$$|\chi(\mathbf{x}', x_n) \leq M|x_n|$$

on Q. Thus

$$\int\limits_{Q_+} u\partial_{x_n}\eta_k\chi d\mathbf{x} = \int\limits_{Q_+} u(\eta_k\partial_{x_n}\chi + \chi\partial_{x_n}\eta_k)d\mathbf{x} \to \int\limits_{Q_+} u\eta_k\partial_{x_n}\chi$$

and thus we obtain in the limit

$$\int\limits_{Q_+} u\partial_{x_n}\chi d\mathbf{x} = -\int\limits_{Q_+} (\partial_{x_n})u\chi d\mathbf{x}.$$

Returning to Q, we obtain

$$\int\limits_{Q} u^*\partial_{x_n}\phi d\mathbf{x} = \int\limits_{Q_+} u\partial_{x_n}\chi d\mathbf{x} = \int\limits_{Q} (\partial_{x_n}u)^\bullet\phi d\mathbf{x}.$$

We also obtain estimates

$$\|u^*\|_{0,Q} \leq 2\|u\|_{0,Q_+} \qquad \|u^*\|_{1,Q} \leq 2\|u\|_{1,Q_+}.$$

Now we can pass to the general result. Let $u \in W_2^1(\Omega)$, Ω bounded with C^1 boundary. Let $\{B_j, H^j\}_{j=1}^N$ be the atlas on the boundary and $\{G_j\}_{j=1}^N$ be the finite subcover constructed in the previous section, that is $G_0 \subset \bar{G}_0 \subset \Omega$, $\bar{G}_j \subset B_j$ with $\partial\Omega \subset \bigcup G_j$ and let $\{\beta\}_{j=1}^N$ be a subordinate partition of unity. Then we take

$$u = \sum_{j=0}^N \beta_j u = \sum_{j=0}^N u_j$$

with $u_0 \in \overset{\circ}{W}{}_1^2(\Omega)$ and $u_j \in W_2^1(\Omega \cap B_j)$. Clearly, $\|u_0\|_{1,\Omega} \leq C_0\|u\|_{1,\Omega}$ and $\|u_j\|_{1,\Omega\cap B_j} \leq C_j\|u\|_{1,\Omega}$, $j = 1, \ldots, n$. The function u_0 can be extended to $\hat{u}_0 \in W_2^1(\mathbb{R}^n)$ by zero in a continuous way. Then $v_j := u_j \circ H^j \in W_2^1(Q_+)$ and we can extend by reflection to $v_j^* \in W_2^1(Q)$. We note that v_j^* has support in Q since the support of u_j only can touch $\partial(B_j \cap \Omega$ at the points of $\partial\Omega$. Again,

$$\|v_j^*\|_{1,Q} \leq 2\|v_j\|_{1,Q_+} \leq C_j''\|u_j\|_{1,\Omega\cap B_j} \leq C_j'\|u\|_{1,\Omega}.$$

Next, we define $w_j = v_j^* \circ (H^j)^{-1} \in W_2^1(B_j)$, again with $\|w_j\|_{1,B_j} \leq C_j''\|u\|_{1,\Omega}$. Moreover, we have $w_j(\mathbf{x}) = u_j(\mathbf{x})$ whenever $\mathbf{x} \in B_j \cap \bar{\Omega}$ as

$$v_j^*((H^j)^{-1}(\mathbf{x})) = v_j((H^j)^{-1}(\mathbf{x})) = u_j(H^j((H^j)^{-1}(\mathbf{x}))) = u_j(\mathbf{x})$$

for such $\mathbf{x}$. We also notice that for each $j = 1, \ldots, N$, support of w_j is contained in B_j and thus can extend w_j by zero to $\mathbb{R}^n$ continuously in $W_2^1(\mathbb{R}^n)$

and denote this extension by $\hat{u}_j$. We note that $\hat{u}_j(\mathbf{x}) = u_j(\mathbf{x})$ for $\mathbf{x} \in \bar{\Omega}$. Indeed, if $\mathbf{x} \in \bar{\Omega}$, for a given j either $\mathbf{x} \in B_j \cap \bar{\Omega}$ and then $\hat{u}_j(\mathbf{x}) = w_j(\mathbf{x}) = u_j(\mathbf{x})$ or $\mathbf{x} \notin B_j \cap \bar{\Omega}$ in which case $\hat{u}_j(\mathbf{x}) = 0$ but then also $u_j(\mathbf{x}) = 0$ by definition. The same argument applies to $j = 0$. Now we define the operator

$$Eu = \hat{u}_0 + \sum_{j=1}^{n} \hat{u}_j$$

and we clearly have

$$Eu(\mathbf{x}) = \hat{u}_0(\mathbf{x}) + \sum_{j=1}^{n} \hat{u}_j(\mathbf{x}) = u_0(\mathbf{x}) + \sum_{j=1}^{n} u_j(\mathbf{x}) = u(\mathbf{x}).$$

Linearity and continuity follows from continuity and linearity of each operation and the fact that the sum is finite.

