Proof. If Ω is bounded then, using Theorem 1.45, we can extend u to a function $Eu \in W_2^1(\mathbb{R}^n)$ with bounded support. The existence of a $C_0^\infty(\mathbb{R}^n)$ sequence converging to u follows from the Friedrichs lemma. If Ω is unbounded (but not equal to $\mathbb{R}^n$), then first we approximate u by a sequence $(\chi_n u)_{n\in\mathbb{N}}$ where χ_n are cut-off functions. Next we construct an extension of $\chi_n u$ to $\mathbb{R}^n$. This is possible as it involves only the part of $\partial\Omega$ intersecting the ball $B(0, 2n+1)$ and χ_n is equal to zero where the sphere intersects $\partial\Omega$. For this extension we pick up an approximating function from $C_0^\infty(\mathbb{R}^n)$.

1.4 Basic applications of the density theorem

1.4.1 Sobolev embedding

In Subsection 1.1.2 we have seen that in one dimension it is possible to identify a $W_2^1(\mathbb{R})$ function. Unfortunately, this is not true in higher dimensions.

Example 1.48. We can consider in $D = \{(x,y) \in \mathbb{R}^2;\ x^2 + y^2 < 1\}$

$$u(x,y) = \left|\frac{1}{2}\ln(x^2+y^2)\right|^{1/3} = (-\ln r)^{1/3}.$$

The function u is not continuous (even not bounded) at $(x,y) = (0,0)$. It is in $L_2(D)$ and for derivatives we have

$$u_x = -\frac{1}{3}(-\ln r)^{-2/3}\frac{x}{r^2}, \qquad u_y = -\frac{1}{3}(-\ln r)^{-2/3}\frac{y}{r^2}$$

and, since

$$\int\limits_D (u_x^2 + u_y^2)dxdy = \frac{2}{9}\int\limits_0^1 \frac{dr}{r(-\ln r)^{4/3}} = \frac{2}{9}\int\limits_1^\infty u^{-4/3}du < \infty$$

we see that $u \in W_2^1(D)$.

However, there is still a link between Sobolev spaces and classical calculus provided we take sufficiently high order of derivatives (or index p in L_p spaces). The link is provided by the Sobolev lemma.

Let Ω be an open and bounded subset of $\mathbb{R}^n$. We say that Ω satisfies the cone condition if there are numbers $\rho > 0$ and $\gamma > 0$ such that each $\mathbf{x} \in \Omega$ is a vertex of a cone $K(\mathbf{x})$ of radius ρ and volume $\gamma\rho^n$. Precisely speaking, if σ_n is the $n-1$ dimensional measure of the unit sphere in $\mathbb{R}^n$, then the volume of a ball of radius ρ is $\sigma_n\rho^n/n$ and then the (solid) angle of the cone is $\gamma n/\omega_n$.

Lemma 1.49. *If Ω satisfies the cone condition, then there exists a constant C such that for any $u \in C^m(\bar{\Omega})$ with $2m > n$ we have*

$$\sup_{\mathbf{x}\in\Omega} |u(\mathbf{x})| \leq C\|u\|_m \tag{1.61}$$

Proof. Let us introduce a cut-off function $\phi \in C_0^\infty(\mathbb{R})$ which satisfies $\phi(t) = 1$ for $|t| \leq 1/2$ and $\phi(t) = 0$ for $|t| \geq 1$. Define $\tau(t) = \phi(t/\rho)$ and note that there are constants A_k, $k = 1, 2, \ldots$ such that

$$\left|\frac{d^k\tau(t)}{dt^k}\right| \leq \frac{A_k}{\rho^k}. \tag{1.62}$$

Let us take $u \in C^m(\bar{\Omega})$ and assume $2m > n$. For $\mathbf{x} \in \bar{\Omega}$ and the cone $K(\mathbf{x})$ we integrate along the ray $\{\mathbf{x} + r\boldsymbol{\omega};\ 0 \leq r \leq \rho, |\boldsymbol{\omega}| = 1$

$$u(\mathbf{x}) = -\int\limits_0^\rho D_r(\tau(r)u(\mathbf{x} + r\boldsymbol{\omega}))dr.$$

Integrating over the surface Γ of the cone we get

$$\int\limits_\Gamma \int\limits_0^\rho D_r(\tau(r)u(\mathbf{x} + \boldsymbol{\omega}))drd\boldsymbol{\omega} = -u(\mathbf{x})\int\limits_C d\boldsymbol{\omega} = -u(\mathbf{x})\frac{\gamma n}{\omega_n}.$$

Next we integrate $m - 1$ times by parts, getting

$$u(\mathbf{x}) = \frac{(-1)^m\omega_n}{\gamma n(m-1)!}\int\limits_C \int\limits_0^\rho D_r^m(\tau(r)u(\mathbf{x} + r\boldsymbol{\omega}))r^{m-1}drd\boldsymbol{\omega}.$$

and changing to Cartesian coordinates and applying Cauchy-Schwarz inequality we obtain

$$\begin{aligned}|u(\mathbf{x})|^2 &\leq \left(\frac{\omega_n}{\gamma n(m-1)!}\int\limits_{K(\mathbf{x})} |D_r^m(\tau u)|r^{m-n}d\mathbf{y}\right)^2 \\ &\leq \left(\frac{\omega_n}{\gamma n(m-1)!}\right)^2 \int\limits_{K(\mathbf{x})} |D_r^m(\tau u)|^2 d\mathbf{y}d\mathbf{y} \int\limits_{K(\mathbf{x})} r^{2(m-n)}d\mathbf{y}.\end{aligned}$$

The last term can be evaluated as

$$\int\limits_{K(\mathbf{x})} r^{2(m-n)}d\mathbf{y} = \int\limits_C \int\limits_0^\rho r^{2m-n-1}drd\boldsymbol{\omega} = \frac{\gamma n\rho^{2m-n}}{\omega_n(2m-n)}$$

so that

$$|u(\mathbf{x})|^2 \leq C(m,n)\rho^{2m-n}\int\limits_{K(\mathbf{x})} |D_r^m(\tau u)|^2 d\mathbf{y}. \tag{1.63}$$

