1.3 Hilbert space methods

One of the most often used theorems of functional analysis is the Riesz representation theorem.

Theorem 1.30 (Riesz representation theorem). *If x^* is a continuous linear functional on a Hilbert space H, then there is exactly one element $y \in H$ such that*

$$< x^*, x >= (x, y). \tag{1.32}$$

1.3.1 To identify or not to identify–the Gelfand triple

Riesz theorem shows that there is a canonical isometry between a Hilbert space H and its dual H^*. It is therefore natural to identify H and H^* and is done so in most applications. There are, however, situations when it cannot be done.

Assume that H is a Hilbert space equipped with a scalar product $(\cdot,\cdot)_H$ and that $V \subset H$ is a subspace of H which is a Hilbert space in its own right, endowed with a scalar product $(\cdot,\cdot)_V$. Assume that V is densely and continuously embedded in H that is $\overline{V} = H$ and $\|x\|_H \le c\|x\|_V$, $x \in V$, for some constant c. There is a canonical map $T : H^* \to V^*$ which is given by restriction to V of any $h^* \in H^*$:

$$< Th^*, v >_{V^*\times V} =< h^*, v >_{H^*\times H}, \quad v \in V.$$

We easily see that

$$\|Th^*\|_{V^*} \le C\|h^*\|_{H^*}.$$

Indeed

$$\begin{aligned}\|Th^*\|_{V^*} &= \sup_{\|v\|_V \le 1} | < Th^*, v >_{V^*\times V} | = \sup_{\|v\|_V \le 1} | < h^*, v >_{H^*\times H} | \\ &\le \|h^*\|_{H^*} \sup_{\|v\|_V \le 1} \|v\|_H \le c\|h^*\|_{H^*}.\end{aligned}$$

Further, T is injective. For, if $Th_1^* = Th_2^*$, then

$$0 =< Th_1^* - Th_2^*, v >_{V^*\times V} =< h_1^* - h_2^*, v >_{H^*\times H}$$

for all $v \in V$ and the statement follows from density of V in H. Finally, the image of TH^* is dense in V^*. Indeed, let $v \in V^{**}$ be such that $< v, Th^* >= 0$ for all $h^* \in H^*$. Then, by reflexivity,

$$0 =< v, Th^* >_{V^{**}\times V^*} =< Th^*, v >_{V^*\times V} =< h^*, v >_{H^*\times H}, \quad h^* \in H^*$$

implies $v = 0$.

Now, if we identify H^* with H by the Riesz theorem and using T as the canonical embedding from H^* into V^*, one writes

$$V \subset H \simeq H^* \subset V^*$$

$V \subset H \subset V^*$

Gelfand triple

and the injections are dense and continuous. In such a case we say that H is the pivot space. Note that the scalar product in H coincides with the duality pairing $< \cdot, \cdot >_{V^* \times V}$:

rigged triplet

$$(f, g)_H = < f, g >_{V^* \times V}, \quad f \in H, g \in V.$$

Remembering now that V is a Hilbert space with scalar product $(\cdot, \cdot)_V$ we see that identifying also V with V^* would lead to an absurd – we would have $V = H = H^* = V^*$. Thus, we cannot identify simultaneously both pairs. In such situations it is common to identify the pivot space H with its dual H^* bur to leave V and V^* as separate spaces with duality pairing being an extension of the scalar product in H.

An instructive example is $H = L_2([0, 1], dx)$ (real) with scalar product

$$(u, v) = \int\limits_0^1 u(x)v(x)dx$$

and $V = L_2([0, 1], wdx)$ with scalar product

$$(u, v) = \int\limits_0^1 u(x)v(x)w(x)dx,$$

where w is a nonnegative ~~bounded~~ unbounded measurable function. Then it is useful to identify $V^* = L_2([0, 1], w^{-1}dx)$ and

$$< f, g >_{V^* \times V} = \int\limits_0^1 f(x)g(x)dx \le \int\limits_0^1 f(x)\sqrt{w(x)}\frac{g(x)}{\sqrt{w(x)}}dx \le \|f\|_V \|g\|_{V^*}.$$

1.3.2 The Radon-Nikodym theorem

Let μ and ν be finite nonnegative measures on the same σ-algebra in Ω. We say that ν is absolutely continuous with respect to μ if every set that has μ-measure 0 also has ν measure 0.

