2

An Overview of Semigroup Theory

In this chapter we are concerned with methods of finding solutions of the Cauchy problem.

Definition 2.1. *Given a Banach space and a linear operator $\mathcal{A}$ with domain $D(\mathcal{A})$ and range $Im\mathcal{A}$ contained in X and also given an element $u_0 \in X$, find a function $u(t) = u(t, u_0)$ such that*

1. *$u(t)$ is continuous on $[0, \infty)$ and continuously differentiable on $(0, \infty)$,*
2. *for each $t > 0$, $u(t) \in D(\mathcal{A})$ and*
$$u'(t) = \mathcal{A}u(t), \quad t > 0, \tag{2.1}$$
3. $$\lim_{t \to 0^+} u(t) = u_0 \tag{2.2}$$
in the norm of X.

A function satisfying all conditions above is called the classical (or strict) solution of (2.1), (2.2).

2.1 What the semigroup theory is about

In the theory of differential equations, one of the first differential equations encountered is
$$u'(t) = \alpha u(t), \qquad \alpha \in \mathbb{C} \tag{2.3}$$
with initial condition $u(0) = u_0$. It is not difficult to verify that $u(t) = e^{t\alpha}u_0$ is a solution of Eq. (2.3).

As early as in 1887, G.P. Peano showed that the system of linear ordinary differential equations with constant coefficients

$$\begin{aligned} u_1' &= \alpha_{11}u_1 + \cdots + \alpha_{1n}u_n, \\ &\vdots \\ u_n' &= \alpha_{n1}u_1 + \cdots + \alpha_{nn}u_n, \end{aligned} \tag{2.4}$$

can be written in a matrix form as

$$u'(t) = Au(t), \tag{2.5}$$

where A is an $n \times n$ matrix $\{\alpha_{ij}\}_{1\le i,j\le n}$ and u is an n-vector whose components are unknown functions, and can be solved using the explicit formula

$$u(t) = e^{tA}u_0, \tag{2.6}$$

where the matrix exponential e^{tA} is defined by

$u_t(x,t) = \int_\Omega k(x,y)u(y,t)\,dy$

$$e^{tA} = I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \cdots . \tag{2.7}$$

Taking a norm on $\mathbb{C}^n$ and the corresponding matrix-norm on $M_n(\mathbb{C})$, the space of all complex $n \times n$ matrices, one shows that the partial sums of the series (2.7) form a Cauchy sequence and converge. Moreover, the map $t \to e^{tA}$ is continuous and satisfies the properties, [79, Proposition I.2.3]:

$u_t = \Delta u$

$x(t, x_0)$

$$\begin{aligned} e^{(t+s)A} &= e^{tA}e^{sA} \qquad \text{for all } t,s \ge 0 \\ e^{0A} &= I. \end{aligned} \tag{2.8}$$

$x(t+s, x_0)$

$= x(t, x(s,x_0))$

Thus the one-parameter family $\{e^{tA}\}_{t\ge0}$ is a homomorphism of the additive semigroup $[0,\infty)$ into a multiplicative semigroup of matrices M_n and forms what is termed a semigroup of matrices.

The representation (2.7) can be used to obtain a solution of the abstract Cauchy problem (2.1-2.2) where $\mathcal{A} : X \to X$ is a bounded linear operator, as in this case the series in (2.7) is still convergent with respect to the norm in the space of linear operators $\mathcal{L}(X)$.

In general, however, the operators coming from applications, such as, for example, differential operators, are not bounded on the whole space X and (2.7) cannot be used to obtain a solution of the abstract Cauchy problem (**??**). This is due to the fact that the domain of the operator $\mathcal{A}$ in such cases is a proper subspace of X and because (2.7) involves iterates of A, their common domain could shrink to the trivial subspace $\{0\}$. For the same reason, another common representation of the exponential function

$$e^{t\mathcal{A}} = \lim_{n\to\infty}\left(1 + \frac{t}{n}\mathcal{A}\right)^n, \tag{2.9}$$

cannot be used. For a large class of unbounded operators a variation of the latter, however, makes the representation (2.6) meaningful with $e^{t\mathcal{A}}$ calculated according to the formula

$$e^{t\mathcal{A}}x = \lim_{n\to\infty}\left(I - \frac{t}{n}\mathcal{A}\right)^{-n}x = \lim_{n\to\infty}\left[\frac{n}{t}\left(\frac{n}{t} - \mathcal{A}\right)^{-1}\right]^n x. \tag{2.10}$$

The aim of the semigroup theory is to find conditions under which such a generalization of the exponential function satisfying (2.8) is possible.