Remark 1.46. Similar argument allows to prove that there is an extension from $W_2^m(\Omega)$ to $W_2^m(\mathbb{R}^n)$ (as well as for $W_p^m(\Omega)$, $1 \leq p \leq \infty$) but this requires the boundary to be a C^m-manifold (so that the flattening preserves the differentiability). However, the extension across the hyperplane $x_n = 0$ is done according to the following reflection

$$u^*(\mathbf{x}', x_n) = \begin{cases} u(\mathbf{x}', x_n) & \text{for } x_n > 0 \\ \lambda_1 u(\mathbf{x}', -x_n) + \lambda_2 u\left(\mathbf{x}', -\frac{x_n}{2}\right) + \ldots + \lambda_m u\left(\mathbf{x}', -\frac{x_n}{m}\right) & \text{for } x_n < 0, \end{cases}$$

where $\lambda_1, \ldots, \lambda_m$ is the solution of the system

$$\begin{aligned} \lambda_1 + \lambda_2 + \ldots + \lambda_m &= 1, \\ -(\lambda_1 + \lambda_2/2 + \ldots + \lambda_m/m) &= 1, \\ &\cdots \\ (-1)^m(\lambda_1 + \lambda_2/2^{m-1} + \ldots + \lambda_m/m^{m-1}) &= 1 \end{aligned}$$

These conditions ensure that the derivatives in the x_n direction are continuous across $x_n = 0$.

An immediate consequence of the extension theorem is

Theorem 1.47. *Let Ω be a bounded set with a C^1 boundary $\partial\Omega$ and $u \in W_2^1(\Omega)$. Then there exits $(u_n)_{n\in\mathbb{N}}$, $u_n \in C_0^\infty(\mathbb{R}^n)$ such that*

$$\lim_{n\to\infty} u_n|_\Omega = u, \qquad \text{in } W_2^1(\Omega).$$

In other words, the set of restriction to Ω of functions from $C_0^\infty(\Omega)$ is dense in $W_2^1(\Omega)$.

Proof. If Ω is bounded then, using Theorem 1.45, we can extend u to a function $Eu \in W_2^1(\mathbb{R}^n)$ with bounded support. The existence of a $C_0^\infty(\mathbb{R}^n)$ sequence converging to u follows from the Friedrichs lemma. If Ω is unbounded (but not equal to $\mathbb{R}^n$), then first we approximate u by a sequence $(\chi_n u)_{n\in\mathbb{N}}$ where χ_n are cut-off functions. Next we construct an extension of $\chi_n u$ to $\mathbb{R}^n$. This is possible as it involves only the part of $\partial\Omega$ intersecting the ball $B(0, 2n+1)$ and χ_n is equal to zero where the sphere intersects $\partial\Omega$. For this extension we pick up an approximating function from $C_0^\infty(\mathbb{R}^n)$.

$\chi(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| \ge 2 \end{cases}$

$\chi_n(x) = \chi\left(\frac{x}{n}\right)$

1.4 Basic applications of the density theorem

1.4.1 Sobolev embedding

In Subsection 1.1.2 we have seen that in one dimension it is possible to identify a $W_2^1(\mathbb{R})$ function. Unfortunately, this is not true in higher dimensions.

Example 1.48.

2 funkcje ciągłe

$\|u\|_m$ norma w $W_2^m(\Omega)$

However, there is still a link between Sobolev spaces and classical calculus provided we take sufficiently high order of derivatives (or index p in L_p spaces). The link is provided by the Sobolev lemma.

$f(x,y) = \ln(x^2+y^2)$ na $B(0,1)$

$f \notin C(B(0,1))$

$f \in L_2$

$\int_0^1 \int_0^{2\pi} \ln r^2 \, r \, dr$ $\quad f \notin W_1^1(B(0,1))$

Let Ω be an open and bounded subset of $\mathbb{R}^n$. We say that Ω satisfies the cone condition if there are numbers $\rho > 0$ and $\gamma > 0$ such that each $\mathbf{x} \in \Omega$ is a vertex of a cone $K(\mathbf{x})$ of radius ρ and volume $\gamma\rho^n$. Precisely speaking, if σ_n is the $n-1$ dimensional measure of the unit sphere in $\mathbb{R}^n$, then the volume of a ball of radius ρ is $\sigma_n \rho^n/n$ and then the (solid) angle of the cone is $\gamma n/\omega_n$.

