1.2.3 Banach–Steinhaus Theorem

Another fundamental theorem of functional analysis is the Banach–Steinhaus theorem, or the Uniform Boundedness Principle. It is based on a fundamental topological results known as the Baire Category Theorem.

Theorem 1.13. *Let X be a complete metric space and let $\{X_n\}_{n\geq 1}$ be a sequence of closed subsets in X. If $Int\, X_n = \emptyset$ for any $n \geq 1$, then $Int \bigcup\limits_{n=1}^{\infty} X_n = \emptyset$. Equivalently, taking complements, we can state that a countable intersection of open dense sets is dense.*

Remark 1.14. Baire's theorem is often used in the following equivalent form: if X is a complete metric space and $\{X_n\}_{n\geq 1}$ is a countable family of closed sets such that $\bigcup\limits_{n=1}^{\infty} X_n = X$, then $Int X_n \neq \emptyset$ at least for one n.

Chaotic dynamical systems

We assume that X is a complete metric space, called the state space. In general, a *dynamical system* on X is just a family of states $(\mathbf{x}(t))_{t\in\mathbb{T}}$ parametrized by some parameter t (time). Two main types of dynamical systems occur in applications: those for which the time variable is discrete (like the observation times) and those for which it is continuous.

Theories for discrete and continuous dynamical systems are to some extent parallel. In what follows mainly we will be concerned with continuous dynamical systems. Also, to fix attention we shall discuss only systems defined for $t \geq 0$, that are sometimes called *semidynamical systems*. Thus by a *continuous dynamical system* we will understand a family of functions (operators) $(\mathbf{x}(t,\cdot))_{t\geq 0}$ such that for each t, $\mathbf{x}(t,\cdot) : X \to X$ is a continuous function, for each $\mathbf{x}_0$ the function $t \to \mathbf{x}(t, \mathbf{x}_0)$ is continuous with $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$. Moreover, typically it is required that the following semigroup property is satisfied (both in discrete and continuous case)

$$\mathbf{x}(t+s, \mathbf{x}_0) = \mathbf{x}(t, \mathbf{x}(s, \mathbf{x}_0)), \qquad t, s \geq 0, \tag{1.24}$$

which expresses the fact that the final state of the system can be obtained as the superposition of intermediate states.

Often discrete dynamical systems arise from iterations of a function

$$\mathbf{x}(t+1, \mathbf{x}_0) = f(\mathbf{x}(t, \mathbf{x}_0)), \qquad t \in \mathbb{N}, \tag{1.25}$$

while when t is continuous, the dynamics are usually described by a differential equation

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = A(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0 \qquad t \in \mathbb{R}_+. \tag{1.26}$$

We say that the dynamical system $(\mathbf{x}(t))_{t\geq 0}$ on a metric space (X, d) (to avoid non-degeneracy we assume that $X \neq \{\mathbf{x}(t, \boldsymbol{p})\}_{t\geq 0}$, for any $\boldsymbol{p} \in X$, that is, the space does not degenerates to a single orbit) is *topologically transitive* if for any two non-empty open sets $U, V \subset X$ there is $t_0 \geq 0$ such that $\mathbf{x}(t, U) \cap V \neq \emptyset$. A *periodic point* of $(\mathbf{x}(t))_{t\geq 0}$ is any point $\boldsymbol{p} \in X$ satisfying

$$\mathbf{x}(T, \boldsymbol{p}) = \boldsymbol{p},$$

for some $T > 0$. The smallest such T is called the period of $\boldsymbol{p}$. The last of ~~Devaney's conditions (~~the observation of which apparently initiated Lorentz's study of chaos) is the so-called *sensitive dependence on initial conditions* and abbreviated as sdic. We say that the system is sdic if there exists $\delta > 0$ such that for every $\boldsymbol{p} \in X$ and a neighbourhood N_p of $\boldsymbol{p}$ there exists a point $\mathbf{y} \in N_p$ and $t_0 > 0$ such that the distance between $\mathbf{x}(t_0, \boldsymbol{p})$ and $\mathbf{x}(t_0, \mathbf{y})$ is larger than δ. This property captures the idea that in chaotic systems minute errors in experimental readings eventually lead to large scale divergence, and is widely understood to be the central idea in chaos.