2.2 Rudiments

2.2.1 Definitions and Basic Properties

If the solution to (2.1), (2.2) is unique, then we can introduce the family of operators $(G(t))_{t\geq 0}$ such that $u(t,u_0) = G(t)u_0$. Ideally, $G(t)$ should be defined on the whole space for each $t > 0$, and the function $t \to G(t)u_0$ should be continuous for each $u_0 \in X$, leading to well-posedness of (2.1), (2.2). Moreover, uniqueness and linearity of $\mathcal{A}$ imply that $G(t)$ are linear operators. A fine-tuning of these requirements leads to the following definition.

Definition 2.2. *A family $(G(t))_{t\geq 0}$ of bounded linear operators on X is called a C_0-semigroup, or a strongly continuous semigroup, if*

(i) $G(0) = I$;
(ii) $G(t+s) = G(t)G(s)$ *for all* $t, s \geq 0$;
(iii) $\underbrace{\lim_{t\to 0^+} G(t)x = x}$ *for any* $x \in X$.

A linear operator A is called the (infinitesimal) generator of $(G(t))_{t\geq 0}$ if

$$Ax = \lim_{h\to 0^+} \frac{G(h)x - x}{h}, \tag{2.11}$$

with $D(A)$ defined as the set of all $x \in X$ for which this limit exists. If we need to use differen generators, then typically the semigroup generated by A will be denoted by $(G_A(t))_{t\geq 0}$, otherwise simply by $(G(t))_{t\geq 0}$.

Why C_0-semigroups?

continuos as $t \to 0^+$

Proposition 2.3. *If $(G(t))_{t\geq 0}$ is uniformly ~~bounded~~, then its generator is bounded.*

Proof. Since $\rho^{-1}\int_0^\rho G(s)ds \to I$ in the uniform operator norm, then there is $\rho > 0$ such that $\|\rho^{-1}\int_0^\rho G(s)ds - I\| < 1$ and thus $\rho^{-1}\int_0^\rho G(s)ds$ and hence $\int_0^\rho G(s)ds$ are invertible.

$A = -I + A + I$

$I + (A - I)$

$$\frac{G(h)-I}{h}\int_0^\rho G(s)ds = \frac{1}{h}\int_0^\rho (G(s+h) - G(s))ds \quad = \frac{1}{h}\int_0^\rho G(h+s)ds - \frac{1}{h}\int_0^\rho G(s)ds$$

$$= \frac{1}{h}\int_\rho^{\rho+h} G(s)\cancel{x}ds - \frac{1}{h}\int_0^h G(s)\cancel{x}ds$$

$$= \frac{1}{h}\int_h^{\rho+h} G(s)ds - \int_0^\rho$$

Thus

$$\int_0^\rho G(s) - \int_0^h G(s)ds - \int_\rho^{\rho+h}$$

$$\frac{G(h)-I}{h} = \left(\frac{1}{h}\int\limits_{\rho}^{\rho+h} G(s)\cancel{x}ds - \frac{1}{h}\int\limits_{0}^{h} G(s)\cancel{x}ds\right)\left(\int\limits_{0}^{\rho} G(s)ds\right)^{-1}.$$

Letting $h \to 0$, we see that $(G(h)-I)/h \to (G(\rho)-I)(\int_0^\rho G(s)ds)^{-1}$ in the uniform norm and thus the generator is bounded.

