$$a(x, ty - x) \geq < \phi, ty - x >_{H^* \times H} .$$

for any $t \in R, v \in H$. Factoring out $t$, we find

$$ta(x, y - xt^{-1}) \geq t < \phi, y - xt^{-1} >_{H^* \times H} .$$

and passing with $t \to \pm\infty$, we obtain

$$a(x, y) \geq < \phi, y >_{H^* \times H}, \qquad a(x, y) \leq < \phi, y >_{H^* \times H} .$$

*Remark 1.38.* Elementary proof of the Lax–Milgram theorem. As we noted earlier

$$a(x, y) = < \phi, y >_{H^* \times H}$$

can be written as the equation

$$(Ax, y) = (f, y)$$

for any $y \in H$, where $A : H \to H$, $\|Ax\| \leq C\|x\|$ and $(Ax, x) \geq \alpha\|x\|^2$. From the latter, $Ax = 0$ implies $x = 0$, hence $A$ is injective. Further, if $y = Ax$, $x = A^{-1}y$ and

$$\|x\|^2 = \|A^{-1}y\|\|x\| \leq \alpha^{-1}(y, x) \leq \alpha^{-1}\|y\|\|x\|$$

so $A^{-1}$ is bounded. This shows that the range of $A$, $R(A)$, is closed. Indeed, if $(y_n)_{n\in\mathbb{N}}$ , $y_n \in R(A)$, $y_n \to y$, then $(y_n)_{n\in\mathbb{N}}$ is Cauchy, but then $(x_n)_{n\in\mathbb{N}}$, $x_n = A^{-1}$ is also Cauchy and thus converges to some $x \in A$. But then, from continuity of $A$, $Ax = y$. On the other hand, $R(A)$ is dense. For, if for some $v \in H$ we have $0 = (Ax, v)$ for any $x \in H$, we can take $v = x$ and

$$0 = (Av, v) \geq \alpha\|v\|^2$$

so $v = 0$ and so $R(A)$ is dense.

### 1.3.6 Dirchlet problem

Let us recall the variational formulation of the Dirichlet problem: find $u \in ?$ such that

$$\int\limits_{\Omega} \nabla u \cdot \nabla \phi dx = \int\limits_{\Omega} f \phi dx. \tag{1.49}$$

for all $C_0^\infty(\Omega)$. We also recall the associated minimization problem for

$$J(u) = \frac{1}{2} \int\limits_{\Omega} |\nabla u|^2 dx - \int\limits_{\Omega} f u dx \tag{1.50}$$

over some closed subspace $K = \{u \in ?\}$.

Let us consider the space $H = L_2(\Omega)$, $\Omega \subset \mathbb{R}^n$ bounded, with the scalar product

$$(u, v)_0 = \int_\Omega u(x)v(x)dx.$$

We know that $\overline{C_0^\infty(\Omega)}^H = H$. The relation (1.49) suggests that we should consider another scalar product, initially on $C_0^\infty(\Omega)$, given by

$$(u, v)_{\cancel{1}} = \int_\Omega \nabla u(x) \nabla v(x) dx.$$

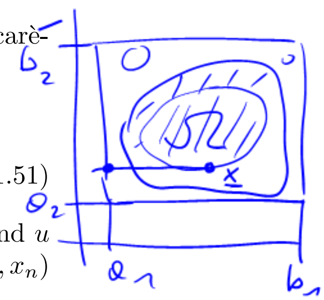
(0,1)

Note that due to the fact that $u, v$ have compact supports, this is a well defined scalar product as

$$0 = (u, u)_{0,1} = \int_\Omega |\nabla u(x)|^2 dx$$

implies $u_{x_i} = 0$ for all $x_i$, $i = 1, \ldots, n$ hence $u = const$ and thus $u \equiv 0$. Note that this is not a scalar product on a space $C^\infty(\bar{\Omega})$.

A fundamental role in the theory is played by the Zaremba - Poincarè-Friedrichs lemma.