With this preliminaries we are able to state Devaney's definition of chaos (as applied to continuous dynamical systems).

Definition 1.15. *Let X be a metric space. A dynamical system $(\mathbf{x}(t))_{t\geq 0}$ in X is said to be chaotic in X if*

1. *$(\mathbf{x}(t))_{t\geq 0}$ is transitive,*
2. *the set of periodic points of $(\mathbf{x}(t))_{t\geq 0}$ is dense in X,*
3. *$(\mathbf{x}(t))_{t\geq 0}$ has sdic.*

To summarize, chaotic systems have three ingredients: indecomposability (property 1), unpredictability (property 3), and an element of regularity (property 2).

It is then a remarkable observation that properties 1. and 2 together imply sdic.

Theorem 1.16. *If $(\mathbf{x}(t))_{t\geq 0}$ is topologically transitive and has dense set of periodic points, then it has sdic.*

We say that X is non-degenerate, if continuous images of a compact intervals are nowhere dense in X.

Lemma 1.17. *Let X be a non-degenerate metric space. If the orbit $O(\boldsymbol{p}) = \{\mathbf{x}(t, \boldsymbol{p})\}_{t\geq 0}$ is dense in X, then also the orbit $O(\mathbf{x}(s, \boldsymbol{p})) = \{\mathbf{x}(t, \boldsymbol{p})\}_{t>s}$ is dense in X, for any $s > 0$.*

Proof. Assume that $O(\mathbf{x}(s, \boldsymbol{p}))$ is not dense in X, then there is an open ball B such that $B \cap \overline{O(\mathbf{x}(s, \boldsymbol{p}))} = \emptyset$. However, each point of the ball is a limit point of the whole orbit $O(\boldsymbol{p})$, thus we must have $\{\mathbf{x}(t, \boldsymbol{p})\}_{0\leq t\leq s} = \overline{\{\mathbf{x}(t, \boldsymbol{p})\}_{0\leq t\leq s}} \supset B$ which contradicts the assumption of nondegeneracy. ■

To fix terminology we say that a semigroup having a dense trajectory is called *hypercyclic*. We note that by continuity $\overline{O(\boldsymbol{p})} = \overline{\{\mathbf{x}(t, \boldsymbol{p})\}_{t\in\mathbb{Q}}}$, where $\mathbb{Q}$

is the set of positive rational numbers, therefore hypercyclic semigroups can exist only in separable spaces.

By X_h we denote the set of hypercyclic vectors, that is,

$$X_h = \{\boldsymbol{p} \in X;\ O(\boldsymbol{p}) \text{ is dense in } X\}$$

Note that if $(\mathbf{x}(t))_{t\geq 0}$ has one hypercyclic vector, then it has a dense set of hypercyclic vectors as each of the point on the orbit $O(\boldsymbol{p})$ is hypercyclic (by the first part of the proof above).

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Theorem 1.18. *Let $(\mathbf{x}(t))_{t\geq 0}$ be a strongly continuous semigroup of continuous operators (possibly nonlinear) on a complete (separable) metric space X. The following conditions are equivalent:*

$\forall t \to x(t,x)$
$x \in X$

1. *X_h is dense in X,*
2. *$(\mathbf{x}(t))_{t\geq 0}$ is topologically transitive.*

Proof. Let as take the set of nonegative rational numbers and enumerate them as $\{t_1, t_2, \ldots\}$. Consider now the sequence ~~family~~ $\{\mathbf{x}(t_n)\}_{n\in\mathbb{N}}$. Clearly, the orbit of $\boldsymbol{p}$ through $(\mathbf{x}(t))_{t\geq 0}$ is dense in X if and only if the set $\{\mathbf{x}(t_n)\boldsymbol{p}\}_{n\in\mathbb{N}}$ is dense.