Theorem 1.31. *If ν is absolutely continuous with respect to μ then there is an integrable function g such that*

$$\nu(E) = \int\limits_E g d\mu, \tag{1.33}$$

for any μ-measurable set $E \subset \Omega$.

Proof. Assume for simplicity that $\mu(\Omega), \nu(\Omega) < \infty$. Let $H = L_2(\Omega, d\mu + d\nu)$ on the field of reals. Schwarz inequality shows that if $f \in H$, then $f \in L_1(d\mu + d\nu)$. The linear functional

$$< x^*, f >= \int\limits_{\Omega} f d\mu$$

To be completed

1.3.3 Projection on a convex set

Corollary 1.32. *Let K be a closed convex subset of a Hilbert* real *space H. For any $x \in H$ there is a unique $y \in K$ such that*

$$\|x - y\| = \inf_{z \in K} \|x - z\|. \tag{1.34}$$

Moreover, $y \in K$ is a unique solution to the variational inequality

$$(x - y, z - y) \le 0 \tag{1.35}$$

for any $z \in K$.

$\|x - z_n\| \le d + \frac{1}{n}$
$\|z_n\| \le \|x\| + d + 1$

Proof. Let $d = \inf_{z \in K} \|x - z\|$. We can assume $x \notin K$ and so $d > 0$. Consider $f(z) = \|x - z\|$, $z \in K$ and consider a minimizing sequence $(z_n)_{n \in \mathbb{N}}$, $z_n \in K$ such that $d \le f(z_n) \le d + 1/n$. By the definition of f, $(z_n)_{n \in \mathbb{N}}$ is bounded and thus it contains a weakly convergent subsequence, say $(\zeta_n)_{n \in \mathbb{N}}$. Since K is closed and convex, by Corollary 1.24, $\zeta_n \rightharpoonup y \in K$. Further we have

$\|x - y\| = \sup_{\|h\|=1} |(h, x-y)|$

$$|(h, x-y)| = \lim_{n \to \infty} |(h, x-\zeta_n)| \le \|h\| \liminf_{n \to \infty} \|x - \zeta_n\| \le \|h\| \liminf_{n \to \infty} d + \frac{1}{n} = \|h\| d$$

for any $h \in H$ and thus, taking supremum over $\|h\| \le 1$, we get $f(y) \le d$ which gives existence of a minimizer.

To prove equivalence of (1.35) and (1.34) assume first that $y \in K$ satisfies (1.34) and let $z \in K$. Then, from convexity, $v = (1-t)y + tz \in K$ for $t \in [0, 1]$ and thus

$$\|x - y\|^2 \le \|x - ((1-t)y + tz)\| = \|(x - y) - t(z - y)\|^2$$

$= ((x-y) - t(z-y), (x-y) - t(z-y))$

and thus

$$\|\cancel{x} - y\|^2 \le \|\cancel{x - y}\|^2 - 2t(x - y, z - y) + t^2 \|z - y\|^2.$$

Hence

$$t\|z - y\|^2 \ge 2(x - y, z - y)$$

for any $t \in (0, 1]$ and thus, passing with $t \to 0$, $(x - y, z - y) \le 0$. Conversely, assume (1.35) is satisfied and consider

$(x,x) - (x,y) - (y,x) + (y,y) - (x,x) + 2(x,z) - (z,z)$

$$\|x-y\|^2 - \|x-z\|^2 = (x-y, x-y) - (x-z, x-z)$$
$$= 2(x,z) - 2(x,y) + 2(y,y) - 2(y,z) + 2(y,z) - (y,y)$$
$$0 \geq\; = 2(x-y, z-y) - (y-z, y-z) \leq 0$$

$= 2(x,z) - 2(x,y) - 2(y,z) + 2(y,y) - (y,y) - (z,z) + 2(y,z)$

hence

$$\|x-y\| \leq \|x-z\|$$

for any $z \in K$.

