambda)x - x$. This implies that for any $x \in D(A)$ and $\lambda > 0$ we have $R(\lambda)x \in D(A)$ and $AR(\lambda) = \lambda R(\lambda) - I$ so

$$(\lambda I - A)R(\lambda) = I. \tag{2.27}$$

On the other hand, for $x \in D(A)$ we have

$$R(\lambda)Ax = \int_0^\infty e^{-\lambda t} G(t)Axdt = A\left(\int_0^\infty e^{-\lambda t} G(t)x\right) dt = AR(\lambda)x$$

by commutativity (Lemma 2.6 (c)) and closedness of A. Thus A and $R(\lambda)$ commute and

$$R(\lambda)(\lambda I - A)x = Ax$$

on $D(A)$. Thus $R(\lambda)$ is the resolvent of A and satisfies the desired estimate.

The converse is more difficult to prove. The starting point of the second part of the proof is the observation that if $(A, D(A))$ is a closed and densely defined operator satisfying $\rho(A) \supset (0, \infty)$ and $\|\lambda R(\lambda, A)\| \leq 1$ for all $\lambda > 0$, then

(i) for any $x \in X$,

$$\lim_{\lambda\to\infty} \lambda R(\lambda, A)x = x. \tag{2.28}$$

Indeed, first consider $x \in D(A)$. Then

$$\|\lambda R(\lambda, A)x - x\| = \|AR(\lambda, A)x\| = \|R(\lambda, A)Ax\| \leq \frac{1}{\lambda}\|Ax\| \to 0$$

as $\lambda \to \infty$. Since $D(A)$ is dense and $\|\lambda R(\lambda, A)\| \leq 1$ then by 3ϵ argument we extend the convergence to X.

(ii) $AR(\lambda, A)$ are bounded operators and for any $x \in D(A)$,

$$\lim_{\lambda\to\infty} \lambda AR(\lambda, A)x = Ax. \tag{2.29}$$

Boundedness follows from $AR(\lambda, A) = \lambda R(\lambda, A) - I$. Eq. (2.29) follows (2.28).

It was Yosida's idea to use the bounded operators

$$A_\lambda = \lambda A R(\lambda, A), \tag{2.30}$$

as an approximation of A for which we can define semigroups uniformly continuous semigroups $(G_\lambda(t))_{t\geq 0}$ via the exponential series. First we note that $(G_\lambda(t))_{t\geq 0}$ are semigroups of contractions and, for any $x \in X$ and $\lambda, \mu > 0$ we have

$$\|G_\lambda(t)x - G_\mu(t)x\| \leq t\|A_\lambda x - A_\mu x\|. \tag{2.31}$$

Indeed, using $A_\lambda = \lambda^2 R(\lambda, A) - \lambda I$ and the series estimates

$$\|G_\lambda(t)x\| \leq e^{-\lambda t} e^{\lambda \|R(\lambda,A)\| t} \leq 1.$$

Further, from the definition operators $G_\lambda(t), G_\mu(t), A_\lambda, A_\mu$ commute with each other. Then

$$\begin{aligned} \|G_\lambda(t)x - G_\mu(t)x\| &= \left\| \int_0^1 \frac{d}{ds} e^{tsA_\lambda} e^{t(1-s)A_\mu} x ds \right\| \\ &\leq t \int_0^1 \|e^{tsA_\lambda} e^{t(1-s)A_\mu} (A_\lambda x - A_\mu x)\| ds \leq t\|A_\lambda x - A_\mu x\|. \end{aligned}$$

Using (2.31) we obtain for $x \in D(A)$

$$\|G_\lambda(t)x - G_\mu(t)x\| \leq t\|A_\lambda x - A_\mu x\| \leq t(\|A_\lambda x - Ax\| + \|Ax - A_\mu x\|).$$

Hence $(G_\lambda(t)x))_\lambda$ strongly converges and the convergence (for each x) is uniform in t on bounded intervals (almost uniform on $\overline{\mathbb{R}_+}$. Since $D(A)$ is dense in X and $\|G_\lambda(t)\| \leq 1$ we get

$$\lim_{\lambda\to\infty} G_\lambda(t)x =: S(t)x$$

for $x \in X$. The convergence is still almost uniform on $\overline{\mathbb{R}_+}$. From the limit we see that $(S(t))_{t\geq 0}$ is a C_0 semigroup of contractions.