Consider now the covering of X by the enumerated sequence of balls B_m centered at points of a countable subset of X with rational radii. Since each $\mathbf{x}(t_m)$ is continuous, the sets

$$G_m = \bigcup_{n\in\mathbb{N}} \mathbf{x}^{-1}(t_n, B_m)$$

are open. Next we claim that

$$\boxed{X_h = \bigcap_{m\in\mathbb{N}} G_m.}$$

In fact, let $\boldsymbol{p} \in X_h$, that is, $\boldsymbol{p}$ is hypercyclic. It means that $\mathbf{x}(t_n, \boldsymbol{p})$ visits each neigbourhood of each point of X for some n. In particular, for each m there must be n such that $\mathbf{x}(t_n, \boldsymbol{p}) \in B_m$ or $\boldsymbol{p} \in \mathbf{x}^{-1}(t_n, B_m)$ which means $\boldsymbol{p} \in \bigcap_{m\in\mathbb{N}} G_m$.

Conversely, if $\boldsymbol{p} \in \bigcap_{m\in\mathbb{N}} G_m$, then for each m there is n such that $\boldsymbol{p} \in \mathbf{x}^{-1}(t_n, B_m)$, that is, $\mathbf{x}(t_n, \boldsymbol{p}) \in B_m$. This means that $\{\mathbf{x}(t_n, \boldsymbol{p})\}_{n\in\mathbb{N}}$ is dense.

The next claim is condition 2. is equivalent to each set G_m being dense in X. If G_m were not dense, then for some B_r, $B_r \cap \mathbf{x}^{-1}(t_n, B_m) = \emptyset$ for any n. But then $\mathbf{x}(t_n, B_r) \cap B_m = \emptyset$ for any n. Since the continuous semigroup is topologically transitive, we know that there is $\mathbf{y} \in B_r$ such that $\mathbf{x}(t_0, \mathbf{y}) \in B_m$ for some t_0. Since B_m is open, $\mathbf{x}(t, \mathbf{y}) \in B_m$ for t from some neighbourhood of t_0 and this neighbourhood must contain rational numbers.

The converse is immediate as for given open U and V we find $B_m \subset V$ and since G_m is dense $U \cap G_m \neq \emptyset$. Thus $U \cap \mathbf{x}^{-1}(t_n, B_m) \neq \emptyset$ for some n, hence $\mathbf{x}(t_n, U) \cap B_m \neq \emptyset$.

So, if $(\mathbf{x}(t))_{t \geq 0}$ is topologically transitive, then X_h is the intersection of a countable collection of open dense sets, and by Baire Theorem in a complete space such an intersection must be still dense, thus X_h is dense.

Conversely, if X_h is dense, then each term of the intersection must be dense, thus each G_m is dense which yields the transitivity. ■

Back to Banach–Steinhaus Theorem

To understand its importance, let us reflect for a moment on possible types of convergence of sequences of operators. Because the space $\mathcal{L}(X, Y)$ can be made a normed space by introducing the norm (1.11), the most natural concept of convergence of a sequence $(A_n)_{n \in \mathbb{N}}$ would be with respect to this norm. Such a convergence is referred to as the *uniform operator convergence*. However, for many purposes this notion is too strong and we work with the pointwise or *strong convergence*: the sequence $(A_n)_{n \in \mathbb{N}}$ is said to converge strongly if, for each $x \in X$, the sequence $(A_n x)_{n \in \mathbb{N}}$ converges in the norm of Y. In the same way we define uniform and strong boundedness of a subset of $\mathcal{L}(X, Y)$.

Note that if $Y = \mathbb{R}$ (or $\mathbb{C}$), then strong convergence coincides with $*$-weak convergence.

$A_n \leq M_x$

After these preliminaries we can formulate the Banach–Steinhaus theorem.