$(x - y_1, y_2 - y_1) \leq 0$

$(x - y_2, y_1 - y_2) \leq 0$

For uniqueness, let y_1, y_2 satisfy

$$(x - y_1, z - y_1) \leq 0, \qquad (x - y_2, z - y_2) \leq 0, \qquad z \in H.$$

Choosing $z = y_2$ in the first inequality and $z = y_1$ in the second and adding them, we get $\|y_1 - y_2\|^2 \leq 0$ which implies $y_1 = y_2$. ■

$-(y_1, y_2 - y_1) - (y_2, y_1 - y_2) \leq 0$

We call the operator assigning to any $x \in K$ the element $y \in K$ satisfying (1.34) the projection onto K and denote it by P_K.

Proposition 1.33. *Let K be a nonempty closed and convex set. Then P_K is non expansive mapping.*

Proof. Let $y_i = P_K x_i$, $i = 1, 2$. We have

$$(x_1 - y_1, z - y_1) \leq 0, \qquad (x_2 - y_2, z - y_2) \leq 0, \qquad z \in H$$

so choosing, as before, $z = y_2$ in the first and $z = y_1$ in the second inequality and adding them together we obtain

$(x_1 - y_1, y_2 - y_1) \leq 0$

$(x_2 - y_2, y_1 - y_2) \leq 0$

$(x_1, y_2 - y_1) + (x_2, y_1 - y_2) - (y_1, y_2 - y_1) + (y_2, y_2 - y_1) \leq 0$

$$\|y_1 - y_2\|^2 \leq (x_1 - x_2, y_1 - y_2),$$

hence $\|P_K x_1 - P_K x_2\| \leq \|x_1 - x_2\|$.

$\|P_K x_1 - P_K x_2\|^2 \leq \|x_1 - x_2\| \, \|P_K x_1 - P_K x_2\|$

$(x_2 - x_1, y_2 - y_1) + (y_2 - y_1, y_2 - y_1) \leq 0$

1.3.4 Theorems of Stampacchia and Lax-Milgram

1.3.5 Motivation

Consider the Dirichlet problem for the Laplace equation in $\Omega \subset \mathbb{R}^n$

$$-\Delta u = f \quad \text{in} \quad \Omega, \tag{1.36}$$
$$u|_{\partial\Omega} = 0. \quad g \tag{1.37}$$

Assume that there is a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$. If we multiply (1.36) by a test function $\phi \in C_0^\infty(\Omega)$ and integrate by parts, then we obtain the problem

$\int_\Omega \Delta u \varphi dx = \int_{\partial\Omega} \varphi \nabla u \cdot n \, ds - \int_\Omega \nabla u \nabla \varphi dx$

$$\int_\Omega \nabla u \cdot \nabla \phi dx = \int_\Omega f \phi dx. \tag{1.38}$$

$J(u) = K\int_\Omega \sqrt{1+|\nabla \varphi|^2}\,dx$

Conversely, if u satisfies (1.38), then it is a distributional solution to (1.37).

Moreover, if we consider the minimization problem for

Cerchio Dirichlet

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega f u dx$$

over $K = \{u \in C^2(\Omega);\ u|_{\partial\Omega} = 0\}$ and if u is a solution to this problem then for any $\epsilon \in \mathbb{R}$ and $\varphi \in C_0^\infty(\Omega)$ we have

$|\nabla u + \varepsilon \nabla \varphi|^2$

$\varepsilon > 0$

$$J(u + \epsilon\phi) \geq J(u),$$

$\varepsilon < 0$ $\quad \varepsilon^2\int|\nabla\varphi|^2 + \varepsilon\int 2\nabla u \nabla\varphi + \varepsilon\int f\varphi \geq 0$

$= |\nabla u|^2 + \varepsilon^2|\nabla\varphi|^2 + \varepsilon 2\nabla u \nabla\varphi$

then we also obtain (1.38). The question is how to obtain the solution.