What remains is to show that $(S(t))_{t\geq 0}$ is generated by A. Let $x \in D(A)$. Then

$$S(t)x - x = \lim_{\lambda\to\infty} (G_\lambda(t)x - x) = \lim_{\lambda\to\infty} \int_0^t e^{sA_\lambda} A_\lambda x ds = \int_0^t S(s)Axds \tag{2.32}$$

where the last equality follows from

$$\begin{aligned} \|e^{sA_\lambda} A_\lambda x - S(s)Ax\| &\leq \|e^{sA_\lambda} A_\lambda x - e^{sA_\lambda} Ax\| + \|e^{sA_\lambda} Ax - S(s)Ax\| \\ &\leq \|A_\lambda x - Ax\| + \|e^{sA_\lambda} Ax - S(s)Ax\|, \end{aligned}$$

by contractivity of $(G_\lambda(t))_{t\geq 0}$, so that the convergence is uniform on bounded intervals. Assume now that $(S(t))_{t\geq 0}$ is generated by B. Dividing (2.32) by t and passing to the limit, we obtain

$$Bx = Ax, \quad x \in D(A)$$

so that $A \subset B$. On the other hand, we know that $I - A$ and $I - B$ are bijections from, resp $D(A)$ and $D(B)$ with $D(A) \subset D(B)$. But then we have $(I - B)D(A) = (I - A)D(A) = X$, that is, $D(A) = (I - B)^{-1}X = D(B)$ so $A = B$.

Corollary 2.12. *A linear operator A is the generator of a C_0 semigroup $(G(t))_{t\geq 0}$ satisfying $\|G(t)\| \leq e^{\omega t}$ if and only if*

(i) A is closed and $\overline{D(A)} = X$;
(ii) $\rho(A) \supset (\omega, \infty)$ and for such λ

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda - \omega}. \tag{2.33}$$

Proof. Follows from the contractive semigroup $S(t) = e^{-\omega t}G(t)$ being generated by $A - \omega I$.

The full version of the Hille-Yosida theorem reads

Theorem 2.13. *$A \in \mathcal{G}(M, \omega)$ if and only if*

(a) A is closed and densely defined,
(b) there exist $M > 0, \omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and for all $n \geq 1, \lambda > \omega$,

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}. \tag{2.34}$$

The proof of this result is based on re-norming of the original space. We have

Lemma 2.14. *Let A be an operator with $\rho(A) \supset (0, \infty)$. If there is M such that*

$$\|\lambda^n (R(\lambda, A))^n\| \leq M \tag{2.35}$$

then there exists a norm $|\cdot|$ that satisfies

$$\|x\| \leq |x| \leq M\|x\|, \quad x \in X \tag{2.36}$$

and

$$|\lambda R(\lambda, A)x| \leq |x|, \quad x \in X, \lambda > 0. \tag{2.37}$$

Proof. For $\mu > 0$ we define first

$$\|x\|_\mu = \sup_{n\geq 0} \|\mu^n (R(\mu, A))^n\|. \tag{2.38}$$

Then obviously

$$\|x\| \leq |x|_\mu \leq M\|x\|, \quad x \in X \tag{2.39}$$

and

$$\|\mu R(\mu, A)x\| \leq \|x\|_\mu. \tag{2.40}$$

Further, using the resolvent identity

$$R(\lambda, A) = R(\mu, A) + (\mu - \lambda)R(\mu, A)R(\mu, A),$$

by (2.40), we obtain for $\lambda \leq \mu$

$$\|R(\lambda, A)x\|_\mu \leq \frac{1}{\mu}\|\mu R(\mu, A)x\|_\mu + \left(1 - \frac{\lambda}{\mu}\right)\|R(\lambda, A)x\|_\mu$$

hence

$$\|\lambda R(\lambda, A)x\|_\mu \leq \|x\|_\mu, \quad 0 < \lambda \leq \mu. \tag{2.41}$$

Thus

$$\|\lambda^n R(\lambda, A)^n x\| \leq \|\lambda^n R(\lambda, A)^n x\|_\mu \leq \|x\|_\mu, \quad 0 < \lambda \leq \mu \tag{2.42}$$

and therefore, by taking supremum over n

$$\|x\|_\lambda \leq \|x\|_\mu, \quad 0 < \lambda \leq \mu.$$

Thus the limit

$$|x| := \lim_{\mu\to\infty} \|x\|_\mu \tag{2.43}$$

exists and finite; it defines a norm on X that is equivalent to $\|\cdot\|$ by (2.39). Finally, taking $n = 1$ in (2.42) we have

$$\|\lambda R(\lambda, A)x\|_\mu \leq \|x\|_\mu$$

and the thesis follows by taking the limit as $\mu \to \infty$.