$\forall_x \exists_{M_x} \|A_n x\|$

Theorem 1.19. *Assume that X is a Banach space and Y is a normed space. Then a subset of $\mathcal{L}(X, Y)$ is uniformly bounded if and only if it is strongly bounded.*

$\exists_M \|A_n\| \leq M$

One of the most important consequences of the Banach–Steinhaus theorem is that a strongly converging sequence of bounded operators is always converging to a linear bounded operator. That is, if for each x there is y_x such that

$$\lim_{n \to \infty} A_n x = y_x,$$

then there is $A \in \mathcal{L}(X, Y)$ satisfying $Ax = y_x$.

Example 1.20. We can use the above result to get a better understanding of the concept of weak convergence and, in particular, to clarify the relation between reflexive and weakly sequentially complete spaces. First, by considering elements of X^* as operators in $\mathcal{L}(X, \mathbb{C})$, we see that every $*$-weakly converging sequence of functionals converges to an element of X^* in $*$-weak topology. On the other hand, for a weakly converging sequence $(x_n)_{n \in \mathbb{N}} \subset X$, such an approach requires that $x_n, n \in \mathbb{N}$, be identified with elements of X^{**} and thus, by the Banach–Steinhaus theorem, a weakly converging sequence always has a limit $x \in X^{**}$. If X is reflexive, then $x \in X$ and X is weakly sequentially complete. However, for nonreflexive X we might have $x \in X^{**} \setminus X$ and then $(x_n)_{n \in \mathbb{N}}$ does not converge weakly to any element of X.

On the other hand, (1.23) implies that a weakly convergent sequence is norm bounded.

We note another important corollary of the Banach–Steinhaus theorem which we use in the sequel.

Corollary 1.21. *A sequence of operators $(A_n)_{n\in\mathbb{N}}$ is strongly convergent if and only if it is convergent uniformly on compact sets.*

Proof. It is enough to consider convergence to 0. If $(A_n)_{n\in\mathbb{N}}$ converges strongly, then by the Banach–Steinhaus theorem, $a = \sup_{n\in\mathbb{N}} \|A_n\| < +\infty$. Next, if $\Omega \subset X$ is compact, then for any ϵ we can find a finite set $N_\epsilon = \{x_1, \ldots, x_k\}$ such that for any $x \in \Omega$ there is $x_i \in N_\epsilon$ with $\|x - x_i\| \le \epsilon/2a$. Because N_ϵ is finite, we can find n_0 such that for all $n > n_0$ and $i = 1, \ldots, k$ we have $\|A_n x_i\| \le \epsilon/2$ and hence

$$\|A_n x\| = \|A_n x_i\| + a\|x - x_i\| \le \epsilon$$

for any $x \in \Omega$. The converse statement is obvious. □

We conclude this unit by presenting a frequently used result related to the Banach–Steinhaus theorem.

Proposition 1.22. *Let X, Y be Banach spaces and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X,Y)$ be a sequence of operators satisfying $\sup_{n\in\mathbb{N}} \|A_n\| \le M$ for some $M > 0$. If there is a dense subset $D \subset X$ such that $(A_n x)_{n\in\mathbb{N}}$ is a Cauchy sequence for any $x \in D$, then $(A_n x)_{n\in\mathbb{N}}$ converges for any $x \in X$ to some $A \in \mathcal{L}(X,Y)$.*

Proof. Let us fix $\epsilon > 0$ and $y \in X$. For this ϵ we find $x \in D$ with $\|x - y\| < \epsilon/M$ and for this x we find n_0 such that $\|A_n x - A_m x\| < \epsilon$ for all $n, m > n_0$. Thus,

$$\|A_n y - A_m y\| \le \|A_n x - A_m x\| + \|A_n(x - y)\| + \|A_m(x - y)\| \le 3\epsilon.$$

Hence, $(A_n y)_{n\in\mathbb{N}}$ is a Cauchy sequence for any $y \in X$ and, because Y is a Banach space, it converges and an application of the Banach–Steinhaus theorem ends the proof. □

Application–limits of integral expressions

Consider an equation describing growth.