Lemma 1.49. *If Ω satisfies the cone condition, then there exists a constant C such that for any $u \in C^m(\bar{\Omega})$ with $2m > n$ we have*

$$\sup_{\mathbf{x}\in\Omega} |u(\mathbf{x})| \le C\|u\|_m \tag{1.61}$$

$f_x = \frac{2x}{x^2+y^2}$

$f_y = \frac{2y}{x^2+y^2}$

$f_x^2 = \frac{x^2}{(x^2+y^2)^2}$

$\frac{r^2 \sin^2\theta}{r^4} \cdot r$

$\frac{r \sin\theta}{r^2} \cdot r$

Proof. Let us introduce a cut-off function $\phi \in C_0^\infty(\mathbb{R})$ which satisfies $\phi(t) = 1$ for $|t| \le 1/2$ and $\phi(t) = 0$ for $|t| \ge 1$. Define $\tau(t) = \phi(t/\rho)$ and note that there are constants A_k, $k = 1, 2, \ldots$ such that

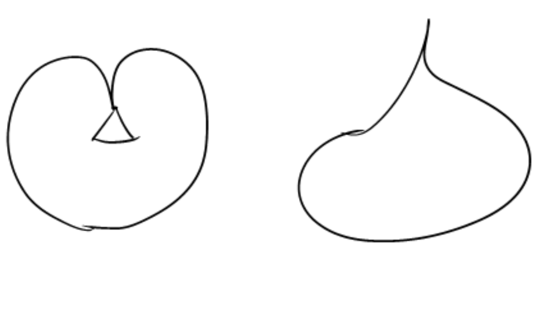
$$\left|\frac{d^k \tau(t)}{dt^k}\right| \le \frac{A_k}{\rho^k}. \tag{1.62}$$

$A_k = \max \phi^{(k)}$

Let us take $u \in C^m(\bar{\Omega})$ and assume $2m > n$. For $\mathbf{x} \in \bar{\Omega}$ and the cone $K(\mathbf{x})$ we integrate along the ray $\{\mathbf{x} + r\mathbf{e};\ 0 \le r \le \rho\}$

$\mathbf{e} \in B(0,1)$

$$u(\mathbf{x}) = -\int_0^\rho D_r(\tau(r)u(\mathbf{x} + r\mathbf{e}))dr.$$



Integrating over the surface of the cone in spherical coordinates we get

$$\int_C \int_0^\rho D_r(\tau(r)u(\mathbf{x} + \overset{\omega}{r}))drd\omega = -u(\mathbf{x}) \int_C d\omega = -u(\mathbf{x})\underline{\frac{\gamma n}{\omega_n}}.$$

Next we integrate $m-1$ times by parts, getting

$$u(\mathbf{x}) = \frac{(-1)^m \omega_n}{\gamma n(m-1)!} \int_C \int_0^\rho D_r^m(\tau(r)u(\mathbf{x} + \overset{\omega}{r}))r^{m-1} \boxed{drd\omega.} \quad r^{n-1}\, r^{-n+1}\, d\mathbf{y}$$

and changing to Cartesian coordinates and applying Cauchy-Schwarz inequality we obtain

$$|u(\mathbf{x})|^2 \le \left(\frac{\omega_n}{\gamma n(m-1)!} \int_{K(\mathbf{x})} |D_r^m(\tau u)| r^{m-n} d\mathbf{y}\right)^2$$

$$\le \left(\frac{\omega_n}{\gamma n(m-1)!}\right)^2 \left(\int_{K(\mathbf{x})} |D_r^m(\tau u)|^2 d\mathbf{y} \cancel{r^{m-n}} d\mathbf{y}\right) \left(\int_{K(\mathbf{x})} r^{2(m-n)} d\mathbf{y}. \right)$$

The last term can be evaluated as

$$\int_{K(\mathbf{x})} r^{2(m-n)} d\mathbf{y} = \int_C \int_0^\rho r^{2m-n-1} drd\omega = \frac{\gamma n \rho^{2m-n}}{\omega_n(2m-n)}$$

$2m-n-1>-1$

$m > \frac{n}{2}$

so that

$$|u(\mathbf{x})|^2 \le C(m,n)\rho^{2m-n} \int_{K(\mathbf{x})} |D_r^m(\tau u)|^2 d\mathbf{y}. \tag{1.63}$$

Let us estimate the derivative. From (1.62) we obtain by the chain rule and the Leibniz formula

$$|D_r^m(\tau u)| = \left| \sum_{k=0}^m \binom{n}{k} D_r^{m-k} \tau D_r^k u \right| \le \sum_{k=0}^m \binom{n}{k} \frac{A_{m-k}}{\rho^{m-k}} \left| D_r^k u \right|,$$

hence

$$|D_r^m(\tau u)|^2 \le C' \sum_{k=0}^m \frac{1}{\rho^{2(m-k)}} \left| D_r^k u \right|^2$$

for some constant C'. With this estimate we can re-write (1.63) as

$$|u(\mathbf{x})|^2 \le C(m,n)C' \sum_{k=0}^m \rho^{2k-n} \int_{K(\mathbf{x})} |D_r^m(u)|^2 d\mathbf{y}. \tag{1.64}$$