Proposition 2.4. *Let $(G(t))_{t\geq 0}$ be a C_0-semigroup. Then there are constants $\omega \geq 0$, $M \geq 1$ such that*

$$\|G(t)\| \leq Me^{\omega t}, \qquad t \geq 0. \tag{2.12}$$

$\forall_x \exists_{M_x} \forall_n \|A_n x\| \leq M_x$

$\Downarrow$

$\exists_M \forall_n \|A_n\| \leq M$

Proof. First we observe that $\|G(t)\|$ is bounded on some interval. Indeed, if not, there is $(t_n)_{n\in\mathbb{N}}$, $t_n \to 0$, $\|G(t_n)\| \geq n$, that is $(G(t_n))$ is unbounded. But, by the Banach-Steinhaus theorem there is an $x \in X$ and a subsequence $(t_{n_k})_{n_k\in\mathbb{N}}$ such that $(G(t_{n_k})x)$ is unbounded, contrary to (iii). So, $\|G(t)\| \leq M$ for $0 \leq t \leq \eta$ for some η and $M \geq 1$ as $G(0) = I$. For any $t \geq 0$ we take $t = n\eta + \delta$, $0 \leq \delta < \eta$ and, by the semigroup property,

$$\|G(t)\| = \|G(\delta)(G(\eta))^n\| \leq MM^n = Me^{(t-\delta)\ln M/\eta} \leq Me^{\omega t}$$

where $\omega = \eta^{-1}\ln M \geq 0$.

As a corollary, we have

Corollary 2.5. *Let $(G(t))_{t\geq 0}$ be a C_0-semigroup. Then for every $x \in X$, $t \to G(t)x \in C(\mathbb{R}_+ \cup \{0\}, X)$.*

Proof. We have for $t, h \geq 0$

$$\|G(t+h)x - G(t)x\| \leq \|G(t)\|\|G(h)x - x\| \leq Me^{\omega t}\|G(h)x - x\|$$

and for $t \geq h \geq 0$

$$\|G(t-h)x - G(t)x\| \leq \|G(t-h)\|\|G(h)x - x\| \leq Me^{\omega t}\|G(h)x - x\|$$

and the statement follows from condition (iii).

Remark 2.6. As we have seen above, for semigroups, the existence of a one-sided limit at some $t_0 > 0$ yields the existence of the limit.

Let $(G(t))_{t\geq 0}$ be a semigroup generated by the operator A. The following properties of $(G(t))_{t\geq 0}$ are frequently used.

Lemma 2.7. *Let $(G(t))_{t\geq 0}$ be a C_0-semigroup generated by A.*

(a) For $x \in X$

$$\lim_{h\to 0}\frac{1}{h}\int\limits_{t}^{t+h} G(s)xds = G(t)x. \tag{2.13}$$

(b) For $x \in X$, $\int_0^t G(s)xds \in D(A)$ and

$$A\int\limits_0^t G(s)xds = G(t)x - x. \tag{2.14}$$

(c) For $x \in D(A)$, $G(t)x \in D(A)$ and

$$\frac{d}{dt}G(t)x = AG(t)x = G(t)Ax. \tag{2.15}$$

(d) For $x \in D(A)$,

$$G(t)x - G(s)x = \int\limits_s^t G(\tau)Axd\tau = \int\limits_s^t AG(\tau)xd\tau. \tag{2.16}$$

Proof. (a) follows from continuity of the semigroup. To prove (b) we consider $x \in X$ and $h > 0$. Then

$$\begin{aligned} \frac{G(h)-I}{h}\int\limits_0^t G(s)xds &= \frac{1}{h}\int\limits_0^t (G(s+h)x - G(s)x)ds \\ &= \frac{1}{h}\int\limits_t^{t+h} G(s)xds - \frac{1}{h}\int\limits_0^h G(s)xds \end{aligned}$$

and the right hand side tends to $G(t)x - x$ by (a) which proves that $\int\limits_0^t G(s)xds \in D(A)$ and (2.14). To prove (c), let $x \in D(A)$ and $h > 0$. As above

$$\frac{G(h)-I}{h}G(t)x = G(t)\left(\frac{G(h)-I}{h}\right)x \to G(t)Ax$$

as $h \to 0$. Thus, $G(t)x \in D(A)$ and $AG(t)x = G(t)Ax$ for $x \in D(A)$. The limit above also shows that

$$\frac{d^+}{dt}G(t)x = AG(t)x = G(t)Ax,$$

that is, the right derivative of $G(t)x$ is $AG(t)$. Take now $t > 0$ and $h \le t$. Then

$$\begin{aligned} &\lim_{h\to 0}\left(\frac{G(t-h)x - G(t)x}{-h} - AG(t)x\right) \\ &\lim_{h\to 0} G(t-h)\left(\frac{G(h)x - x}{h} - Ax\right) + \lim_{h\to 0}(G(t-h)Ax - G(t)Ax) \end{aligned}$$

and we see that both limits are 0 by uniform boundedness of $(G(t))_{t\ge 0}$, strong continuity and $x \in D(A)$.