Proof of Theorem 2.13. Similarly to Corollary 2.12, the problem can be reduced to the one for uniformly bounded semigroups: $\|G(t)\| \leq M$. Then the resolvent estimate becomes

$$\|\lambda^n (R(\lambda, A))^n\| \leq M, \quad \lambda > 0, n = 1, 2, \ldots. \tag{2.44}$$

The necessity part is reduced to the contractive case by renorming the space using

$$|x| = \sup_{t\geq 0} \|G(t)x\|. \tag{2.45}$$

Then

$$\|x\| \le |x| \le M\|x\| \tag{2.46}$$

and hence $|\cdot|$ is an equivalent norm on X. In this norm

$$|G(t)x| = \sup_{s\ge 0}\|G(t)G(s)x\| \le \sup_{\tau\ge 0}\|G(\tau)x\| = |x|$$

by the semigroup property. Then there is the generator A, densely defined and closed, satisfying $|\lambda R(\lambda, A)x| \le |x|$. But then, by (2.46),

$$\|\lambda^n (R(\lambda, A))^n x\| \le |\lambda^n (R(\lambda, A))^n x| \le |x| \le M\|x\|.$$

For sufficiency, we renorm the space using (2.43) so that $|\lambda R(\lambda, A)x| \le |x|$ and in this norm A generates a contractive semigroup $(G(t))_{t\ge 0}$. But then, by (2.36),

$$\|G(t)x\| \le |G(t)x| \le |x| \le M\|x\|$$

and the theorem follows.

2.2.3 Relation with the exponential formula

Theorem 2.15. *Let $(G_A(t))_{t\ge 0}$ be a C_0-semigroup on X generated by A. Then*

$$G_A(t)x = \lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} x = \lim_{n\to\infty}\left(\frac{n}{t}R\left(\frac{n}{t}, A\right)\right)^n x, \quad x \in X, \tag{2.47}$$

and the limit is uniform in t on any bounded interval.

Proof. Let $\|G_A(t)\| \le Me^{\omega t}$. For $\lambda > \omega$ the resolvent $R(\lambda, A)$ is an analytic function satisfying

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda s} G_A(s)x ds, \quad x \in X. \tag{2.48}$$

Then

$$\frac{d^n}{d\lambda^n} R(\lambda, A)x = \int_0^\infty (-s)^n e^{-\lambda s} G_A(s)x ds$$

that, upon substitution $\lambda = t/n$, gives

$$\frac{d^n}{d\lambda^n} R\left(\frac{t}{n}, A\right) x = (-1)^n \int_0^\infty s^n e^{-n\frac{s}{t}} G_A(s)x ds$$

and, substituting $v = s/t$,

$$\frac{d^n}{d\lambda^n}R\left(\frac{t}{n},A\right)x=(-1)^n t^{n+1}\int\limits_0^\infty v^n e^{-nv}G_A(tv)xdv.$$

Then, by (2.24),

$$\left(\frac{n}{t}R\left(\frac{t}{n},A\right)\right)^{n+1}x=\frac{n^{n+1}}{n!}\int\limits_0^\infty v^n e^{-nv}G_A(tv)xdv.$$

However, from the definition of the Γ function

$$\frac{n^{n+1}}{n!}\int\limits_0^\infty v^n e^{-nv}dv=\frac{1}{n!}\int\limits_0^\infty r^n e^{-r}dr=\frac{\Gamma(n+1)}{n!}=1,$$

thus we can write

$$\left(\frac{n}{t}R\left(\frac{t}{n},A\right)\right)^{n+1}x-G_A(t)x=\frac{n^{n+1}}{n!}\int\limits_0^\infty v^n e^{-nv}(G_A(tv)x-G_A(t)x)dv.$$