Since by the chain rule

$$|D_r^m(u)|^2 \le C'' \sum_{|\alpha| \le k} |D^\alpha u|^2$$

by extending the integral to Ω we obtain

$$\sup_{\mathbf{x} \in \Omega} |u(\mathbf{x})| \le C\|u\|_m$$

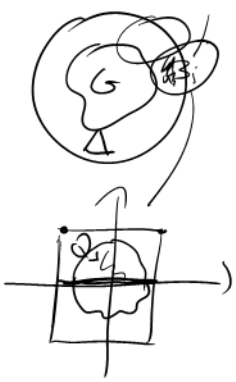
$\sup |u(x)| \le C\|u\|_m$ dla $u \in W_2^m(\Omega)$

which is (1.61).

Theorem 1.50. *Assume that Ω is a bounded open set with C^m boundary and let $m > k + n/2$ where m and k are integers. Then the embedding*

$$W_2^m(\Omega) \subset C^k(\bar{\Omega})$$

is continuous.



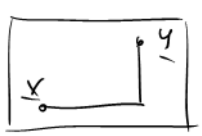
Proof. Under the assumptions, the problem can be reduced to the set $G_0 \Subset \Omega$ consisting of internal point, separated from the boundary by a fixed positive distance, and points in the boundary strip, covered by sets $\bar{\Omega} \cap B_j$ which are transformed onto $Q_+ \cup Q_0$. Any point in G_0 satisfies the cone conditions. Points on $Q_0 \cup Q_+$ also satisfy the condition so, if $u \in W_2^m(\Omega)$, then extending the boundary components of Λu to Q we obtain functions in $W_2^1(\Omega)$ and $W_2^1(Q)$ with compact supports in respective domains. By Friedrichs lemma, restrictions to Ω and Q of $C^\infty(\mathbb{R}^n)$ functions are dense in, respectively, $W_2^m(\Omega)$ and $W_2^m(Q)$ and therefore the estimate (1.61) can be extended by density to $W_2^m(\Omega)$ showing that the canonical injection into $C(\bar{\Omega})$ is continuous. To obtain the result for higher derivatives we substitute higher derivatives of u for u in (1.61). Thus, all components of Λu are they are C^k functions. Transferring them back, we see that $u \in C^k(\bar{\Omega})$, by regularity of the local atlas and $m > k$, we obtain the thesis.

1.4.2 Compact embedding and Rellich–Kondraschov theorem

Lemma 1.51. *let $Q = \{\mathbf{x};\ a_j \le x_j \le b_j\}$ be a cube in $\mathbb{R}^n$ with edges of length $d > 0$. If $u \in C^1(\bar{Q})$, then*

$$\|u\|_{0,Q}^2 \le d^{-n} \left(\int_Q u d\mathbf{x} \right)^2 + \frac{nd^2}{2} \sum_{j=1}^n \|\partial_j u\|_{0,Q}^2 \qquad (1.65)$$

Proof. For any $\mathbf{x}, \mathbf{y} \in Q$ we can write



$$u(\mathbf{x}) - u(\mathbf{y}) = \sum_{j=1}^n \int_{y_j}^{x_j} \partial_j u(y_1, \ldots, y_{j-1}, s, x_{j+1}, \ldots, x_n) ds.$$

$\|\tau_h u - u\|_{L^2(\Omega')} \le |h| \, \|D_h\|$

$\Omega' \subset\subset \Omega$

$\mathcal{F} = \{u\} \subset L_p$ jest wzgl. zwarty jeśli

$\|\tau_h u - u\|_{L_p(\Omega)} \to 0$, $h \to 0$, jednostajnie po $u \in \mathcal{F}$

$\tau_h u(x) = u(x+h)$

Squaring this identity and using Cauchy-Schwarz inequality we obtain

$$u^2(\mathbf{x})+u^2(\mathbf{y})-2u(\mathbf{x})u(\mathbf{y}) \le nd\sum_{j=1}^{n} n\int_{a_j}^{b_j}(\partial_j u)^2(y_1,\ldots,y_{j-1},s,x_{j+1},\ldots,x_n)ds.$$

Integrating the above inequality with respect to all variables, we obtain

$$2d^n\|u\|^2_{0,Q} \le 2\left(\int_Q u d\mathbf{x}\right)^2 + nd^{n+2}\sum_{j=1}^{n}\|\partial_j u\|^2_{0,Q}$$

as required.