Part (d) is obtained by integrating (2.15).

From (2.15) and condition (iii) of Definition 2.2 we see that if A is the generator of $(G(t))_{t\geq 0}$, then for $x \in D(A)$ the function $t \to G(t)x$ is a classical solution of the following Cauchy problem,

$$\partial_t u(t) = A(u(t)), \quad t > 0, \tag{2.17}$$

$$\lim_{t\to 0^+} u(t) = x. \tag{2.18}$$

We note that ideally the generator A should coincide with $\mathcal{A}$ but in reality very often it is not so.

Remark 2.8. We noted above that for $x \in D(A)$ the function $u(t) = G(t)x$ is a classical solution to (2.17), (2.18). For $x \in X \setminus D(A)$, however, the function $u(t) = G(t)x$ is continuous but, in general, not differentiable, nor $D(A)$-valued, and, therefore, not a classical solution. Nevertheless, from (2.14), it follows that the integral $v(t) = \int_0^t u(s)ds \in D(A)$ and therefore it is a strict solution of the integrated version of (2.17), (2.18):

$$\begin{aligned} \partial_t v &= Av + x, \quad t > 0 \\ v(0) &= 0, \end{aligned} \tag{2.19}$$

or equivalently,

$$u(t) = A\int_0^t u(s)ds + x. \tag{2.20}$$

We say that a function u satisfying (2.19) (or, equivalently, (2.20)) is a *mild solution* or *integral solution* of (2.17), (2.18).

Corollary 2.9. *If $(G(t))_{t\geq 0}$ is a C_0-semigroup generated by A, then A is a closed densely defined linear operator.*

Proof. For $x \in X$ we set $x_t = t^{-1}\int_0^t G(s)xds$. By (b), $x_t \in D(A)$ and by (a), $x_t \to x$ as $t \to 0$. To prove closedness, let $D(A) \ni x_n \to x \in X$ and let $Ax_n \to y \in X$. From (d) we have

$$G(t)x_n - x_n = \int_0^t G(s)Ax_n ds.$$

By local boundedness of $(G(t))_{t\geq 0}$ we have that $G(s)Ax_n \to G(s)y$ uniformly on bounded intervals, hence, by letting $n \to \infty$,

$$\underbrace{G(t)x - x}_{\in} = \int_0^t G(s)yds.$$

Thus, using (a), $x \in D(A)$ and $Ax = y$.

Thus, if we have a semigroup, we can identify the Cauchy problem of which it is a solution. Usually, however, we are interested in the reverse question, that is, in finding the semigroup for a given equation.

A first step in this direction is

Theorem 2.10. *Let $(G_A(t))_{t\geq 0}$ and $(G_B(t))_{t\geq 0}$ be C_0 semigroups generated by, respectively, A and B. If $A = B$, then $G_A(t) = G_B(t)$.*

Proof. Let $x \in D(A) = D(B)$. Consider the function

$$s \to G_A(t-s)G_B(s)x, \quad 0 \leq s \leq t$$

is continuous on $[0,t]$. Writing, for appropriate s, h

$$\begin{aligned}
&\frac{G_A(t-(s+h))G_B(s+h)x - G_A(t-s)G_B(s)x}{h} \\
&= \frac{G_A(t-(s+h))G_B(s+h)x - G_A(t-(s+h))G_B(s)x}{h} \\
&\quad + \frac{G_A(t-(s+h))G_B(s)x - G_A(t-s)G_B(s)x}{h}
\end{aligned}$$

$= G_A(t-(s+h))\left(\frac{G_B(s+h)x - G_B(s)x}{h}\right)$

$G(t)x$

we see that by local boundedness both terms converge and, by (c), we obtain

B

$$\begin{aligned}
\frac{d}{ds}G_A(t-s)G_B(s)x &= -\cancel{A}G_A(t-s)G_B(s)x + G_A(t-s)\cancel{A}G_B(s)x \\
&= -G_A(t-s)AG_B(s)x + G_A(t-s)BG_B(s)x = 0.
\end{aligned}$$

Thus $G_A(t-s)G_B(s)x$ is constant and, in particular, evaluating at $s = 0$ and $s = t$ we get $G_A(t)x = G_B(t)x$ for any t and $x \in D(A)$. From density, we obtain the equality on X.