From the strong continuity of $(G_A(t))_{t\geq 0}$, for any $\epsilon>0, 0<t_0<\infty$ we can find $0<a<1<b<\infty$ such that for any $t\in[0,t_0], v\in[a,b]$ we have

$$\|G_A(tv)x-G_A(t)x\|\leq\epsilon.$$

Next, we observe that $v\to v^n e^{-nv}$ is strictly increasing on $[0,1)$ attains maximum e^{-1} at $v=1$ and is strictly decreasing on $(0,\infty)$. Now, we write

$$\frac{n^{n+1}}{n!}\int\limits_0^\infty v^n e^{-nv}(G_A(tv)x-G_A(t)x)dv=I_1+I_2+I_3,$$

where

$$\begin{aligned}\|I_1\|&\leq\left\|\frac{n^{n+1}}{n!}\int\limits_0^a v^n e^{-nv}(G_A(tv)x-G_A(t)x)dv\right\|\\&\leq\frac{n^{n+1}}{n!}a^n e^{-na}\int\limits_0^a\|(G_A(tv)x-G_A(t)x)\|dv\\\|I_2\|&\leq\left\|\frac{n^{n+1}}{n!}\int\limits_a^b v^n e^{-nv}(G_A(tv)x-G_A(t)x)dv\right\|\leq\epsilon\frac{n^{n+1}}{n!}\int\limits_a^b v^n e^{-nv}dx\leq\epsilon\\\|I_3\|&\leq\left\|\frac{n^{n+1}}{n!}\int\limits_b^\infty v^n e^{-nv}(G_A(tv)x-G_A(t)x)dv\right\|\end{aligned}$$

where we used the monotonicity of $v \to v^n e^{-nv}$ on $[0,1)$ for the estimate of I_1. Further, for a given $a < 1$, $ae^{-a} = q < e^{-1}$ and using the Strirling formula

$$\Gamma(n+1) = n! = O\left(\frac{\sqrt{n}n^n}{e^n}\right)$$

we obtain

$$\frac{n^{n+1}}{n!}\int\limits_0^a v^n e^{-nv} = O(\sqrt{n}(eq)^n)$$

which shows $\|I_1\| \to 0$ as $n \to \infty$. Now, for I_3 we observe $\|G_A(tv)x - G_A(t)x\| \leq M\|x\|(e^{\omega tv} + e^{\omega t}) \leq 2M\|x\|e^{\omega tv}$ (as $v \geq b > 1$) and hence the integral is finite if we take $n > \omega t$. For a given t_0, let us fix $n_0 > \omega t_0$. Then we can write

$$\|I_3\| \leq 2M\frac{n^{n+1}}{n!}\int\limits_b^\infty v^n e^{-(n-n_0)v} e^{-(n_0-\omega t)v} dv.$$

The maximum of $v \to v^n e^{-(n-n_0)v}$ is attained at $v_{max} = \frac{n}{n-n_0}$ and thus satisfies $1 < v_{max} < b$ for sufficiently large n. Hence $v \to v^n e^{-(n-n_0)v}$ is strictly decreasing on $[b, \infty)$ for large n and we can write

$$\|I_3\| \leq 2M\frac{n^{n+1}}{n!}b^n e^{-nb}e^{n_0 b}\int\limits_b^\infty e^{-(n_0-\omega t)v} dv.$$

Therefore $\|I_3\| \to 0$ as $n \to \infty$ uniformly on $[0, t_0]$ by the same argument as for I_1. Thus, for any $\epsilon > 0$ there is N such that for any $n > N$

$$\left\|\left(\frac{n}{t}R\left(\frac{n}{t}, A\right)\right)^{n+1} x - G_A(t)x\right\| \leq \epsilon;$$

that is,

$$\lim_{n\to\infty}\left(\frac{n}{t}R\left(\frac{n}{t}, A\right)\right)^{n+1} x = G_A(t)x$$

uniformly on $[0, t_0]$ for any $t_0 < \infty$. However, by (2.28) and the Banach-Steinhaus theorem (uniform boundedness of strongly convergent sequence of operators), we obtain (2.47).

2.2.4 Dissipative operators and the Lumer-Phillips theorem

Let X be a Banach space (real or complex) and X^* be its dual. From the Hahn–Banach theorem, Theorem 1.12 for every $x \in X$ there exists $x^* \in X^*$ satisfying

$$<x^*, x> = \|x\|^2 = \|x^*\|^2.$$

Therefore the *duality set*

$$\mathcal{J}(x) = \{x^* \in X^*;\ <x^*, x>= \|x\|^2 = \|x^*\|^2\} \tag{2.49}$$

is nonempty for every $x \in X$.