In a similar way, we consider the obstacle problem, to minimize J over $K = \{u \in C^2(\Omega);\ u|_{\partial\Omega} = 0,\ u \geq g\}$ over some continuous function g satisfying $g|_{\partial\Omega} < 0$. Note that K is convex. Again, if $u \in K$ is a solution then for any $\epsilon > 0$ and $\phi \in K$ we obtain that $u + \epsilon(\phi - u) = (1-\epsilon)u + \epsilon\phi$ is in K and therefore

$$J(u + \epsilon(\phi - u)) \geq J(u).$$

Here, we obtain only

$1 = \alpha + \beta$

$$\int_\Omega \nabla u \cdot \nabla(\phi - u) dx \geq \int_\Omega f(\phi - u) dx. \tag{1.39}$$

$\alpha u + \beta v - 1g =$
$= \alpha u + \beta v - \alpha g - \beta g$
$= \alpha(u-g) + \beta(v-g)$
≥ 0

for any $\phi \in K$. For twice differentiable u we obtain

$$-\int_\Omega \Delta u(\phi - u) dx \geq \int_\Omega f(\phi - u) dx$$

$u = g$ na $\partial\Omega$

and choosing $\phi = u + \psi$, $\psi \in C_0^\infty(\Omega)$ we get

≥ 0

$$-\Delta u \geq f$$

almost everywhere on Ω. As u is continuous, the set $N = \{x \in \Omega;\ u(x) > g(x)\}$ is open. Thus, taking $\psi \in C_0^\infty(N)$, we see that for sufficiently small $\epsilon > 0$, $u \pm \epsilon\phi \in K$. Then, on N

$$\underbrace{-\Delta u = f}$$

Summarizing, for regular solutions the minimizer satisfies

$$-\Delta u \geq f$$
$$u \geq g$$
$$(\Delta u + f)(u - g) = 0$$

on Ω.

Hilbert space theory

We begin with the following definition.

Definition 1.34. *Let H be a Hilbert space. A bilinear form $a : H \times H \to \mathbb{R}$ is said to be*

(i) continuous of there is a constant C such that

$$|a(x,y)| \leq C\|x\|\|y\|, \qquad x, y \in H;$$

coercive if there is a constant $\alpha > 0$ such that

$$a(x,x) \geq \alpha\|x\|^2.$$

Note that in the complex case, coercivity means $|a(x,x)| \geq \alpha\|x\|^2$.

Stampecchia

Theorem 1.35. *Assume that $a(\cdot,\cdot)$ is a continuous coercive bilinear form on a Hilbert space H. Let K be a nonempty closed and convex subset of H. Then, given any $\phi \in H^*$, there exists a unique element $x \in K$ such that for any $y \in K$*

$$a(x, y-x) \geq < \phi, y-x >_{H^* \times H} \tag{1.40}$$

Moreover, if a is symmetric, then x is characterized by the property

$$x \in K \quad \text{and} \quad \frac{1}{2}a(x,x) - < \phi, x >_{H^* \times H} = \min_{y \in K} \frac{1}{2}a(y,y) - < \phi, y >_{H^* \times H} . \tag{1.41}$$

Proof. First we note that from Riesz theorem, there is $f \in H$ such that $< \phi, y >_{H^* \times H} = (f, y)$ for all $y \in H$. Now, if we fix $x \in H$, then $y \to a(x,y)$ is a continuous linear functional on H. Thus, again by the Riesz theorem, there is an operator $A : H \to H$ satisfying $a(x,y) = (Ax, y)$. Clearly, A is linear and satisfies

$$\|Ax\| \leq C\|x\|, \tag{1.42}$$

$$a(x,x) = (Ax, x) \geq \underset{= \sup|a(x,y)|}{\alpha}\|x\|^2. \tag{1.43}$$

Indeed,

$$\|Ax\| = \sup_{\|y\|=1} |(Ax, y)| \leq C\|x\| \sup_{\|y\|=1} \|y\|,$$

and (1.43) is obvious.