The final answer is given by the Hille–Yosida theorem (or, more properly, the Feller–Miyadera–Hille–Phillips–Yosida theorem). Before, however, we need to discuss the concept of resolvent.

$\left(\left(I - \frac{t}{n}A\right)^{-1}\right)^n$

Let A be any operator in X. The *resolvent set* of A is defined as

$$\rho(A) = \{\lambda \in \mathbb{C};\ \lambda I - A : D(A) \to X \text{ is invertible}\}. \tag{2.21}$$

We call $(\lambda I - A)^{-1}$ the resolvent of A and denote it by $R(\lambda, A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$. The complement of $\rho(A)$ in $\mathbb{C}$ is called the *spectrum* of A and denoted by $\sigma(A)$.

$y = (\lambda I - A)x_\lambda$

$y = (\mu I - A)x_\mu$

The resolvent of any operator A satisfies the *resolvent identity*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A), \qquad \lambda, \mu \in \rho(A), \tag{2.22}$$

✓

$x_\lambda = R(\lambda, A)y$

$x_\mu = R(\mu, A)y$

from which it follows, in particular, that $R(\lambda, A)$ and $R(\mu, A)$ commute. Writing

$$R(\mu, A) = R(\lambda, A)(I - (\mu - \lambda)R(\mu, A))$$

$R(\lambda, A) = (I - (\mu-\lambda)R(\mu, A))^{-1} R(\mu, A)$

$(\lambda I - A)x_\lambda = (\mu I - A)x_\mu$

$x_\lambda = (\lambda I - A)^{-1}(\mu I - A)x_\mu =$

$= (\lambda I - A)^{-1}(\lambda I - A + (\mu - \lambda)I)x_\mu$

$= x_\mu + R(\lambda)[(\mu - \lambda)I]x_\mu = (\mu - \lambda)R(\lambda, A)R(\mu, A)$

we see by the Neuman expansion that $R(\lambda, A)$ can be written as the power series

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^{n+1} \tag{2.23}$$

for $|\mu - \lambda| < \|R(\mu, A)\|^{-1}$ so that $\rho(A)$ is open and $\lambda \to R(\lambda, A)$ is an analytic function in $\rho(A)$. The iterates of the resolvent and its derivatives are related by

$$\frac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1}. \tag{2.24}$$

$\|G(t)\| \le Me^{\omega t}$

2.2.2 The Hille–Yosida Theorem

We begin with the simplest case of contractive semigroups. A C_0 semigroup $(G_A(t))_{t\geq 0}$ is called contractive if

$$\|G_A(t)\| \leq 1$$

$u' = Au$

$\lambda u = Au + f$

Theorem 2.11. *A is the generator of a contractive semigroup $(G_A(t))_{t\geq 0}$ if and only if*

(a) A is closed and densely defined,
(b) $(0, \infty) \subset \rho(A)$ and for all $\lambda > 0$,

$R(\lambda, A) = (\lambda I - A)^{-1}$

$(\lambda I - A)u = f \quad u = (\lambda I - A)^{-1} f$

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}. \tag{2.25}$$

Proof. (Necessity) If A is the generator of a C_0 semigroup $(G_A(t))_{t\geq 0}$, then it is densely defined and closed. Let us define

$u' = \alpha u$

$Au = \alpha u$

$$R(\lambda)x = \int_0^{\infty} e^{-\lambda t} G(t) x dt \tag{2.26}$$

$\boxed{\frac{1}{\lambda - \alpha} = R(\lambda, A)}$

is valid for all $x \in X$. Since $(G_A(t))_{t\geq 0}$ is contractive, the integral exists for $\lambda > 0$ as an improper Riemann integral and defines a bounded linear operator $R(\lambda)x$ (by the Banach-Steinhaus theorem). $R(\lambda)$ satisfeis