Let us estimate the derivative. From (1.62) we obtain by the chain rule and the Leibniz formula

$$|D_r^m(\tau u)| = \left|\sum_{k=0}^{m}\binom{n}{k}D_r^{m-k}\tau D_r^k u\right| \leq \sum_{k=0}^{m}\binom{n}{k}\frac{A_{m-k}}{\rho^{m-k}}\left|D_r^k u\right|,$$

hence

$$|D_r^m(\tau u)|^2 \leq C'\sum_{k=0}^{m}\frac{1}{\rho^{2(m-k)}}\left|D_r^k u\right|^2$$

for some constant C'. With this estimate we can re-write (1.63) as

$$|u(\mathbf{x})|^2 \leq C(m,n)C'\sum_{k=0}^{m}\rho^{2k-n}\int\limits_{K(\mathbf{x})}|D_r^m(u)|^2 d\mathbf{y}. \tag{1.64}$$

Since by the chain rule

$$|D_r^m u|^2 \leq C''\sum_{|\alpha|\leq k}|D^\alpha u|^2$$

by extending the integral to Ω we obtain

$$\sup_{\mathbf{x}\in\Omega}|u(\mathbf{x})| \leq C\|u\|_m$$

which is (1.61).

Theorem 1.50. *Assume that Ω is a bounded open set with C^m boundary and let $m > k + n/2$ where m and k are integers. Then the embedding*

$$W_2^m(\Omega) \subset C^k(\bar{\Omega})$$

is continuous.

Proof. Under the assumptions, the problem can be reduced to the set $G_0 \Subset \Omega$ consisting of internal point, separated from the boundary by a fixed positive distance, and points in the boundary strip, covered by sets $\bar{\Omega}\cap B_j$ which are transformed onto $Q_+ \cup Q_0$. Any point in G_0 satisfies the cone conditions. Points on $Q_0 \cup Q_+$ also satisfy the condition so, if $u \in W_2^m(\Omega)$, then extending the boundary components of Λu to Q we obtain functions in $W_2^1(\Omega)$ and $W_2^1(Q)$ with compact supports in respective domains. By Friedrichs lemma, restrictions to Ω and Q of $C^\infty(\mathbb{R}^n)$ functions are dense in, respectively, $W_2^m(\Omega)$ and $W_2^m(Q)$ and therefore the estimate (1.61) can be extended by density to $W_2^m(\Omega)$ showing that the canonical injection into $C(\bar{\Omega})$ is continuous. To obtain the result for higher derivatives we substitute higher derivatives of u for u in (1.61). Thus, all components of Λu are they are C^k functions. Transferring them back, we see that $u \in C^k(\bar{\Omega})$, by regularity of the local atlas and $m > k$, we obtain the thesis.

1.4.2 Compact embedding and Rellich–Kondraschov theorem

Lemma 1.51. *let $Q = \{\mathbf{x};\ a_j \le x_j \le b_j\}$ be a cube in $\mathbb{R}^n$ with edges of length $d > 0$. If $u \in C^1(\bar{Q})$, then*

$$\|u\|_{0,Q}^2 \le d^{-n}\left(\int\limits_Q u d\mathbf{x}\right)^2 + \frac{nd^2}{2}\sum_{j=1}^n \|\partial_{x_j} u\|_{0,Q}^2 \tag{1.65}$$

Proof. For any $\mathbf{x}, \mathbf{y} \in Q$ we can write

$$u(\mathbf{x}) - u(\mathbf{y}) = \sum_{j=1}^n \int\limits_{y_j}^{x_j} \partial_{x_j} u(y_1, \ldots, y_{j-1}, s, x_{j+1}, \ldots, x_n) ds.$$

Squaring this identity and using Cauchy-Schwarz inequality we obtain

$$u^2(\mathbf{x}) + u^2(\mathbf{y}) - 2u(\mathbf{x})u(\mathbf{y}) \le nd \sum_{j=1}^n \int\limits_{a_j}^{b_j} (\partial_j u)^2 (y_1, \ldots, y_{j-1}, s, x_{j+1}, \ldots, x_n) ds.$$

Integrating the above inequality with respect to all variables, we obtain

$$2d^n \|u\|_{0,Q}^2 \le 2\left(\int\limits_Q u d\mathbf{x}\right)^2 + nd^{n+2}\sum_{j=1}^n \|\partial_j u\|_{0,Q}^2$$

as required.

Theorem 1.52. *Let Ω be open and bounded. If the sequence $(u_k)_{k\in\mathbb{N}}$ of elements of $\overset{\circ}{W}{}_2^1(\Omega)$ is bounded, then there is a subsequence which converges in in $L_2(\Omega)$. In other words, the injection $\overset{\circ}{W}{}_2^1(\Omega) \subset L_2(\Omega)$ is compact.*

Proof. By density, we may assume $u_k \in C_0^\infty$. Let $M = \sup_k\{\|u_k\|_1\}$. We enclose Ω in a cube Q; we may assume the edges of Q to be of unit length. Further, we extend each u_k by zero to $Q \setminus \Omega$.

We decompose Q into N^n cubes of edges of length $1/N$. Since clearly $(u_k)_{k\in\mathbb{N}}$ is bounded in $L_2(Q)$ it contains a weakly convergent subsequence (which we denote again by $(u_k)_{k\in\mathbb{N}}$). For any ϵ' there is n_0 such that

$$\left|\int\limits_{Q_j} (u_k - u_l) d\mathbf{x}\right| < \epsilon', \qquad k, l \ge n_0 \tag{1.66}$$

for each $j = 1, \ldots, N^n$. Now, we apply (1.65) on each Q_j and sum over all j getting

$$\|u_k - u_l\|_{0,Q}^2 \leq N^n \epsilon' + \frac{n}{2N^2} 2M^2.$$

Now, we see that for a fixed ϵ we can find N large that $nM^2/N^2 < e$ and, having fixed N, for $\epsilon' = \epsilon/2N^n$ we can find n_0 such that (1.66) holds. Thus $(u_k)_{k\in\mathbb{N}}$ is Cauchy in $L_2(\Omega)$.