Theorem 1.52. *Let Ω be open and bounded. If the sequence $(u_k)_{k\in\mathbb{N}}$ of elements of $\mathring{W}^2_1(\Omega)$ is bounded, then there is a subsequence which converges in in $L_2(\Omega)$. In other words, the injection $\mathring{W}^2_1(\Omega) \subset L_2(\Omega)$ is compact.*

Proof. By density, we may assume $u_k \in C_0^\infty$. Let $M = \sup_k\{\|u_k\|_1\}$. We enclose Ω in a cube Q; we may assume the edges of Q to be of unit length. Further, we extend each u_k by zero to $Q\setminus\Omega$.

We decompose Q into N^n cubes of edges of length $1/N$. Since clearly $(u_k)_{k\in\mathbb{N}}$ is bounded in $L_2(Q)$ it contains a weakly convergent subsequence (which we denote again by $(u_k)_{k\in\mathbb{N}}$). For any ϵ' there is n_0 such that

$\int_{Q_1} |\partial_{x_n} u|^2 dx$

$$\overset{// \langle 1,\, u_k - u_l\rangle}{\left|\int_{Q_j}(u_k-u_l)d\mathbf{x}\right|} < \epsilon', \qquad k,l \ge n_0 \tag{1.66}$$

$\int_{Q_2} |\partial_{x_n} u|^2 dx$

for each $j = 1,\ldots,N^n$. Now, we apply (1.65) on each Q_j and sum over all j getting

$$\|u_k-u_l\|^2_{0,Q} \le N^n\epsilon' + \frac{n}{2N^2}2M^2 \qquad \|\partial_{x_j}u_k - \partial_{x_j}u_l\|^2 \le 2M^2$$

$< d^2$

$\int_{Q_1\cup Q} |\partial_{x_n} u|^2 dx$

Now, we see that for a fixed ϵ we can find N large that $nM^2/N^2 < e$ and, having fixed N, for $\epsilon' = \epsilon/2N^n$ we can find n_0 such that (1.66) holds. Thus $(u_k)_{k\in\mathbb{N}}$ is Cauchy in $L_2(\Omega)$.

Corollary 1.53. *If Ω is a bounded open subset of $\mathbb{R}^n$, then the embedding $\mathring{W}^m_2(\Omega) \subset \mathring{W}^{m-1}_2(\Omega)$ is compact.*

Proof. Applying the previous theorem to the sequences of derivatives, we see that the derivatives form bounded sequences in $W^1_2(\Omega)$ and thus contain subsequences converging in $L_2(\Omega)$. Selecting common subsequence we get convergence in $W^1_2(\Omega)$ etc, (by closedness of derivatives).

$ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$

Theorem 1.54. *If $\partial\Omega$ is a C^m boundary of a bounded open set Ω. Then the embedding $W^m_2(\Omega) \subset W^{m-1}_2(\Omega)$ is compact.*

$\epsilon a \frac{1}{\epsilon} b \le$

$\le \frac{1}{2}\epsilon^2 a^2 + \frac{1}{2}\frac{1}{\epsilon^2} b^2$

Proof. The result follows by extension to $\mathring{W}^m_2(\Omega')$ where Ω' is a bounded set containing Ω.

1.4.3 Trace theorems

We know that if $u \in W_2^m(\Omega)$ with $m > n/2$ then u can be represented by a continuous function and thus can be assigned a value at the boundary of Ω (or, in fact, at any point). The requirement on m is, however, too restrictive — we have solved the Dirichlet problem, which requires a boundary value of the solution, in $\overset{\circ}{W}{}_1^2(\Omega)$. In this space, unless $n = 1$, the solution need not be continuous. It turns out that it is possible to give a meaning to the operation of taking the boundary value of a function even if it is not continuous.

We begin with the simplest (nontrivial) case when $\Omega = \mathbb{R}^n_+ := \{\mathbf{x};\ \mathbf{x} = (\mathbf{x}', x_n), 0 < x_n\}$.