$\lambda u - \alpha u = f$

$$\|R(\lambda)x\| \leq \frac{1}{\lambda}\|x\|.$$

Furthermore, $h > 0$,

$$\begin{aligned}
\frac{G_A(h) - I}{h} R(\lambda)x &= \frac{1}{h} \int_0^{\infty} e^{-\lambda t}(G_A(t+h)x - G_A(t)x)dt \\
&= \frac{1}{h}\left(\int_h^{\infty} e^{-\lambda(t-h)} G_A(t)xdt - \int_0^{\infty} e^{-\lambda t} G_A(t)xdt \right) \\
&= \frac{e^{\lambda h} - 1}{h} \int_h^{\infty} e^{-\lambda t} G_A(t)xdt - \frac{e^{\lambda h}}{h} \int_0^{h} e^{-\lambda t} G_A(t)xdt
\end{aligned}$$

$G_\alpha(t) = e^{t\alpha}$

$\alpha \leq 1$

$u'(t) = \alpha u(t)$

$\lambda \hat{u} = \alpha \hat{u} + \mathring{u}$

$\hat{u} = \frac{\mathring{u}}{\lambda - \alpha}$

$\frac{e^{\lambda h}}{h}\int_0^h \quad -\frac{e^{\lambda h}}{h}\int_0^h$

A - lin $\quad \int_0^n f \in D(A) \quad \cancel{f \in} A \int_0^n A f <$

$A\int_0^\infty = \int_0^\infty A \quad \int_0^\infty \in D(A)$

$\int_0^n f \to \int_0^\infty f$

$A\int_0^n f = \int_0^n Af \xrightarrow{\infty} \int Af$

$\int_0^\infty \in D(A)$

By strong continuity of G_A, the right hand side converges to $\lambda R(\lambda)x - x$. This implies that for any $x \in D(A)$ and $\lambda > 0$ we have $R(\lambda)x \in D(A)$ and $AR(\lambda) = \lambda R(\lambda) - I$ so

$$(\lambda I - A)R(\lambda) = I. \tag{2.27}$$

$\int_0^n G^{-1} \in D(A)$

$\int_0^t G(s)x\,ds \in D(A)$

On the other hand, for $x \in D(A)$ we have

$$R(\lambda)Ax = \int_0^\infty e^{-\lambda t}G(t)Ax dt = A\left(\int_0^\infty e^{-\lambda t}G(t)x\right) dt = AR(\lambda)x$$

by commutativity (Lemma 2.7 (c)) and closedness of A. Thus A and $R(\lambda)$ commute and

$$R(\lambda)(\lambda I - A)x = \cancel{A}x \qquad \lambda R(\lambda)x = x + AR(\lambda)x$$

on $D(A)$. Thus $R(\lambda)$ is the resolvent of A and satisfies the desired estimate.

The converse is more difficult to prove. The starting point of the second part of the proof is the observation that if $(A, D(A))$ is a closed and densely defined operator satisfying $\rho(A) \supset (0, \infty)$ and $\|\lambda R(\lambda, A)\| \le 1$ for all $\lambda > 0$, then

(i) for any $x \in X$,
$$\lim_{\lambda\to\infty} \lambda R(\lambda, A)x = x. \tag{2.28}$$
Indeed, first consider $x \in D(A)$. Then
$$\|\lambda R(\lambda, A)x - x\| = \|AR(\lambda, A)x\| = \|R(\lambda, A)Ax\| \le \frac{1}{\lambda}\|Ax\| \to 0$$
as $\lambda \to \infty$. Since $D(A)$ is dense and $\|\lambda R(\lambda, A)\| \le 1$ then by 3ϵ argument we extend the convergence to X.

(ii) $AR(\lambda, A)$ are bounded operators and for any $x \in D(A)$,
$$\lim_{\lambda\to\infty} \lambda AR(\lambda, A)x = Ax. \tag{2.29}$$
Boundedness follows from $AR(\lambda, A) = \lambda R(\lambda, A) - I$. Eq. (2.29) follows (2.28).