$\frac{dm}{dt} = g(m)$

$$\frac{\partial N}{\partial t} + \frac{\partial (g(m)N)}{\partial m} = -\mu(m)N(t,m) + \cancel{PN(t,m)}, \tag{1.27}$$

with the boundary condition

$m \in [0,1]$

$$g(0)N(t,0) = 0 \tag{1.28}$$

and with the initial condition

$N \in L_1([0,1])$

$$N(0,m) = N_0(m) \quad \text{for } m \in [0,1]. \tag{1.29}$$

Consider the 'formal' equation for the stationary version of the equation (the resolvent equation)

$u = e^{\lambda t} N(m)$

$$\lambda N(m) + (g(m)N(m))' + \mu(m)N(m) = f(m), \quad \in L_1([0,1])$$

whose solution is given by

$$g(m) N_\lambda(m) = \frac{e^{-\lambda G(m) - Q(m)}}{\cancel{g(m)}} \int_a^m e^{\lambda G(s)+Q(s)} f(s)\, ds \tag{1.30}$$

where $G(m) = \int_a^m (1/g(s))\, ds$ and $Q(m) = \int_a^m (\mu(s)/g(s))\, ds$. To shorten notation we denote

$$e_{-\lambda}(m) := e^{-\lambda G(m) - Q(m)}, \qquad e_\lambda(m) := e^{\lambda G(m)+Q(m)}.$$

Our aim is to show that $g(m)N_\lambda(m) \to 0$ as $m \to 1^-$ provided $1/g$ or μ is not integrable close to 1. If the latter condition is satisfied, then $e_\lambda(m) \to \infty$ and $e^{-\lambda}(m) \to 0$ as $m \to 1^-$.

Indeed, consider the family of functionals $\{\xi_m\}_{m\in[1-\epsilon,1)}$ for some $\epsilon > 0$ defined by

$$\xi_m f = e_{-\lambda}(m) \int_0^m e_\lambda(s) f(s)\, ds$$

for $f \in L^1[0,1]$. We have

$$|\xi_m f| \le e_{-\lambda}(m) \int_0^m e_\lambda(s)|f(s)|\, ds \le \int_0^1 |f(s)|\, ds$$

on account of monotonicity of e_λ. Moreover, for f with support in $[0, 1-\delta]$ with any $\delta > 0$ we have $\lim_{m\to 1^-} \xi_m f = 0$ and, by Proposition 1.22, the above limit extends by density for any $f \in L^1[0,1]$.

1.2.4 Weak compactness

In finite dimensional spaces normed spaces we have Bolzano-Weierstrass theorem stating that from any bounded sequence of elements of X_n one can select a convergent subsequence. In other words, a closed unit ball in X_n is compact.

There is no infinite dimensional normed space in which the unit ball is compact.

Weak compactness comes to the rescue. Let us begin with (separable) Hilbert spaces.

Theorem 1.23. *Each bounded sequence (u_n) in a separable Hilbert space X has a weakly convergent subsequence.*

Proof. Let $\{v_k\}_{k\in\mathbb{N}}$ be dense in X and consider numerical sequences $((u_n, v_k))_{n\in\mathbb{N}}$ for any k. From Banach-Steinhaus theorem and

$$|(u_n, v_k)| \leq \|u_n\|\|v_k\|$$

we see that for each k these sequences are bounded and hence each has a convergent subsequence. We use the diagonal procedure: first we select $(u_{1n})_{n\in\mathbb{N}}$ such that $(u_{1n}, v_1) \to a_1$, then from $(u_{1n})_{n\in\mathbb{N}}$ we select $(u_{2n})_{n\in\mathbb{N}}$ such that $(u_{2n}, v_2) \to a_2$ and continue by induction. Finally, we take the diagonal sequence $w_n = u_{nn}$ which has the property that $(w_n, v_k) \to a_k$. This follows from the fact that elements of $(w_n)_{n\in\mathbb{N}}$ belong to (u_{kn}) for $n \geq k$. Since $\{v_k\}_{k\in\mathbb{N}}$ is dense in X and $(u_n)_{n\in\mathbb{N}}$ is norm bounded, Proposition 1.22 implies $((w_n, v))_{n\in\mathbb{N}}$ converges to, say, $a(v)$ for any $v \in X$ and $v \to a(v)$ is a bounded (anti) linear functional on X. By the Riesz representation theorem, there is $w \in X$ such that $a(v) = (v, w)$ and thus $w_n \rightharpoonup w$. $\square$