Theorem 1.55. *The trace operator* $\gamma_0 : C^1(\overline{\mathbb{R}^n_+}) \cap W_2^1(\mathbb{R}^n_+) \to C^0(\mathbb{R}^{n-1})$ *defined by*

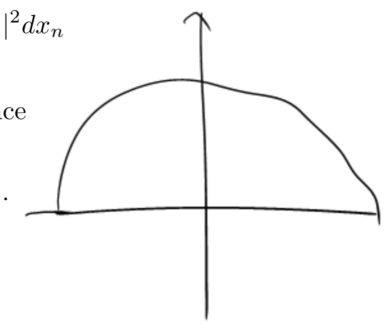
$$(\gamma_0\phi)(\mathbf{x}') = \phi(\mathbf{x}', 0), \qquad \phi \in C^1(\overline{\mathbb{R}^n_+}) \cap W_2^1(\mathbb{R}^n_+), \mathbf{x}' \in \mathbb{R}^{n-1},$$

has a unique extension to a continuous linear operator $\gamma_0 : W_2^1(\mathbb{R}^n_+) \to L_2(\mathbb{R}^{n-1})$ *whose range in dense in* $L_2(\mathbb{R}^{n-1})$. *The extension satisfies*

$$\gamma_0(\beta u) = \gamma_0(\beta)\gamma_0(u), \qquad \beta \in C^1(\overline{\mathbb{R}^n_+}) \cap L_\infty(\mathbb{R}^n_+), u \in W_2^1(\mathbb{R}^n_+).$$

Proof. Let $\phi \in C^1(\overline{\mathbb{R}^n_+}) \cap W_2^1(\mathbb{R}^n_+)$. Then, by Fubini's theorem, for almost any $\mathbf{x}'$, $\partial_{x_n}|\phi(\mathbf{x}', x_n)|^2 \in L_2(\mathbb{R}_+)$ we can write

$$|\phi(\mathbf{x}', r)|^2 - |\phi(\mathbf{x}', 0)|^2 = \int\limits_0^r \partial_{x_n}|\phi(\mathbf{x}', x_n)|^2 dx_n$$



and thus $|\phi(\mathbf{x}', r)|^2$ has a limit which must equal 0. Hence

No $C_0^\infty(\mathbb{R}^n)$

$$|\phi(\mathbf{x}', 0)|^2 = -\int\limits_0^\infty \partial_{x_n}|\phi(\mathbf{x}', x_n)|^2 dx_n.$$

$\|\varphi\|_{L_2(\mathbb{R}^{n-1})} \le C\|\varphi\|_{W_2^1(\mathbb{R}^n_+)}$

Integrating over $\mathbb{R}^{n-1}$ we obtain

$u \in W_2^1(\mathbb{R}^n_+)$

$$\|\phi(\mathbf{x}', 0)\|^2_{0,\mathbb{R}^{n-1}} \le 2\int\limits_{\mathbb{R}^n_+} \partial_{x_n}\phi(\mathbf{x})\phi(\mathbf{x})d\mathbf{x}$$
$$\le 2\|\partial_{x_n}\phi\|_{0,\mathbb{R}^n_+}\|\phi\|_{0,\mathbb{R}^n_+} \le \|\partial_{x_n}\phi\|^2_{0,\mathbb{R}^n_+} + \|\phi\|^2_{0,\mathbb{R}^n_+}.$$

$\varphi_n \to u$ $W_2^1(\mathbb{R}^n_+)$

Hence, by density, the operation of taking value at $x_n = 0$ extends to $W_2^1(\mathbb{R}^n_+)$.

If $\phi \in C_0^\infty(\mathbb{R}^{n-1})$ and τ is a truncation function $\tau(t) = 1$ for $|t| \le 1$ and $\tau(t) = 0$ for $|t| \ge 0$ then $\phi(\mathbf{x}) = \psi(\mathbf{x}')\tau(x_n) \in C^1(\overline{\mathbb{R}^n_+}) \cap W_2^1(\mathbb{R}^n_+)$ and $\gamma_0(\phi) = \psi$ so that the range of the trace operator contains $C_0^\infty(\mathbb{R}^{n-1})$ and thus is dense. The last identity follows from continuity of the trace operator and of the operator of multiplication by bounded differentiable functions in $W_2^1(\mathbb{R}^n_+)$.

$\|\varphi_n - \varphi_k\|_{L_2(\mathbb{R}^{n-1})} \le C\|\varphi_n - \varphi_k\|_{W_2^1(\mathbb{R}^n_+)}$

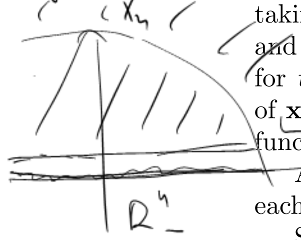
Theorem 1.56. *Let* $u \in W_2^1(\mathbb{R}^n_+)$. *Then* $u \in \overset{\circ}{W}{}_2^1(\mathbb{R}^n_+)$ *if an only if* $\gamma_0(u) = 0$,

Proof. If $u \in \overset{\circ}{W}{}_2^1(\mathbb{R}^n_+)$, then u is the limit of a sequence $(\phi_k)_{k\in\mathbb{N}}$ from $C_0^\infty(\mathbb{R}^n_+)$ in $W_2^1(\mathbb{R}^n_+)$. Since $\gamma_0(\phi_k) = 0$ for any k, we obtain $\gamma_0(u) = 0$.