It was Yosida's idea to use the bounded operators

$$A_\lambda = \lambda AR(\lambda, A), = \lambda^2 R(\lambda, A) - \lambda I \tag{2.30}$$

$G_\lambda(t)x = e^{tA_\lambda x}$

as an approximation of A for which we can define semigroups uniformly continuous semigroups $(G_\lambda(t))_{t\ge0}$ via the exponential series. First we note that $(G_\lambda(t))_{t\ge0}$ are semigroups of contractions and, for any $x \in X$ and $\lambda, \mu > 0$ we have

$$\|G_\lambda(t)x - G_\mu(t)x\| \le t\|A_\lambda x - A_\mu x\|. \tag{2.31}$$

Indeed, using $A_\lambda = \lambda^2 R(\lambda, A) - \lambda I$ and the series estimates

$$\|e^{tA}\| = \left\|\sum \frac{A^n t^n}{n!}\right\| \le e^{\|A\|t}$$

$\|G_\lambda(t)x\| = \|e^{tA_\lambda}\| = e^{-t\lambda} e^{t\lambda^2 R(\lambda,A)}$

$\|R(\lambda, A)\| \le \frac{1}{\lambda}$

$$\|G_\lambda x\| \le e^{-\lambda t} e^{\lambda \|R(\lambda,A)\| t} \le 1.$$

Further, from the definition operators $G_\lambda(t), G_\mu(t), A_\lambda, A_\mu$ commute with each other. Then

e^{ts}

$$\|G_\lambda(t)x - G_\mu(t)x\| = \left\| \int_0^1 \frac{d}{ds} e^{tsA_\lambda} e^{t(1-s)A_\mu} x ds \right\|$$
$$\le t \int_0^1 \|e^{tsA_\lambda} e^{t(1-s)A_\mu}(A_\lambda x - A_\mu x)\| ds \le t\|A_\lambda x - A_\mu x\|.$$

Using (2.31) we obtain for $x \in D(A)$

$$\|G_\lambda(t)x - G_\mu(t)x\| \le t\|A_\lambda x - A_\mu x\| \le t(\|A_\lambda x - Ax\| + \|Ax - A_\mu x\|).$$

Hence $(G_\lambda(t)x))_\lambda$ ~~strongly~~ converges and the convergence (for each x) is uniform in t on bounded intervals (almost uniform on $\overline{\mathbb{R}_+}$. Since $D(A)$ is dense in X and $\|G_\lambda(t)\| \le 1$ we get

$G_\lambda(t+s)x \to S(t+s)$

$G_\lambda(t)G(s)x \to S(t)S(s)x$

$x \in D(A)$

$G(t)x \to S(t)x$

$$\lim_{\lambda\to\infty} G_\lambda(t)x =: S(t)x$$

$G_\lambda(0)x = Ix$

for $x \in X$. The convergence is still almost uniform on $\overline{\mathbb{R}_+}$. From the limit we see that $(S(t))_{t\geq 0}$ is a C_0 semigroup of contractions.

$\|G(t)y - S(t)y\|$

$y \in X$

What remains is to show that $(S(t))_{t\geq 0}$ is generated by A. Let $x \in D(A)$. Then

$\|G(t)y - G(t)x\| \le \|G(t)\| \|y - x\|$

$+ \|G(t)x -$

$+ \|S(t)x$

$$\frac{G(t)x - x}{t} = \lim_{\lambda\to\infty} (G_\lambda(t)x - x) = \lim_{\lambda\to\infty} \int_0^t e^{sA_\lambda} A_\lambda x ds = \frac{1}{t}\int_0^t G(s)Axds \quad (2.32)$$

Ax

where the last equality follows from

$$\|e^{sA_\lambda} A_\lambda x - G(s)Ax\| \le \|e^{sA_\lambda} A_\lambda x - e^{sA_\lambda} Ax\| + \|e^{sA_\lambda} Ax - G(s)Ax\|$$
$$\le \|A_\lambda x - Ax\| + \|e^{sA_\lambda} Ax - G(s)Ax\|,$$

by contractivity of $(G_\lambda(t))_{t\geq 0}$, so that convergence is uniform on bounded intervals. Assume now that $(G(t))_{t\geq 0}$ is generated by B. Dividing (2.32) by t and passing to the limit, we obtain

$$Bx = Ax, \quad x \in D(A)$$

so that $A \subset B$. On the other hand, we know that $I - A$ and $I - B$ are bijections from, resp $D(A)$ and $D(B)$ with $D(A) \subset D(B)$. But then we have $(I - B)D(A) = (I - A)D(A) = X$, that is, $D(A) = (I - B)^{-1}X = D(B)$ so $A = B$.