Problem (1.40) amounts to finding $x \in K$ satisfying, for all $y \in K$,

$$\rho\,(Ax, y-x) \geq \rho\,(f, y-x). \tag{1.44}$$

$(\rho Ax, y-x) \geq (\rho f, y-x)$

Let us fix a constant ρ to be determined later. Then, multiplying both sides of (1.44) by ρ and moving to one side, we find that (1.44) is equivalent to

$(u - x, y - x) \leq 0$

$$\forall_{y \in K} \quad (\overbrace{\rho f - \rho Ax + x}^{u} - x, y - x) \leq 0. \tag{1.45}$$

Here we recognize the equivalent formulation of the projection problem (1.35), that is, we can write

$$x = P_K(\rho f - \rho Ax + x) \tag{1.46}$$

This is a fixed point problem for x in K. Denote $Sy = P_K(\rho f - \rho Ay + y)$. Clearly $S : K \to K$ as it is a projection onto K and K, being closed, is a complete metric space in the metric induced from H. Since P_K is nonexpansive, we have

$$\|Sy_1 - Sy_2\| \leq \|(y_1 - y_2) - \rho(Ay_1 - Ay_2)\|$$

and thus $\quad (A(y_1 - y_2), y_1 - y_2) \geq \alpha \|y_1 - y_2\|^2$

$$\begin{aligned} \|Sy_1 - Sy_2\|^2 &= \|y_1 - y_2\|^2 - 2\rho(Ay_1 - Ay_2, y_1 - y_2) + \rho^2\|Ay_1 - Ay_2\|^2 \\ &\leq \|y_1 - y_2\|^2(1 - 2\rho\alpha + \rho^2 C^2) \qquad 0 < \rho < \frac{2\alpha}{C^2} \end{aligned}$$

We can choose ρ in such a way that $k^2 = \cancel{1} - 2\rho\alpha + \rho^2 C^2 < \cancel{1}$ we see that S has a unique fixed point in K.

Assume now that a is symmetric. Then $(x, y)_1 = a(x, y)$ defines a new scalar product which defines an equivalent norm $\|x\|_1 = \sqrt{a(x, x)}$ on H. Indeed, by continuity and coerciveness

$$\|x\|_1 = \sqrt{a(x, x)} \leq \sqrt{C}\|x\|$$

and

$$\|x\|_1 = \sqrt{a(x, x)} \geq \sqrt{\alpha}\|x\|.$$

Using again Riesz theorem, we find $g \in H$ such that

$$< \phi, y >_{H^* \times H} = a(g, y)$$

and then (1.40) amounts to finding $x \in K$ such that

$$(g - x, y - x)_1 = a(g - x, y - x) \leq 0$$

for all $y \in K$ but this is nothing else but finding projection x onto K with respect to the new scalar product. Thus, there is a unique $x \in K$

$$\sqrt{a(g - x, g - x)} = \min_{y \in K} \sqrt{a(g - y, g - y)}.$$

$$\|g - x\|_1 = \min_{y \in K} \|g - y\|$$

However, expanding, this is the same as finding minimum of the function

$$y \to a(g - y, g - y) = a(g, g) + a(y, y) - 2a(g, y) = \underbrace{a(y, y)}\underbrace{-2 < \phi, y >_{H^* \times H}} + \underbrace{a(g, g)}.$$