Corollary 1.53. *If Ω is a bounded open subset of $\mathbb{R}^n$, then the embedding $\overset{\circ}{W}{}_2^m(\Omega) \subset \overset{\circ}{W}{}_2^{m-1}(\Omega)$ is compact.*

Proof. Applying the previous theorem to the sequences of derivatives, we see that the derivatives form bounded sequences in $W_2^1(\Omega)$ and thus contain subsequences converging in $L_2(\Omega)$. Selecting common subsequence we get convergence in $W_2^1(\Omega)$ etc, (by closedness of derivatives).

Theorem 1.54. *If $\partial\Omega$ is a C^m boundary of a bounded open set Ω. Then the embedding $W_2^m(\Omega) \subset W_2^{m-1}(\Omega)$ is compact.*

Proof. The result follows by extension to $\overset{\circ}{W}{}_2^m(\Omega')$ where Ω' is a bounded set containing Ω.

1.4.3 Trace theorems

We know that if $u \in W_2^m(\Omega)$ with $m > n/2$ then u can be represented by a continuous function and thus can be assigned a value at the boundary of Ω (or, in fact, at any point). The requirement on m is, however, too restrictive — we have solved the Dirichlet problem, which requires a boundary value of the solution, in $\overset{\circ}{W}{}_2^1(\Omega)$. In this space, unless $n = 1$, the solution need not be continuous. It turns out that it is possible to give a meaning to the operation of taking the boundary value of a function even if it is not continuous.

We begin with the simplest (nontrivial) case when $\Omega = \mathbb{R}_+^n := \{\mathbf{x};\ \mathbf{x} = (\mathbf{x}', x_n), 0 < x_n\}$.

Theorem 1.55. *The trace operator $\gamma_0 : C^1(\overline{\mathbb{R}_+^n}) \cap W_2^1(\mathbb{R}_+^n) \to C^0(\mathbb{R}^{n-1})$ defined by*

$$(\gamma_0\phi)(\mathbf{x}') = \phi(\mathbf{x}', 0), \qquad \phi \in C^1(\overline{\mathbb{R}_+^n}) \cap W_2^1(\mathbb{R}_+^n), \mathbf{x}' \in \mathbb{R}^{n-1},$$

has a unique extension to a continuous linear operator $\gamma_0 : W_2^1(\mathbb{R}_+^n) \to L_2(\mathbb{R}^{n-1})$ whose range in dense in $L_2(\mathbb{R}^{n-1})$. The extension satisfies

$$\gamma_0(\beta u) = \gamma_0(\beta)\gamma_0(u), \qquad \beta \in C^1(\overline{\mathbb{R}_+^n}) \cap L_\infty(\mathbb{R}_+^n), u \in W_2^1(\mathbb{R}_+^n).$$

Proof. Let $\phi \in C^1(\overline{\mathbb{R}_+^n}) \cap W_2^1(\mathbb{R}_+^n)$. Then, from continuity, for any $\mathbf{x}'$, $\partial_{x_n}|\phi(\mathbf{x}', x_n)|^2 \in L_2(\mathbb{R}_+)$ we can write

$$|\phi(\mathbf{x}', r)|^2 - |\phi(\mathbf{x}', 0)|^2 = \int\limits_0^r \partial_{x_n}|u(\mathbf{x}', x_n)|^2 dx_n$$

and thus $|\phi(\mathbf{x}', r)|^2$ has a limit which must equal 0. Hence

$$|\phi(\mathbf{x}', 0)|^2 = -\int_0^\infty \partial_{x_n} |\phi(\mathbf{x}', x_n)|^2 dx_n.$$

Integrating over $\mathbb{R}^{n-1}$ we obtain

$$\begin{aligned}\|\phi(\mathbf{x}', 0)\|^2_{0,\mathbb{R}^{n-1}} &\leq 2 \int_{\mathbb{R}^n_+} \partial_{x_n}\phi(\mathbf{x})\phi(\mathbf{x}) d\mathbf{x} \\ &\leq 2\|\partial_{x_n}\phi\|_{0,\mathbb{R}^n_+} \|\phi\|_{0,\mathbb{R}^n_+} \leq \|\partial_{x_n}\phi\|^2_{0,\mathbb{R}^n_+} + \|\phi\|^2_{0,\mathbb{R}^n_+}.\end{aligned}$$

Hence, by density, the operation of taking value at $x_n = 0$ extends to $W^1_2(\mathbb{R}^n_+)$.

If $\phi \in C_0^\infty(\mathbb{R}^{n-1})$ and τ is a truncation function $\tau(t) = 1$ for $|t| \leq 1$ and $\tau(t) = 0$ for $|t| \geq 0$ then $\phi(\mathbf{x}) = \psi(\mathbf{x}')\tau(x_n) \in C^1(\overline{\mathbb{R}^n_+}) \cap W^1_2(\mathbb{R}^n_+)$ and $\gamma_0(\phi) = \psi$ so that the range of the trace operator contains $C_0^\infty(\mathbb{R}^{n-1})$ and thus is dense. The last identity follows from continuity of the trace operator and of the operator of multiplication by bounded differentiable functions in $W^1_2(\mathbb{R}^n_+)$.

Theorem 1.56. *Let $u \in W^1_2(\mathbb{R}^n_+)$. Then $u \in \overset{\circ}{W}{}^1_2(\mathbb{R}^n_+)$ if an only if $\gamma_0(u) = 0$,*

Proof. If $u \in \overset{\circ}{W}{}^1_2(\mathbb{R}^n_+)$, then u is the limit of a sequence $(\phi_k)_{k\in\mathbb{N}}$ from $C_0^\infty(\mathbb{R}^n_+)$ in $W^1_2(\mathbb{R}^n_+)$. Since $\gamma_0(\phi_k) = 0$ for any k, we obtain $\gamma_0(u) = 0$.

Conversely, let $u \in W^1_2(\mathbb{R}^n_+)$ with $\gamma_0 u = 0$. By using the truncating functions, we may assume that u has compact support in $\overline{\mathbb{R}^n_+}$.