**Lemma 1.39.** *There is a constant $d$ such that for any $u \in C_0^\infty(\Omega)$*

$$\|u\|_0 \leq d\|u\|_{0,1}. \tag{1.51}$$

*Proof.* Let $R$ be a box $[a_1, b_1] \times \ldots \times [a_n, b_n]$ such that $\bar{\Omega} \subset R$ and extend $u$ by zero to $R$. Since $u$ vanishes at the boundary of $R$, for any $\mathbf{x} = (x_1, \ldots, x_n)$ we have

$$u(\mathbf{x}) = \int_{a_i}^{x_i} u_{x_i}(x_1, \ldots, t, \ldots)dt$$

and, by Schwarz inequality,

$$u^2(\mathbf{x}) = \left( \int_{a_i}^{x_i} u_{x_i}(x_1, \ldots, t, \ldots, x_n)dt \right)^2 \leq \left( \int_{a_i}^{x_i} 1dt \right) \left( \int_{a_i}^{x_i} u_{x_i}^2(x_1, \ldots, t, \ldots, x_n)dt \right)$$

$$\leq (b_i - a_i) \int_{a_i}^{b_i} u_{x_i}^2(x_1, \ldots, t, \ldots, x_n)dt$$

for any $\mathbf{x} \in R$. Integrating over $R$ we obtain

$$\int_R u^2(\mathbf{x})d\mathbf{x} \leq (b_i - a_i)^2 \int_R u_{x_i}^2(\mathbf{x})d\mathbf{x}.$$

This can be re-written

$$\int\limits_{\Omega} u^2(\mathbf{x})d\mathbf{x} \leq (b_i - a_i)^2 \int\limits_{\Omega} u^2_{x_i}(\mathbf{x})d\mathbf{x} \leq c \int\limits_{\Omega} |\nabla u(\mathbf{x})|^2 d\mathbf{x}$$

We see that the lemma remains valid if $\Omega$ is bounded just in one direction. Let us define $\overset{\circ}{W}{}^1_2(\Omega)$ as the completion of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{0,1}$. We have

**Theorem 1.40.** *The space $\overset{\circ}{W}{}^1_2(\Omega)$ is a separable Hilbert space which can be identified with a subspace continuously and densely embedded in $L_2(\Omega)$. Every $v \in \overset{\circ}{W}{}^1_2(\Omega)$ has generalized derivatives $D_{x_i}v \in L_2(\Omega)$. Furthermore, the distributional integration by parts formula*

$$\int\limits_{\Omega} D_{x_i} v u d\mathbf{x} = -\int\limits_{\Omega} v D_{x_i} u d\mathbf{x} \tag{1.52}$$

*is valid for any $u, v \in \overset{\circ}{W}{}^1_2(\Omega)$.*

*Proof.* . The completion in the scalar product gives a Hilbert space. By Lemma 1.39, every equivalence class of the completion in the norm $\|\cdot\|_{0,1}$ is also an equivalence class in $\|\cdot\|_0$ and thus can be identified with the element of $\overline{C_0^\infty(\Omega)}^{\|\cdot\|_0}$ and thus with an element $v \in L_2(\Omega)$. This identification is one-to-one. Density follows from $C_0^\infty(\Omega) \subset \overset{\circ}{W}{}^1_2(\Omega) \subset L_2(\Omega)$ and continuity of injection from Lemma 1.39.

If $(v_n)_{n\in\mathbb{N}}$ of $C_0^\infty(\Omega)$ functions converges to $v \in \overset{\circ}{W}{}^1_2(\Omega)$ in $\|\cdot\|_{0,1}$, then $v_n \to v$ in $L_2(\Omega)$ and $D_{x_i}v_n \to v^i$ in $L_2(\Omega)$ for some functions $v^i \in L_2(\Omega)$. Taking arbitrary $\phi \in C_0^\infty(\Omega)$, we obtain

$$\int\limits_{\Omega} D_{x_i} v_n \phi d\mathbf{x} = -\int\limits_{\Omega} v_n D_{x_i} \phi d\mathbf{x}$$

and we can pass to the limit

$$\int\limits_{\Omega} v^i \phi d\mathbf{x} = -\int\limits_{\Omega} v D_{x_i} \phi d\mathbf{x}$$