Taking into account that $a(g, g)$ is a constant, we see that x is the unique minimizer of

$$y \to \frac{1}{2}a(y, y) - < \phi, y >_{H^* \times H} .$$

Twierdzenie Laxa - Milgrama

Corollary 1.36. *Assume that $a(\cdot,\cdot)$ is a continuous coercive bilinear form on a Hilbert space H. Then, given any $\phi \in H^*$, there exists a unique element $x \in H$ such that for any $y \in H$* $\int \nabla x \nabla y = \int \phi y$ $\{ H = H^1(\Omega)$, $W^1_2(\Omega)$

$$a(x,y) = <\phi, y>_{H^* \times H} \tag{1.47}$$

Moreover, if a is symmetric, then x is characterized by the property

$$x \in H \quad \text{and} \quad \frac{1}{2}a(x,x) - <\phi, x>_{H^*\times H} = \min_{y\in H} \frac{1}{2}a(y,y) - <\phi, y>_{H^*\times H}\ . \tag{1.48}$$

Proof. We use the Stampacchia theorem with $K = H$. Then there is a unique element $x \in H$ satisfying $\|Ax\| \geq \alpha \|x\|$

$\forall\, y \in H$

$$a(x, y - x) \geq <\phi, y - x>_{H^*\times H}\ .$$

$\forall\, a(x,y) = \langle \varphi, y \rangle$, $y \in H$

$|a(x,y)| \leq C\|x\|\cdot\|y\|$

$a(x,x) \geq \alpha \|x\|^2$

$(Ax,x) \geq$

$\forall\, y$ $(Ax,y) = (f,y)$

$\|Ax\| \leq C\|x\|$

$(Ax,y) = a(x,y)$

$Ax = 0$

Using linearity, this must hold also for

$$a(x, ty - x) \geq <\phi, ty - x>_{H^*\times H}\ .$$

for any $t \in R, y \in H$. Factoring out t, we find

$$\cancel{t}a(x, y - \cancel{xt^{-1}}) \overset{\leq}{\geq} \cancel{t} <\phi, y - \cancel{xt^{-1}}>_{H^*\times H}\ .$$

and passing with $t \to \pm\infty$, we obtain $t \to -\infty$

y anihiluje $R(A)$, $x \in H$

$$a(x,y) \geq <\phi, y>_{H^*\times H}, \qquad a(x,y) \leq <\phi, y>_{H^*\times H}\ .$$

$0 = (y, Ax) = (Ax, y) = 0$, $= a(x,y) \geq \alpha\|y\|^2$, $x = y$, $y = 0$

$0 = a(x,x) \geq \alpha\|x\|^2$

$y \in R(A)$, $y = Ax$, $x = A^{-1}y$ | $\alpha\|A^{-1}y\|\,\|x\| \Rightarrow 0$, $\leq a(x,x) \leq C\|x\|^2$

1.3.6 Adjoint Operators

An important role in functional analysis is played by the operation of taking *operator adjoint.* If $A \in \mathcal{L}(X,Y)$, then the adjoint operator A^* is defined as

$$<y^*, Ax> = <A^*y^*, x> \tag{1.49}$$

and it can be proved that it belongs to $\mathcal{L}(Y^*, X^*)$ with $\|A^*\| = \|A\|$. If A is an unbounded operator, then the situation is more complicated. In general, A^* may not exist as a single-valued operator. In other words, there may be many operators B satisfying

$$<y^*, Ax> = <By^*, x>, \qquad x \in D(A),\ y^* \in D(B). \tag{1.50}$$

Operators A and B satisfying (1.50) are called *adjoint to each other.*

However, if $D(A)$ is dense in X, then there is a unique maximal operator A^* adjoint to A; that is, any other B such that A and B are adjoint to each other, must satisfy $B \subset A^*$. This A^* is called the *adjoint operator* to A. It can be constructed in the following way. The domain $D(A^*)$ consists of all elements y^* of Y^* for which there exists $f^* \in X^*$ with the property