Next we use the truncating functions $\eta_k \in C^\infty(\mathbb{R})$, as in Theorem 1.45, by taking function η which satisfies $\eta(t) = 1$ for $t \geq 1$ and $\eta(t) = 0$ for $t \leq 1/2$ and define $\eta_k(x_n) = \eta(kx_n)$. To simplify notation, we assume that $0 \leq \eta' \leq 3$ for $t \in [1/2, 1]$ so that $0 \leq \eta'_k(x_n) \leq 3k$. Then the extension by 0 to $\mathbb{R}^n_-$ of $\mathbf{x} \to \eta_k(x_n)u(\mathbf{x}', x_n)$ is in $W^1_2(\mathbb{R}^n)$ and can be approximated by $C_0^\infty(\mathbb{R}^n_+)$ functions in $W^1_2(\mathbb{R}^n_+)$. Hence, we have to prove that $\eta_k u \to u$ in $W^1_2(\mathbb{R}^n_+)$.

As in the proof of Theorem 1.45 we can prove $\eta_k u \to u$ in $L_2(\mathbb{R}^n_+)$ and for each $i = 1, \ldots, n-1$, $\partial_{x_i}(\eta_k u) = \eta_k \partial_{x_i} u \to \partial_{x_i} u$ in $L_2(\mathbb{R}^n_+)$ as $k \to \infty$.

Since

$$\partial_{x_n}(\eta_k u) = u\partial_{x_n}\eta_k + \eta_k \partial_{x_n} u$$

we see that we have to prove that $u\partial_{x_n}\eta_k \to 0$ in $L_2(\mathbb{R}^n_+)$ as $k \to \infty$. For this, first we prove that if $\gamma_0(u) = 0$, then

$$u(\mathbf{x}', s) = \int_0^s \partial_{x_n} u(\mathbf{x}', t) dt \tag{1.67}$$

almost everywhere on $\mathbb{R}^n_+$. Indeed, let u_r be a bounded support C^1 function approximating u in $W^1_2(\mathbb{R}^n_+)$. Then $\int_0^s \partial_{x_n} u_r(\mathbf{x}', t)dt \to \int_0^s \partial_{x_n} u(\mathbf{x}', t)dt$ in

$L_2(\mathbb{R}^n_+)$. This follows from $\partial_{x_n} u_r \to \partial_{x_n} u$ in $L_2(\mathbb{R}^n_+)$ and, taking Q to be the box enclosing support of all u_r, u, with edges of length at most d

$$\int\limits_Q \left| \int\limits_0^s \partial_{x_n} u_r(\mathbf{x}', t)dt - \int\limits_0^s \partial_{x_n} u(\mathbf{x}', t)dt \right|^2 d\mathbf{x}$$
$$\leq d^2 \int\limits_Q |\partial_{x_n} u_r(\mathbf{x}', t) - \partial_{x_n} u(\mathbf{x}', t)|^2 \, d\mathbf{x}$$

Then we have, for any s, $0 \leq s \leq d$

$$\int\limits_Q \left| \int\limits_0^s \partial_{x_n} u_r(\mathbf{x}', t)dt - u_r(\mathbf{x}', s) \right|^2 d\mathbf{x} = \int\limits_Q |u_r(\mathbf{x}', 0)|^2 d\mathbf{x} = d \int\limits_{\mathbb{R}^{n-1}} |u_r(\mathbf{x}', 0)|^2 d\mathbf{x}'$$

and, since the right hand side goes to zero as $r \to \infty$, we obtain (1.67). Then, by Cauchy-Schwarz inequality

$$|u(\mathbf{x}', s)|^2 \leq s \int\limits_0^s |\partial_{x_n} u(x', t)|^2 dt$$

and therefore

$$\int\limits_0^\infty |\eta_k'(s) u(\mathbf{x}', s)|^2 ds \leq 9k^2 \int\limits_0^{2/k} s \int\limits_0^s |\partial_{x_n} u(\mathbf{x}', t)|^2 dt ds$$
$$18k \int\limits_0^{2/k} \int\limits_0^s |\partial_{x_n} u(\mathbf{x}', t)|^2 dt ds = 18k \int\limits_0^{2/k} \int\limits_t^{2/k} |\partial_{x_n} u(\mathbf{x}', t)|^2 ds dt$$
$$\leq 36 \int\limits_0^{2/k} |\partial_{x_n} u(\mathbf{x}', t)|^2 dt.$$

Integration over $\mathbb{R}^{n-1}$ gives

$$\|\eta_k' u\|^2_{0,\mathbb{R}^n_+} \leq 36 \int\limits_{\mathbb{R}^{n-1} \times 2/k} |\partial_{x_n} u|^2 d\mathbf{x}$$

which tends to 0.

The consideration above can be extended to the case where Ω is an open bounded region in $\mathbb{R}^n$ lying locally on one side of its C^1 boundary. Using the partition of unity, we define

$$\gamma_0(u) := \sum_{j=1}^{N} (\gamma_0((\beta_j u) \circ H^j)) \circ (H^j)^{-1}$$

It is clear that if $u \in C^1(\bar{\Omega})$, then $\gamma_0 u$ is the restriction of u to $\partial\Omega$. Thus, we have the following result

Theorem 1.57. *Let Ω be a bounded open subset of $\mathbb{R}^n$ which lies on one side of its boundary $\partial\Omega$ which is assumed to be a C^1 manifold. Then there exists a unique continuous and linear operator $\gamma_0 : W_2^1(\Omega) \to L_2(\partial\Omega)$ such that for each $u \in C^1(\bar{\Omega})$, γ_0 is the restriction of u to $\partial\Omega$. The kernel of γ_0 is equal to $\overset{\circ}{W}{}_2^1(\Omega)$ and its range is dense in $L_2(\partial\Omega)$.*

$\omega\ \Omega$

$-\Delta u = f$

$u|_{\partial\Omega} = 0$

$u_{x_1x_1} \cdots u_{x_nx_n}$

1.4.4 Regularity of variational solutions to the Dirichlet problem

From Subsection 1.3.6 we know that there is a unique variational solution $u \in \overset{\circ}{W}{}_2^1(\Omega)$ of the problem

$$\int_\Omega \nabla u \nabla v d\mathbf{x} = < f, v >_{(\overset{\circ}{W}{}_2^1(\Omega))^* \times \overset{\circ}{W}{}_2^1(\Omega)}, \qquad v \in \overset{\circ}{W}{}_2^1(\Omega).$$

Moreover, now we can say that $\gamma_0 u = 0$ on $\partial\Omega$ (provided $\partial\Omega$ is C^1).