Definition 2.16. *We say that an operator* $(A, D(A))$ *is* dissipative *if for every* $x \in D(A)$ *there is* $x^* \in \mathcal{J}(x)$ *such that*

$$\Re <x^*, Ax> \leq 0. \tag{2.50}$$

If X is a real space, then the real part in the above definition can be dropped.

Theorem 2.17. *A linear operator* A *is dissipative if and only if for all* $\lambda > 0$ *and* $x \in D(A)$,

$$\|(\lambda I - A)x\| \geq \lambda \|x\|. \tag{2.51}$$

Proof. Let A be dissipative, $\lambda > 0$ and $x \in D(A)$. If $x^* \in \mathcal{J}$ and $\Re < Ax, x^* >\leq 0$, then

$$\|\lambda x - Ax\|\|x\| \geq |(\lambda x - Ax, x^* > | \geq \Re < \lambda x - Ax, x^* >\geq \lambda \|x\|^2$$

so that we get (2.51).

Conversely, let $x \in D(A)$ and $\lambda\|x\| \leq \|\lambda x - Ax\|$ for $\lambda > 0$. Consider $y^*_\lambda \in \mathcal{J}(\lambda x - Ax)$ and $z^*_\lambda = y^*_\lambda / \|y^*_\lambda\|$.

$$\begin{aligned}\lambda\|x\| \leq \|\lambda x - Ax\| = \|\lambda x - Ax\|\|z^*_\lambda\| = \|y^*_\lambda\|^1 \|\lambda x - Ax\|\|y^*_\lambda\| &= \|y^*_\lambda\|^1 < \lambda x - Ax, y^*_\lambda > \\ &= < \lambda x - Ax, z^*_\lambda >= \lambda \Re < x, z^*_\lambda > - \Re < Ax, z^*_\lambda > \\ &\leq \lambda \|x\| - \Re < Ax, z^*_\lambda >\end{aligned}$$

for every $\lambda > 0$. From this estimate we obtain that $\Re < Ax, z^*_\lambda >\leq 0$ and, by $|\alpha| \geq \Re\alpha$,

$$\lambda \Re < x, z^*_\lambda >= \lambda\|x\| + \Re < Ax, z^*_\lambda >\geq \lambda \|x\| - |\Re < Ax, z^*_\lambda > | \geq \lambda\|x\| - \|Ax\|$$

or $\Re < x, z^*_\lambda >\geq \|x\| - \lambda^{-1}\|Ax\|$. Now, the unit ball in X^* is weakly-$*$ compact and thus there is a sequence $(z^*_{\lambda_n})_{n\in\mathbb{N}}$ converging to z^* with $\|z^*\| = 1$. From the above estimates, we get

$$\Re < Ax, z^* >\leq 0$$

and $\Re < x, z^* >\geq \|x\|$. Hence, also, $| < x, z^* > | \geq \|x\|$ On the other hand, $Re < x, z^* >\leq | < x, z^* > | \leq \|x\|$ and hence $< x, z^* >= \|x\|$. Taking $x^* = z^*\|x\|$ we see that $x^* \in \mathcal{J}(x)$ and $\Re < Ax, x^* >\leq 0$ and thus A is dissipative.

Theorem 2.18. *Let* A *be a linear operator with dense domain* $D(A)$ *in* X.

(a) If A is dissipative and there is $\lambda_0 > 0$ such that the range $Im(\lambda_0 I - A) = X$, then A is the generator of a C_0-semigroup of contractions in X.

(b) If A is the generator of a C_0 semigroup of contractions on X, then $Im(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, for every $x \in D(A)$ and every $x^* \in \mathcal{J}(x)$ we have $\Re < Ax, x^* > \leq 0$.