$u \in W_2^1(\mathbb{R}^n)$

$\gamma u = 0$

Conversely, let $u \in W_2^1(\mathbb{R}^n_+)$ with $\gamma_0 u = 0$. By using the truncating functions, we may assume that u has compact support in $\overline{\mathbb{R}^n_+}$.



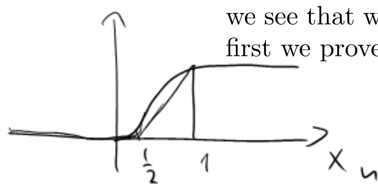
Next we use the truncating functions $\eta_k \in C^\infty(\mathbb{R})$, as in Theorem 1.45, by taking function η which satisfies $\eta(t) = 1$ for $t \geq 1$ and $\eta(t) = 0$ for $t \leq 1/2$ and define $\eta_k(x_n) = \eta(kx_n)$. To simplify notation, we assume that $0 \leq \eta' \leq 3$ for $t \in [1/2, 1]$ so that $0 \leq \eta_k'(x_n) \leq 3k$. Then the extension by 0 to $\mathbb{R}^n_-$ of $\mathbf{x} \to \eta_k(x_n)u(\mathbf{x}', x_n)$ is in $\overset{\circ}{W}{}_2^1(\mathbb{R}^n)$ and can be approximated by $C_0^\infty(\mathbb{R}^n_+)$ functions in $W_2^1(\mathbb{R}^n_+)$. Hence, we have to prove that $\eta_k u \to u$ in $W_2^1(\mathbb{R}^n_+)$. Ω

As in the proof of Theorem 1.45 we can prove $\eta_k u \to u$ in $L_2(\mathbb{R}^n_+)$ and for each $i = 1, \ldots, n-1$, $\partial_{x_i}(\eta_k u) = \eta_k \partial_{x_i} u \to \partial_{x_i} u$ in $L_2(\mathbb{R}^n_+)$ as $k \to \infty$.

Since

$$\partial_{x_n}(\eta_k u) = u\partial_{x_n}\eta_k + \eta_k \partial_{x_i} u \to \partial_{x_n} u$$

we see that we have to prove that $u\partial_{x_n}\eta_k \to 0$ in $L_2(\mathbb{R}^n_+)$ as $k \to \infty$. For this, first we prove that if $\gamma_0(u) = 0$, then



$$u(\mathbf{x}', s) = \int_0^s \partial_{x_n} u(\mathbf{x}', t)dt \tag{1.67}$$

almost everywhere on $\mathbb{R}^n_+$. Indeed, let u_r be a bounded support C^1 function approximating u in $W_2^1(\mathbb{R}^n_+)$. Then $\int_0^s \partial_{x_n} u_r(\mathbf{x}', t)dt \to \int_0^s \partial_{x_n} u(\mathbf{x}', t)dt$ in $L_2(\mathbb{R}^n_+)$. This follows from $\partial_{x_n} u_r \to \partial_{x_n} u$ in $L_2(\mathbb{R}^n_+)$ and, taking Q to be the box enclosing support of all u_r, u, with edges of length at most d

$$\int_Q \left| \int_0^s \partial_{x_n} u_r(\mathbf{x}', t)dt - \int_0^s \partial_{x_n} u(\mathbf{x}', t)dt \right|^2 d\mathbf{x}$$

$$\leq d^2 \int_Q |\partial_{x_n} u_r(\mathbf{x}', t)\cancel{dt} - \partial_{x_n} u(\mathbf{x}', t)\cancel{dt}|^2 d\mathbf{x}$$

Then we have, for any s, $0 \leq s \leq d$

$$\int_Q \left| \int_0^s \partial_{x_n} u_r(\mathbf{x}', t)dt - u_r(\mathbf{x}', s) \right|^2 d\mathbf{x} = \int_Q |u_r(\mathbf{x}', 0)|^2 d\mathbf{x} = d \int_{\mathbb{R}^{n-1}} |u_r(\mathbf{x}', 0)|^2 d\mathbf{x}'$$

and, since the left hand side goes to zero as $r \to \infty$, we obtain (1.67). Then, by Cauchy-Schwarz inequality

$$|u(\mathbf{x}', s)|^2 \leq s \int_0^s |\partial_{x_n} u(x', t)|^2 dt$$

and therefore

$$\int_0^\infty |\eta_k'(s) u(\mathbf{x}', s)|^2 ds \leq 9k^2 \int_0^{2/k} s \int_0^s |\partial_{x_n} u(\mathbf{x}', t)|^2 dt ds$$

$$18k \int_0^{2/k} \int_0^s |\partial_{x_n} u(\mathbf{x}', t)|^2 dt ds = 18k \int_0^{2/k} \int_t^{2/k} |\partial_{x_n} u(\mathbf{x}', t)|^2 ds dt$$

$$\leq 36 \int_0^{2/k} |\partial_{x_n} u(\mathbf{x}', t)|^2 dt.$$

Integration over $\mathbb{R}^{n-1}$ gives

$$\|\eta_k' u\|^2_{0,\mathbb{R}^n_+} \leq 36 \int_{\mathbb{R}^{n-1} \times 2/k} |\partial_{x_n} u|^2 d\mathbf{x}$$

which tends to 0.