We have the following theorem

Theorem 1.58. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with C^2 boundary (or $\Omega = \mathbb{R}^n_+$). Let $f \in L_2(\Omega)$ and let $u \in \overset{\circ}{W}{}_2^1(\Omega)$ satisfy*

$$\int_\Omega \nabla u \nabla v d\mathbf{x} = (f, v), \qquad v \in \overset{\circ}{W}{}_2^1(\Omega). \tag{1.68}$$

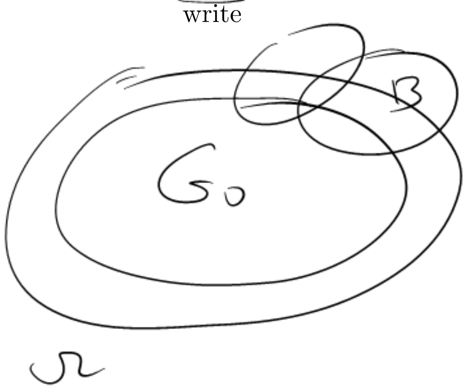
Then $u \in W_2^2(\Omega)$ and $\|u\|_{2,\Omega} \le C\|f\|_{0,\Omega}$ where C is a constant depending only on Ω. Furthermore, if Ω is of class C^{m+2} and $f \in W_2^m(\Omega)$, then

$$u \in W_2^{m+2}(\Omega) \quad \text{and} \quad \|u\|_{m+2,\Omega} \le C\|f\|_{m,\Omega}.$$

In particular, if $m \ge n/2$, then $u \in C^2(\bar{\Omega})$ is a classical solution.

Moreover, if Ω is bounded, then the solution operator $G : L_2(\Omega) \to \overset{\circ}{W}{}_2^1(\Omega)$ is self-adjoint and compact.

Proof. The proof naturally splits into two cases: interior estimates and boundary estimates. Let Ω be bounded with at least C^1 boundary and consider the partition of unity $\{\beta_j\}_{j=0}^N$ subordinated to the covering $\{G_j\}_{j=0}^N$. For the interior estimates let us consider $u_0 = \beta_0 u$ and let $v \in \overset{\circ}{W}{}_2^1(\Omega)$. Then we can write



$-\beta \Delta u = f' \beta_0$

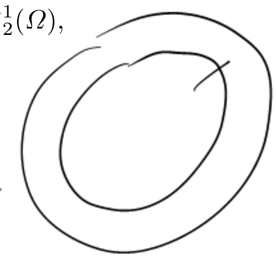
$\int_\Omega D u D(v) = \int_\Omega f(v)$

$$
\begin{aligned}
\int_\Omega \nabla(\beta_0 u)\nabla v d\mathbf{x} &= \int_\Omega \beta_0 \nabla u \nabla v d\mathbf{x} + \int_\Omega u\nabla\beta_0\nabla v d\mathbf{x} \\
&= \int_\Omega \nabla u \nabla(\beta_0 v) d\mathbf{x} - \int_\Omega v\nabla u\nabla\beta_0 d\mathbf{x} + \int_\Omega u\nabla v\nabla\beta_0 d\mathbf{x} \\
&= \int_\Omega \nabla u \nabla(\beta_0 v) d\mathbf{x} - \int_\Omega v\nabla u\nabla\beta_0 d\mathbf{x} - \int_\Omega \nabla(u\nabla\beta_0) v d\mathbf{x} \\
&= \int_\Omega \nabla u \nabla(\beta_0 v) d\mathbf{x} - 2\int_\Omega v\nabla u\nabla\beta_0 d\mathbf{x} - \int_\Omega uv\Delta\beta_0 d\mathbf{x} \\
&= \int_\Omega (f\beta_0 - \Delta\beta_0 u - 2\nabla u\nabla\beta_0) v d\mathbf{x} = \int_\Omega F v d\mathbf{x}, \qquad v \in \overset{\circ}{W}{}_2^1(\Omega),
\end{aligned}
$$

$\int D_v(\ )$

$u \in W_2^1$

W_2^2



$\|\tau_h \partial_x u - \partial_x u\| \le |h| C$

where $F \in L_2(\Omega)$ and we used $v \in \overset{\circ}{W}{}_2^1(\Omega)$ to get

$$
\int_\Omega u\nabla v\nabla\beta_0 d\mathbf{x} = -\int_\Omega \nabla(u\nabla\beta_0) v d\mathbf{x}.
$$

$\int D w D v = \int F v \quad \forall\, v \in \overset{\circ}{W}{}_2^1(\Omega)$

Hence, the function $w = \beta_0 u$ is the variational solution to the above problem in $\mathbb{R}^n$. Let us define $D_h u = |\mathbf{h}|^{-1}(\tau_h u - u)$ and take $v = D_{-h}(D_h w)$. It is possible since w has compact support in Ω and thus $v \in \overset{\circ}{W}{}_2^1(\Omega)$ for sufficiently small $\mathbf{h}$. Thus we obtain

$$
\int_\Omega |\nabla D_h w|^2 d\mathbf{x} = \int_\Omega F D_{-h}(D_h w) d\mathbf{x},
$$

that is,

$$
\|D_h w\|_{1,\Omega}^2 \le \|F\|_{0,\Omega}\|D_{-h}(D_h w)\|_{0,\Omega}. \tag{1.69}
$$