Proof. Let $\lambda > 0$, then dissipativeness of A implies $\|\lambda x - Ax\| \geq \lambda \|x\|$ for $x \in D(A), \lambda > 0$. This gives injectivity and, since by assumption, the $Im(\lambda_0 I - A)D(A) = X$, $(\lambda_0 I - A)^{-1}$ is a bounded everywhere defined operator and thus closed. But then $\lambda_0 I - A$, and hence A, are closed. We have to prove that $Im(\lambda I - A)D(A) = X$ for all $\lambda > 0$. Consider the set $\Lambda = \{\lambda > 0;\ Im(\lambda I - A)D(A) = X\}$. Let $\lambda \in \Lambda$. This means that $\lambda \in \rho(A)$ and, since $\rho(A)$ is open, Λ is open in the induced topology. We have to prove that Λ is closed in the induced topology. Assume $\lambda_n \to \lambda$, $\lambda > 0$. For every $y \in X$ there is $x_n \in D(A)$ such that

$$\lambda_n x_n - Ax_n = y.$$

From (**??**), $\|x_n\| \leq \frac{1}{\lambda_n}\|y\| \leq C$ for some $C > 0$. Now

$$\begin{aligned}\lambda_m \|x_n - x_m\| &\leq \|\lambda_m(x_n - x_m) - A(x_n - x_m)\| \\ &= \| - \lambda_m x_n + \lambda_m x_m - \lambda_n x_n + \lambda_n x_n - Ax_n + Ax_m\| \\ &= |\lambda_n - \lambda_m| \|x_n\| \leq C|\lambda_n - \lambda_m|\end{aligned}$$

Thus, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $x_n \to x$, then $Ax_n \to \lambda x - y$. Since A is closed, $x \in D(A)$ and $\lambda x - Ax = y$. Thus, for this λ, $Im(\lambda I - A)D(A) = X$ and $\lambda \in \Lambda$. Thus Λ is also closed in $(0, \infty)$ and since $\lambda_0 \in \Lambda$, $\Lambda \neq \emptyset$ and thus $\Lambda = (0, \infty)$ (as the latter is connected). Thus, the thesis follows from the Hille-Yosida theorem.

On the other hand, if A is the generator of a semigroup of contractions $(G(t))_{t\geq 0}$, then $(0, \infty) \subset \rho(A)$ and $Im(\lambda I - A)D(A) = X$ for all $\lambda > 0$. Furthermore, if $x \in D(A), x^* \in \mathcal{J}(x)$, then

$$| < G(t)x, x^* > | \leq \|G(t)x\| \|x^*\| \leq \|x\|^2$$

and therefore

$$\Re < G(t)x - x, x^* >= \Re < G(t)x, x^* > - \|x\|^2 \leq 0$$

and, dividing the left hand side by t and passing with $t \to \infty$, we obtain

$$< Ax, x^* > \leq 0.$$

Since this holds for every $x^* \in \mathcal{J}(x)$, the proof is complete.

Adjoint operators

Before we move to an important corollary, let as recall the concept of the adjoint operator. If $A \in \mathcal{L}(X, Y)$, then the adjoint operator A^* is defined as

$$<y^*, Ax> = <A^* y^*, x> \tag{2.52}$$

and it can be proved that it belongs to $\mathcal{L}(Y^*, X^*)$ with $\|A^*\| = \|A\|$. If A is an unbounded operator, then the situation is more complicated. In general, A^* may not exist as a single-valued operator. In other words, there may be many operators B satisfying

$$<y^*, Ax> = <By^*, x>, \qquad x \in D(A),\ y^* \in D(B). \tag{2.53}$$

Operators A and B satisfying (2.53) are called *adjoint to each other*.

However, if $D(A)$ is dense in X, then there is a unique maximal operator A^* adjoint to A; that is, any other B such that A and B are adjoint to each other, must satisfy $B \subset A^*$. This A^* is called the *adjoint operator* to A. It can be constructed in the following way. The domain $D(A^*)$ consists of all elements y^* of Y^* for which there exists $f^* \in X^*$ with the property

$$<y^*, Ax> = <f^*, x> \tag{2.54}$$

for any $x \in D(A)$. Because $D(A)$ is dense, such element f^* can be proved to be unique and therefore we can define $A^* y^* = f^*$. Moreover, the assumption $\overline{D(A)} = X$ ensures that A^* is a closed operator though not necessarily densely defined. In reflexive spaces the situation is better: if both X and Y are reflexive, then A^* is closed and densely defined with

$$\overline{A} = (A^*)^*; \tag{2.55}$$

see [**?**, Theorems III.5.28, III.5.29].