showing that $v^i = D_{x_i}v$ in generalized sense. Furthermore, we can pass to the limit in $\|\cdot\|_{0,1}$ with $\phi \to u \in \overset{\circ}{W}{}^1_2(\Omega)$ and, by the above, $D_{x_i}\phi \to D_{x_i}u$ in $L_2(\Omega)$, giving (1.52). This also shows that $\overset{\circ}{W}{}^1_2(\Omega)$ can be identified with a closed subspace of $(L_2(\Omega))^n$ (the graph of gradient) and thus it is a separable space.

$X$ - ośrodkowa

$Y$ - domknięta

$\{x_n\} \subset X$

$B(x_n, r_n)$ $r_n$ - wymierne.

$\forall_{r_n} \exists_{x_n} B(x_n, r_n) \cap Y \neq \emptyset$

$y_n \in Y$

$y_n \in B(x_n, r_n) \cap Y$

$-\Delta u + \underline{b} \cdot Du + cu$



$Au = \sum_i \partial_{x_i}(a_{ij} \partial_{x_j} u)$

$-\sum_{i,j} a_{ij} \xi_i \xi_j \geq \gamma|\xi|^2$

Consider now on $\overset{\circ}{W}{}^1_2(\Omega)$ the bilinear form

$$a(u,v) = \int_\Omega \nabla u \nabla v d\mathbf{x}.$$

Clearly, by Schwarz inequality

$$|a(u,v)| \leq \|u\|_{0,1}\|v\|_{0,1}$$

and

$$a(u,u) = \int_\Omega \nabla u \nabla u d\mathbf{x} = \|u\|^2_{0,1}$$

and thus $a$ is a continuous and coercive bilinear form on $\overset{\circ}{W}{}^1_2(\Omega)$. Thus, if we take $f \in (\overset{\circ}{W}{}^1_2(\Omega))^* \supset L_2(\Omega)$ then there is a unique $u \in \overset{\circ}{W}{}^1_2(\Omega)$ satisfying

$$\int_\Omega \nabla u \nabla v d\mathbf{x} = < f, v >_{(\overset{\circ}{W}{}^1_2(\Omega))^* \times \overset{\circ}{W}{}^1_2(\Omega)}$$

for any $v \in \overset{\circ}{W}{}^1_2(\Omega)$ or, equivalently, minimizing the functional

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 d\mathbf{x} - < f, v >_{(\overset{\circ}{W}{}^1_2(\Omega))^* \times \overset{\circ}{W}{}^1_2(\Omega)}$$

over $K = \overset{\circ}{W}{}^1_2(\Omega)$.

The question is what this solution represents. Clearly, taking $v \in C_0^\infty(\Omega)$ we obtain

$$-\Delta u = f$$

in the sense of distribution. However, to get a deeper understanding of the meaning of the solution, we have investigate the structure of $\overset{\circ}{W}{}^1_2(\Omega)$.

### 1.3.7 Sobolev spaces

Let $\Omega$ be a nonempty open subset of $\mathbb{R}^n$, $n \geq 1$ and let $m \in \mathbb{N}$. The Sobolev space $W_2^m(\Omega)$ consists of all $u \in L_2(\Omega)$ for which all generalized derivatives $D^\alpha u$ exist and belong to $L_2$. $W_2^m(\Omega)$ is equipped with the scalar product

$$(u,v)_m = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u D^\alpha v d\mathbf{x}. \tag{1.53}$$

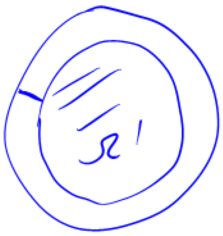
In particular,

$$(u,v)_1 = \int_\Omega (uv + \nabla u \nabla v) d\mathbf{x}.$$

We obtain

**Proposition 1.41.** *The space $W_2^m(\Omega)$ is a separable Hilbert space.*

*Proof.* The proof follows since the generalized differentiation is a closed operator in $L_2(\Omega)$.