$\int \varphi D_{-h} \psi = \int \varphi(x) \frac{1}{|h|}[\psi(x-h)$

On the other hand, from Friedrichs lemma, for any $v \in W_2^1(\Omega)$ with compact support

$$
\|D_{-h} v\|_{0,\Omega} \le \|\nabla v\|_{0,\Omega}. \tag{1.70}
$$

Applying this to $v = D_h u$, we obtain

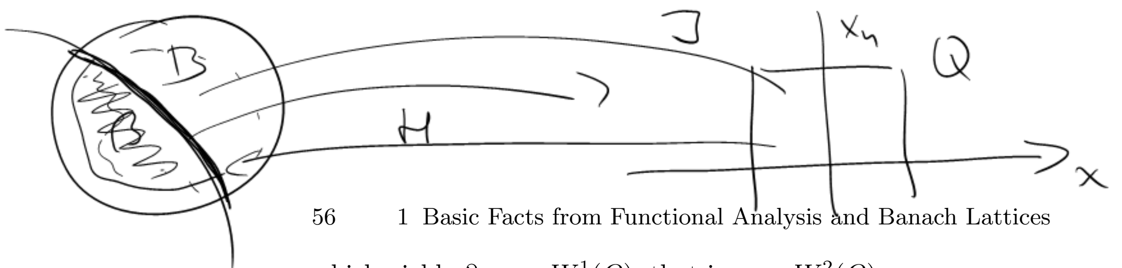
$$
\|D_h w\|_{1,\Omega}^2 \le \|F\|_{0,\Omega}\|\nabla D_h w\|_{0,\Omega} \le \|F\|_{0,\Omega}\|D_h w\|_{1,\Omega},
$$

that is,

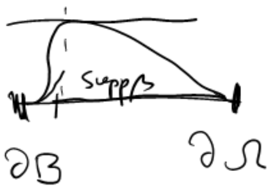
$$
\boxed{\|D_h w\|_{1,\Omega} \le \|F\|_{0,\Omega}.}
$$

In particular, we obtain

$$
\|D_h \partial_{x_i} w\|_{0,\Omega} \le \|F\|_{0,\Omega}, \quad i = 1,\dots,n,
$$



which yields $\partial_{x_i} w \in W_2^1(\Omega)$, that is, $w \in W_2^2(\Omega)$.



In the next step, we shall move to estimates close to the boundary. Let us fix some some set B_j and corresponding function β_j, $1 \le j \le N$ from the partition of unity and drop the index j. Then we have a C^2 diffeomorphism $H : Q \to B$ the inverse of which we denote $J = H^{-1}$ so that $H(Q_+) = \Omega \cap B$ and $H(Q_0) = \partial\Omega \cap B$. We denote $\mathbf{x} = H(\mathbf{y})$, $\mathbf{y} \in Q$ and $\mathbf{y} = J(\mathbf{x})$. As before, we see that $w = \beta u$ is a variational solution to

$$\int\limits_{\Omega\cap B} \nabla w \nabla v d\mathbf{x} = \int\limits_{\Omega\cap B} (f\beta - u\Delta\beta - 2\nabla u \nabla \beta) v d\mathbf{x} = \int\limits_{\Omega\cap B} g v d\mathbf{x}, \qquad v \in \overset{\circ}{W}{}_2^1(\Omega) \tag{1.71}$$

where the Green's formula

$\int_{\partial(\Omega\cap B)} v \cdot u \cdot \nabla\beta_0$

$$\int\limits_{\Omega\cap B} u \nabla v \nabla \beta_0 d\mathbf{x} = -\int\limits_{\Omega\cap B} \nabla(u \nabla \beta_0) v d\mathbf{x}.$$

can be justified by noting that the integration is actually carried out over the domain $G \Subset B$ and we can use a function χv, where χ is equal to 1 on G and has support in B, instead of v. Function $\chi v \in \overset{\circ}{W}{}_2^1(\Omega \cap B)$ (as v can be approximated by ϕ compactly supported in Ω and χv can be approximated by $\chi\phi$ compactly supported in $\Omega \cap B$).

Now we transfer (1.71) to Q_+. We have $z(\mathbf{y}) = w(H(\mathbf{y}))$ for $\mathbf{y} \in Q_+$ or $w(\mathbf{x}) = z(J(\mathbf{x}))$ for $\mathbf{x} \in \Omega \cap B$. Let $\psi \in \overset{\circ}{W}{}_2^1(Q_+)$ and $\phi(\mathbf{x}) = \psi(J(\mathbf{x}))$. Then $\phi \in \overset{\circ}{W}{}_2^1(\Omega \cap B)$ and we have

$$\partial_{x_j} w = \sum_{k=1}^n \partial_{y_k} z \partial_{x_j} J_k, \qquad \partial_{x_j}\phi = \sum_{l=1}^n \partial_{y_l}\psi \partial_{x_j} J_l$$

and hence

$$\int\limits_{\Omega\cap B} \nabla w \nabla \phi d\mathbf{x} = \int\limits_{Q_+} \sum_{k,j,l=1}^n \partial_{x_j} J_k \partial_{x_j} J_l \partial_{y_k} z \partial_{y_l}\psi |\mathrm{det}\mathcal{J}_H| d\mathbf{y} = \int\limits_{Q_+} \sum_{k,l=1}^n a_{k,l} \partial_{y_k} z \partial_{y_l}\psi d\mathbf{y}$$

where $\mathcal{J}$ is the Jacobi matrix of H. We note that we can write

$$a_{k,l} = |\mathrm{det}\mathcal{J}_H| \mathcal{J}_J \mathcal{J}_J^T$$

and thus we have

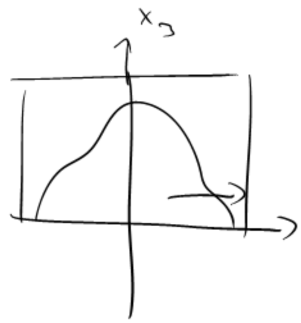
$|\mathrm{det}\, \mathcal{J}_H|\, |\mathcal{J}_J^T \xi|^2$

$$\sum_{k,l=1}^n a_{k,l}\xi_k\xi_l = |\mathrm{det}\mathcal{J}_H| (\mathcal{J}_J^T \boldsymbol{\xi}, \mathcal{J}_J^T \boldsymbol{\xi}) \ge \alpha |\boldsymbol{\xi}|^2 \tag{1.72}$$

for all $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ since both Jacobi matrices $\mathcal{J}_H, \mathcal{J}_J$ are nonsingular. Also