The consideration above can be extended to the case where Ω is an open bounded region in $\mathbb{R}^n$ lying locally on one side of its C^1 boundary. Using the partition of unity, we define

$$\gamma_0(u) := \sum_{j=1}^N (\gamma_0((\beta_j u) \circ H^j)) \circ (H^j)^{-1}$$

It is clear that if $u \in C^1(\bar{\Omega})$, then $\gamma_0 u$ is the restriction of u to $\partial\Omega$. Thus, we have the following result

Theorem 1.57. *Let Ω be a bounded open subset of $\mathbb{R}^n$ which lies on one side of its boundary $\partial\Omega$ which is assumed to be a C^1 manifold. Then there exists a unique continuous and linear operator $\gamma_0 : W_2^1(\Omega) \to L_2(\partial\Omega)$ such that for each $u \in C^1(\bar{\Omega})$, γ_0 is the restriction of u to $\partial\Omega$. The kernel of γ_0 is equal to $\overset{\circ}{W}{}_2^1(\Omega)$ and its range is dense in $L_2(\partial\Omega)$.*

1.4.4 Regularity of variational solutions to the Dirichlet problem

From Subsection 1.3.6 we know that there is a unique variational solution $u \in \overset{\circ}{W}{}_1^2(\Omega)$ of the problem

$$\int_{\Omega} \nabla u \nabla v d\mathbf{x} = < f, v >_{(\overset{\circ}{W}{}_1^2(\Omega))^* \times \overset{\circ}{W}{}_1^2(\Omega)}, \qquad v \in \overset{\circ}{W}{}_1^2(\Omega).$$

Moreover, now we can say that $\gamma_0 u = 0$ on $\partial\Omega$ (provided $\partial\Omega$ is C^1).

We have the following theorem

Theorem 1.58. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with C^2 boundary (or $\Omega = \mathbb{R}^n_+$). Let $f \in L_2(\Omega)$ and let $u \in \overset{\circ}{W}{}_1^2(\Omega)$ satisfy*

$$\int_{\Omega} \nabla u \nabla v d\mathbf{x} = (f, v), \qquad v \in \overset{\circ}{W}{}_1^2(\Omega). \tag{1.68}$$

Then $u \in W_2^2(\Omega)$ and $\|u\|_{2,\Omega} \leq C\|f\|_{0,\Omega}$ where C is a constant depending only on Ω. Furthermore, if Ω is of class C^{m+2} and $f \in W_2^m(\Omega)$, then

$$u \in W_2^{m+2}(\Omega) \quad \text{and} \quad \|u\|_{m+2,\Omega} \leq C\|f\|_{m,\Omega}.$$

In particular, if $m \geq n/2$, then $u \in C^2(\bar{\Omega})$ is a classical solution.

Moreover, if Ω is bounded, then the solution operator $G : L_2(\Omega) \to \overset{\circ}{W}{}_1^2(\Omega)$ is self-adjoint and compact.

Proof. The proof naturally splits into two cases: interior estimates and boundary estimates. Let Ω be bounded with at least C^1 boundary and consider the partition of unity $\{\beta_j\}_{j=0}^N$ subordinated to the covering $\{G_j\}_{j=0}^N$. For the interior estimates let us consider $u_0 = \beta_0 u$ and let $v \in \overset{\circ}{W}{}_1^2(\Omega)$. Then we can write

$$\begin{aligned}
\int_{\Omega} \nabla(\beta_0 u) \nabla v d\mathbf{x} &= \int_{\Omega} \beta_0 \nabla u \nabla v d\mathbf{x} + \int_{\Omega} u \nabla \beta_0 \nabla v d\mathbf{x} \\
&= \int_{\Omega} \nabla u \nabla(\beta_0 v) d\mathbf{x} - 2\int_{\Omega} \nabla u v \nabla \beta_0 d\mathbf{x} - \int_{\Omega} u v \Delta \beta_0 d\mathbf{x} \\
&= \int_{\Omega} (f - \Delta\beta_0 u - 2\nabla u \nabla \beta_0) v d\mathbf{x} = \int_{\Omega} F v d\mathbf{x}.
\end{aligned}$$