We note that $\overset{\circ}{W}{}_2^1(\Omega)$ is a closed subspace of $W_2^1(\Omega)$ as the norms $\|\cdot\|_{0,1}$ and $\|\cdot\|_1$ ~~coincide there~~. are equivalent there.

We shall focus on the case $m = 1$. A workhorse of the theory is the Friedrichs lemma.

**Lemma 1.42.** *Let $u \in W_2^1(\Omega)$. Then there exists a sequence $(u_k)_{k\in\mathbb{N}}$ from $C_0^\infty(\mathbb{R}^n)$ such that*

$$u_k|_\Omega \to u \quad \text{in} \quad L_2(\Omega) \tag{1.54}$$

*and for any $\Omega' \Subset \Omega$*

$\bar\Omega' \subset \Omega$

$$\nabla u_k|_{\Omega'} \to \nabla u \quad \text{in} \quad L_2(\Omega) \tag{1.55}$$

*If $\Omega = \mathbb{R}^n$, then both convergences occur in $\mathbb{R}^n$.*

Dla dow zbioru otwartego

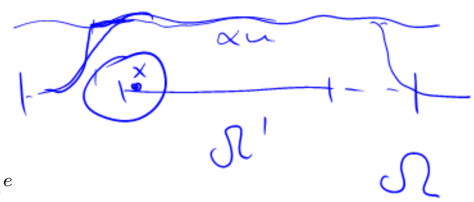
$\Omega^\varepsilon = \bigcup_{x\in\Omega} B(x,\varepsilon)$

*Proof.* Set

$$u^e(x) = \begin{cases} u(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \notin \Omega \end{cases}$$

and define $v_\epsilon = u^e * \omega_\epsilon$. We know $v_\epsilon \in C^\infty(\mathbb{R}^n)$ and $v_\epsilon \to u$ in $L_2(\Omega)$. Let us take $\Omega' \Subset \Omega$ and fix a function $\alpha \in C_0^\infty(\Omega)$ which equals 1 on a neighbourhood of $\Omega'$. Then, for sufficiently small $\epsilon$, we have

$$\omega_\epsilon * (\alpha u)^\epsilon = \omega_\epsilon * u^\epsilon$$



on $\Omega'$. Then, by Proposition 1.6,

$$\partial_j(\omega_\epsilon * (\alpha u)^\epsilon) = \omega_\epsilon * (\alpha\partial_j u + \partial_j\alpha u)^e$$

$\varphi = \begin{cases} 1 & \text{na } \Omega^\varepsilon \\ 0 & \text{na } \mathbb{R}^n \setminus \Omega^{2\varepsilon} \end{cases}$

hence

$$\partial_j(\omega_\epsilon * (\alpha u)^\epsilon) \to (\cancel{\alpha}\partial_j u + \cancel{\partial_j\alpha u})^e$$

in $L_2(\Omega)$ and, in particular,

$$\partial_j(\omega_\epsilon * (\alpha u)^\epsilon) \to \partial_j u$$

$\varphi = \omega_\eta * \chi_{\Omega^\varepsilon}$

in $L_2(\Omega')$. But on $\Omega'$ we can discard $\alpha$ to get

$$\partial_j(\omega_\epsilon * u^\epsilon) \to \partial_j u.$$

$\varphi(x) = \int_{\mathbb{R}^n} \omega_\varepsilon(x-y)\,\chi_{\Omega^\varepsilon}(y)\,dy$

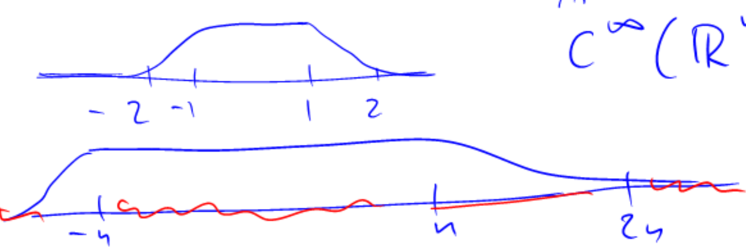
$x \in \Omega$

If $v_k$ do not have compact support (e.g. when $\Omega$ is not bounded), then we multiply $v_k$ by a sequence of smooth cut-off functions $\zeta_k = \zeta(x/k)$ where $\zeta(x) = 1$ for $|x| \le 1$ and $\zeta(x) = 0$ for $|x| \ge 2$.