$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} = f$

$$\int\limits_{\Omega\cap B} g\phi d\mathbf{x} = \int\limits_{Q_+} (g\circ H)\psi|\det\mathcal{J}_H|d\mathbf{y} = \int\limits_{Q_+} G\psi d\mathbf{y}$$



where $G \in L_2(Q_+)$ so that $z \in \overset{\circ}{W}{}^1_2(Q)$ is a solution to the (elliptic) variational problem

$$\int\limits_{Q_+} \sum_{k,l=1}^{n} a_{k,l}\partial_{y_k} z \partial_{y_l}\psi d\mathbf{y} = \int\limits_{Q_+} G\psi d\mathbf{y}, \qquad \psi \in \overset{\circ}{W}{}^1_2(Q_+). \tag{1.73}$$

Next the process is split into two cases. First we shall consider the method of finite differences, as in the G_0 case but only in the directions parallel to the boundary. Thus, we take $\psi = D_{-h}(D_h z)$ for $|\mathbf{h}|$ small enough to still have $\psi \in \overset{\circ}{W}{}^1_2(Q_+)$. Then, as above

$$\int\limits_{Q_+} D_h\left(\sum_{k,l=1}^{n} a_{k,l}\partial_{y_k} z\right)\partial_{y_l}(D_h z)d\mathbf{y} = \int\limits_{Q_+} GD_{-h}(D_h z)d\mathbf{y}.$$

Since $D_h x \in \overset{\circ}{W}{}^1_2(Q_+)$, we can use Friedrichs lemma to estimate

$$\int\limits_{Q_+} GD_{-h}(D_h z)d\mathbf{y} \leq \|G\|_{0,Q_+}\|D_{-h}(D_h z)\|_{0,Q_+} \leq \|G\|_{0,Q_+}\|\nabla(D_h z)\|_{0,Q_+}.$$

Then, using $\tau_h(fg) - fg = \tau_h f(\tau_h g - g) + (\tau_h f - f)g$, we find

$$D_h\left(\sum_{k,l=1}^{n} a_{k,l}\partial_{y_k} z\right)(\mathbf{y}) = a_{k,l}(\mathbf{y}+\mathbf{h})\partial_{y_k} D_h z(\mathbf{y}) + (D_h a_{k,l})(\mathbf{y})\partial_{y_k}(\mathbf{y})$$

and thus we can write, be the reverse Cauchy-Schwarz inequality

$$\begin{aligned}
&\int\limits_{Q_+} D_h\left(\sum_{k,l=1}^{n} a_{k,l}\partial_{y_k} z\right)\partial_{y_l}(D_h z)d\mathbf{y}\\
&= \int\limits_{Q_+}\sum_{k,l=1}^{n}(\tau_h a_{k,l})\partial_{y_k}(D_h z)\partial_{y_l}(D_h z)d\mathbf{y} + \int\limits_{Q_+}\sum_{k,l=1}^{n}(D_h a_{k,l})\partial_{y_k} z\partial_{y_l}(D_h z)d\mathbf{y}\\
&\geq \alpha\|\nabla(D_h z)\|^2_{0,Q_+} - C\|\nabla z\|_{0,Q_+}\|\nabla(D_h z)\|_{0,Q_+}
\end{aligned}$$

where C depends on the C^1 norm of $a_{k,l}$ (and thus C^2 norm of the local atlas). Thus

$$\begin{aligned}
\|\nabla(D_h z)\|^2_{0,Q_+} &\leq \alpha^{-1}\left(\|G\|_{0,Q_+}\|\nabla(D_h z)\|_{0,\Omega} + C\|z\|_{1,\Omega}\|\nabla(D_h z)\|_{0,Q_+}\right)\\
&\leq C'\|G\|_{0,Q_+}\|\nabla(D_h z)\|_{0,Q_+},
\end{aligned} \tag{1.74}$$

$\alpha\|\nabla D_h z\|^2 - C\|\nabla z\|\|\nabla D_h z\| \leq \ldots.$

$\Rightarrow \leq \|G\|\|\nabla(D_h z)\|$

where we have used the $W_2^1(\Omega)$ estimates for solutions to (1.73): for $\psi = z \in \overset{\circ}{W}{}_2^1(Q_+)$

$$\alpha\|\nabla z\|^2 \leq \int\limits_{Q_+} \sum_{k,l=1}^{n} a_{k,l}\partial_{y_k} z \partial_{y_l} z d\mathbf{y} = \int\limits_{Q_+} G z d\mathbf{y} \leq \|G\|_{0,Q_+}\|\nabla z\|_{0,Q_+}.$$

O, Q_+

Note that in the last inequality we used the Poincarè inequality as $z \in \overset{\circ}{W}{}_2^1(Q_+)$ and the constant in this inequality can be taken 1.

Thus we have

$$\|\nabla(D_h z)\|_{0,Q_+} \leq C'\|G\|_{0,Q_+}, \tag{1.75}$$

for any $\mathbf{h}$ which is parallel to Q_0. Let $j = 1,\ldots,n$, $\mathbf{h} = |\mathbf{h}|\mathbf{e}_k$, $k = 1,\ldots,n-1$ and $\phi \in C_0^\infty(Q_+)$. Then we can write

$$\int\limits_{Q_+} D_h\partial_{y_j} z\phi d\mathbf{y} = -\int\limits_{Q_+} \partial_{y_j} z D_{-h}\phi d\mathbf{y}$$

and, by (1.75),

$u \in W_2^1$, L^2

$= \int f\varphi$

$\varphi \to \int u \frac{\partial \varphi}{\partial x_i} \leq C\|\varphi\|_0$

$$\left|\int\limits_{Q_+} \partial_{y_j} z D_{-h}\phi d\mathbf{y}\right| = \left|\int\limits_{Q_+} D_h\partial_{y_j} z\phi d\mathbf{y}\right| \leq C'\|G\|_{0,Q_+}\|\phi\|_{0,Q_+}$$

which, passing to the limit as $|h| \to 0$ gives for any $\cancel{(j,k) \neq (n,n)}$

$j = 1,\ldots n$
$k = 1,\ldots n-1$

$$\left|\int\limits_{Q_+} \partial_{y_j} z\partial_{y_k}\phi d\mathbf{y}\right| \leq C'\|G\|_{0,Q_+}\|\phi\|_{0,Q_+}. \tag{1.76}$$