Corollary 2.19. *Let A be a densely defined closed linear operator. If both A and A^* are dissipative, then A is the generator of a C_0-semigroup of contractions on X.*

Proof. It suffices to prove that, e.g., $Im(I - A) = X$. Since A is dissipative and closed, $Im(\lambda I - A)$ is a closed subspace of X. Indeed, if $y_n \to y$, $y_n \in Im(I - A)$, then, by dissipativity, $\|x_n - x_m\| \le \|(x_n - x_m) - (Ax_n - Ax_m)\| = \|y_n - y_m\|$ and $(x_n)_{n\in\mathbb{N}}$ converges. But then $(Ax_n)_{n\in\mathbb{N}}$ converges and, by closedness, $x \in D(A)$ and $x - Ax = y \in Im(I - A)$. Assume $Im(I - A) \neq X$, then by H-B theorem, there is $0 \neq x^* \in X^*$ such that $< x^*, x - Ax >= 0$ for all $x \in D(A)$. But then $x^* \in D(A^*)$ and, by density of $D(A)$, $x^* - A^*x^* = 0$ but dissipativeness of A^* gives $x^* = 0$.

The Cauchy problem for the heat equation

Let $C = \Omega \times (0, \infty), \Sigma = \partial\Omega \times (0, \infty)$ where Ω is an open set in $\mathbb{R}^n$. We consider the problem

$$\partial_t u = \Delta u, \quad \text{in} \Omega \times [0, T], \tag{2.56}$$
$$u = 0, \quad \text{on} \Sigma, \tag{2.57}$$
$$u = u_0, \quad \text{on} \Omega. \tag{2.58}$$

Theorem 2.20. *Assume that $u_0 \in L_2(\Omega)$ where Ω is bounded and has a C^2 boundary. Then there exists a unique function u satisfying (2.58)–(1.30) such that $u \in C([0, \infty); L_2(\Omega)) \cap C([0, \infty); W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega))$,*

Proof. The strategy is to consider (2.58–(1.30) as the abstract Cauchy problem

$$u' = Au, \quad u(0) = u_0$$

in $X = L_2(\Omega)$ where A is the unbounded operator defined by

$$Au = \Delta u$$

for

$$u \in D(A) = \{u \in \overset{\circ}{W}{}_2^1(\Omega); \Delta u \in L_2(\Omega)\} = W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)).$$

First we observe that A is densely defined as $C_0^\infty(\Omega) \subset \overset{\circ}{W}{}_2^1(\Omega)$ and $\Delta C_0^\infty(\Omega) \subset L_2(\Omega)$. Next, A is dissipative. For $u \in L_2(\Omega)$, $\mathcal{J}u = u$ and

$$(Au, u) = -\int\limits_\Omega |\nabla u|^2 d\mathbf{x} \leq 0$$

Further, we consider the variational problem associated with $I - A$, that is, to find $u \in \overset{\circ}{W}{}_2^1(\Omega)$ to

$$a(u, v) = \int\limits_\Omega \nabla u \nabla v d\mathbf{x} + \int\limits_\Omega uv d\mathbf{x} = \int\limits_\Omega fv d\mathbf{x}, \quad v \in \overset{\circ}{W}{}_2^1(\Omega)$$

where $f \in L_2(\Omega)$ is given. Clearly, $a(u, u) = \|u\|_{1,\Omega}^2$ and thus is coercive. Hence there is a unique solution $u \in \overset{\circ}{W}{}_2^1$ which, by writing

$$\int\limits_\Omega \nabla u \nabla v d\mathbf{x} = \int\limits_\Omega fv d\mathbf{x} - \int\limits_\Omega uv d\mathbf{x} = \int\limits_\Omega (f - u)v d\mathbf{x},$$

can be shown to be in $W_2^2(\Omega)$. This ends the proof of generation.

If we wanted to use the Hille-Yosida theorem instead, then to find the resolvent, we would have to solve

$$a(u,v) = \int_\Omega \nabla u \nabla v d\mathbf{x} + \lambda \int_\Omega uv d\mathbf{x} = \int_\Omega fv d\mathbf{x}, \quad v \in \overset{\circ}{W}{}^1_2(\Omega)$$

for $\lambda > 0$. The procedure is the same and we get in particular for the solution

$$\|\nabla u_\lambda\|^2_{0,\Omega} + \lambda \|u_\lambda\|^2_{0,\Omega} \leq \|f\|_{0,\Omega} \|u_\lambda\|_{0,\Omega}.$$

Since $u_\lambda = R(\lambda, A)f$ we obtain

$$\lambda \|R(\lambda, A)f\|_{0,\Omega} \leq \lambda^{-1} \|f\|_{0,\Omega}.$$

Closedness follows from continuous invertibility.