As an immediate application we show

$u_k \longrightarrow u \quad W_2^1(\mathbb{R}^n)$

$u_k \in C^\infty(\mathbb{R}^n)$



$\zeta_k u_k \longrightarrow u$

$\zeta_k u_k' + \zeta_k' u_k$

$$\overset{\circ}{W}{}^1_2(\mathbb{R}^n) = W^1_2(\mathbb{R}^n)$$



**Proposition 1.43.** *(i) Let* $u, v \in W_2^1(\Omega) \cap L_\infty(\Omega)$*. Then* $uv \in W_2^1(\Omega) \cap L_\infty(\Omega)$ *with*

$$\partial_j(uv) = \partial_j uv + u\partial_j v, \qquad i = 1, \ldots, n \tag{1.56}$$

*Let* $\Omega, \Omega_1$ *be two open sets in* $\mathbb{R}^n$ *and let* $H : \Omega_1 \to \Omega$ *be a* $C^1(\bar{\Omega})$ *diffeomorphism. If* $u \in W_2^1(\Omega)$ *then* $u \circ H \in W_2^1(\Omega')$ *and*

$$\int\limits_{\Omega_1} (u \circ H)\partial_j \phi d\mathbf{y} = -\int\limits_{\Omega_1} \sum_{i=1}^{n} (\partial_i u \circ H))\partial_j H_i \phi d\mathbf{y} \tag{1.57}$$

*Proof.* Using Friedrichs lemma, we find sequences $(u_k)_{k\in\mathbb{N}}$, $(v_k)_{k\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ such that

$$u_k \to u, \qquad v_k \to v$$

in $L_2(\Omega)$ and for any $\Omega' \Subset \Omega$ we have

$$\nabla u_k \to \nabla u, \qquad \nabla v_k \to \nabla v$$

in $L_2(\Omega')$. Moreover, from the construction of the mollifiers we get

$$\|u_k\|_{L_\infty(\Omega)} \leq \|u\|_{L_\infty(\Omega)} \qquad \|v_k\|_{L_\infty(\Omega)} \leq \|v\|_{L_\infty(\Omega)}.$$

On the other hand

$$\int\limits_{\Omega} u_n v_k \partial_j \phi d\mathbf{x} = -\int\limits_{\Omega} (\partial_j u_n v_k + u_k \partial_j v_k)\phi d\mathbf{x}$$

for any $\phi \in C_0^\infty(\Omega)$. Thanks to the compact support of $\phi$, the integration actually occurs over compact subsets of $\Omega$ and we can use $L_2$ convergence of $\nabla u_k, \nabla v_k$. Thus

$$\int\limits_{\Omega} uv\partial_j \phi d\mathbf{x} = -\int\limits_{\Omega} (\partial_j uv + u\partial_j v)\phi d\mathbf{x}$$

and the fact that $uv \in W_2^1(\Omega)$ follows from $\partial_j u, \partial_j v \in W_2^1(\Omega)$ and $u, v \in L_\infty(\Omega)$. The proof of the second statement follows similarly. We select sequence $(u_k)_{k\in\mathbb{N}}$ as above; then clearly $u_k \circ H \to u \circ H$ in $L_2(\Omega_1)$ and

$$(\partial_i u_k \circ H)\partial_j H_i \to (\partial_i u \circ H)\partial_j H_i$$

in $L_2(\Omega_1')$ for any $\Omega_1' \Subset \Omega$. For any $\psi \in C_0^\infty(\Omega_1)$ we get

$$\int\limits_{\Omega_1} (u_k \circ H)\partial_j \phi d\mathbf{y} = -\int\limits_{\Omega_1} \sum_{i=1}^{k} (\partial_i u_k \circ H))\partial_j H_i \phi d\mathbf{y}$$

and in the limit we obtain (1.57).