To conclude, we have to show also the above estimate for $k = n$. First we observe that $a_{nn} \geq \alpha$ on Q_+. This follows from (1.72) by taking $\boldsymbol{\xi} = (1,0,\ldots,0)$. Thus, we can replace in (1.73) ψ by ψ/a_{nn}. Then we rewrite (1.73) as

$$\int\limits_{Q_+} a_{n,n}\partial_{y_n} z\underbrace{\partial_{y_n}(a_{n,n}^{-1}\psi)d\mathbf{y}} = \int\limits_{Q_+} a_{n,n}G(a_{n,n}^{-1}\psi)d\mathbf{y}$$
$$-\int\limits_{Q_+} \sum_{(k,l)\neq(n,n)} a_{k,l}\partial_{y_k} z\partial_{y_l}(a_{n,n}^{-1}\psi)d\mathbf{y},$$

and differentiating on the left hand side

$$\int\limits_{Q_+} \partial_{y_k} z \partial_{y_k} \psi d\mathbf{y} = \int\limits_{Q_+} a_{n,n}^{-1} \psi \partial_{y_n} a_{n,n} \partial_{y_k} z d\mathbf{y} + \int\limits_{Q_+} a_{n,n} G \cdot (a_{n,n}^{-1}\psi) d\mathbf{y}$$
$$- \int\limits_{Q_+} \sum_{(k,l)\neq(n,n)} (a_{n,n}^{-1}\psi) \partial_{y_l} a_{k,l} \partial_{y_k} z d\mathbf{y}$$
$$+ \int\limits_{Q_+} \sum_{(k,l)\neq(n,n)} \partial_{y_k} z \partial_{y_l} (a_{n,n}^{-1} a_{k,l}\psi) d\mathbf{y},$$

Applying now (1.78), we get

$$\left| \int\limits_{Q_+} \partial_{y_k} z \partial_{y_k} \psi d\mathbf{y} \right| \leq C(\|G\|_{0,Q_+} + \|z\|_{1,Q_+}) \|\psi\|_{0,Q}. \tag{1.77}$$

This shows that

$$\left| \int\limits_{Q_+} \partial_{y_j} z \partial_{y_k} \phi d\mathbf{y} \right| \leq C' \|G\|_{0,Q_+} \|\phi\|_{0,Q_+}. \tag{1.78}$$

for any $j, k = 1, \ldots n$ and thus, by Proposition 1.44, each first derivative of z belongs to $W_2^1(Q_+)$ and thus $z \in W_2^2(Q_+)$. Using the first part of the proof and transferring the solution back to Ω shows that $u \in W_2^2(\Omega)$.

Let us consider higher derivatives. As before, we split u according to the partition of unity and separately argue argue in $G_0 \Subset \Omega$ and in Q_+. Let us begin with $u \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ and consider $w = \beta_0 u$. Let $f \in W_2^1(\Omega)$ and consider any derivative ∂u, $i = 1, \ldots, n$. We know that $\partial u \in W_2^1(\Omega)$. Then we can use $\phi \in C_0^\infty$ and take $\partial\phi$ as the test function in (1.68) so that , integrating by parts

$$- \int\limits_{\Omega} \partial f \phi d\mathbf{x} = \int\limits_{\Omega} f \partial\phi d\mathbf{x} = \int\limits_{\Omega} \nabla u \nabla \partial \phi d\mathbf{x} = - \int\limits_{\Omega} \nabla \partial u \nabla \phi d\mathbf{x}$$

so that ∂u is a variational solution with square integrable right hand side and thus $\partial u \in W_2^2(\Omega)$ and $u \in W_2^3(\Omega)$. Then we can proceed by induction.

Let us consider $z \in W_2^2(Q_+) \cap \overset{\circ}{W}{}_2^1$ and let ∂u be any derivative in direction tangential to Q_0. We claim that $\partial z \in \overset{\circ}{W}{}_2^1$. First, we note that $D_h z \in \overset{\circ}{W}{}_2^1$ (Q_+) if $\mathbf{h}$ is parallel to Q_0 for sufficiently small $|\mathbf{h}|$. By (1.75), $D_h z$ is bounded in $\cancel{W_0^1(Q)}$ $\overset{\circ}{W}{}^1(Q)$ and thus we have a subsequence $\mathbf{h}_n$ such that $D_{h_n} z \rightharpoonup g \in \overset{\circ}{W}{}_2^1(Q)$. Clearly, $D_{h_n} z$ converges weakly in $L_2(Q_+)$ and thus for any $\phi \in C_0^\infty(Q_+)$

$$\int\limits_{Q_+} (D_{h_n} z) \phi d\mathbf{y} = \int\limits_{Q_+} z D_{-h_n} \phi d\mathbf{y}$$

and thus passing to the limit

$$\int_{Q_+} g\phi d\mathbf{y} = -\int_{Q_+} z\partial\phi d\mathbf{y}$$

and thus $\partial z \in \overset{\circ}{W}{}^1_2(Q_+)$. Then, as before

$$\int_{\Omega} \partial G\psi d\mathbf{y} = \int_{\Omega} \sum_{k,l=1}^{n} \partial_{y_k}(\partial z)\partial_{y_l}\psi d\mathbf{y} \tag{1.79}$$

for any $\phi \in \overset{\circ}{W}{}^1_2(Q_+)$. We argue by induction in m. Let $f \in W_2^{m+1}(Q_+)$. From induction assumption, we have $z \in W^{m+2}(Q_+)$. Also ∂z in any tangential direction ~~derivative~~ is in $\overset{\circ}{W}{}^1_2(Q_+)$ and satisfies (1.79). By induction assumption ~~to~~ on ∂z and ∂G we see that $\partial z \in W_2^{m+2}(Q_+)$. Finally we can write

(applied to ∂z)

C^{m+2}

$$\partial^2_{x_n x_n} u = \frac{1}{a_{n,n}}\left(-G - \int_{\Omega} \sum_{(k,l)\neq(n,n)} \partial_{y_k}(\partial z)\partial_{y_l}\psi d\mathbf{y}\right)$$

so that the claim follows.

$\partial z \in W^2_2(\Omega)$

$\partial^2_{x_n x_n} z = -\frac{1}{a_{nn}}(\quad) + G$