If X is not separable, then we can consider $Y = \overline{\mathcal{L}in\{u_n\}_{n\in\mathbb{N}}}$ which is separable and apply the above theorem in Y getting an element $w \in Y$ for which

$$(w_n, v) \to (w, v), \qquad v \in Y.$$

Let now $z \in X$. By orthogonal decomposition, $z = v + v^\perp$ by linearity and continuity (as $w \in Y$)

$$(w_n, z) = (w_n, v) \to (w, v) = (w, z)$$

and so $w_n \rightharpoonup w$ in X.

Corollary 1.24. *Closed unit ball in X is weakly sequentially compact.*

Proof. We have

$$(v, w_n) \to (v, w), \quad n \to \infty$$

for any v. We can assume $w = 0$ We prove that for any k there are indices $n_1, \ldots, n_k$ such that $(w_1, w_4) \to 0$

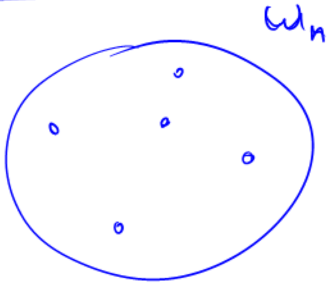
$$k^{-1}(w_{n_1} + \ldots + w_{n_k}) \to 0$$

$(w_{n_2}, w_n) \to 0$

in X. Since $(w_1, w_n) \to 0$, we set $n_1 = 1$ and select n_2 such that $|(w_{n_1}, w_{n_2})| \leq 1/2$. Then we select n_3 such that $|(w_{n_1}, w_{n_3})| \leq 1/2$ and $|(w_{n_2}, w_{n_3})| \leq 1/2$ and further, n_k such that $|(w_{n_1}, w_{n_k})| \leq 1/(k-1), \ldots, |(w_{n_{k-1}}, w_{n_k})| \leq 1/(k-1)$. Since $\|w_n\| \leq C$, we obtain

$$\begin{aligned}\|k^{-1}(w_{n_1} + \ldots + w_{n_k})\|^2 &\leq k^{-2}\left(\sum_{j=1}^{k}\|w_{n_j}\|^2 + 2\sum_{j=1}^{k-1}(w_{n_j}, w_{n_k}) + 2\sum_{j=1}^{k-2}(w_{n_j}, w_{n_{k-1}}) + \ldots\right)\\ &\leq k^{-2}(kC^2 + 2(k-1)(k-1)^{-1} + 2(k-2)(k-2)^{-1} + \ldots 2)\\ &\leq k^{-1}(C^2 + 2)\end{aligned}$$

$w_n \rightharpoonup w$



$\|w_n\| \leq 1$

$\frac{1}{k}(w_{n_1} + \ldots + w_{n_k}) \to w \in B(0,1)$

$= \tilde{w}_k \in B(0,1)$

w normie

What about other spaces?

A Banach space is reflexive if and only if the closed unit ball is weakly sequentially compact.

Helly's theorem: If X is a separable Banach space and $U = X^*$, then the closed unit ball in U is weak* sequentially compact. Alaoglu removed separability.

1.2.5 The Open Mapping Theorem

The Open Mapping Theorem is fundamental for inverting linear operators. Let us recall that an operator $A : X \to Y$ is called *surjective* if $ImA = Y$ and *open* if the set $A\Omega$ is open for any open set $\Omega \subset X$.

Theorem 1.25. *Let X, Y be Banach spaces. Any surjective $A \in \mathcal{L}(X, Y)$ is an open mapping.*

One of the most often used consequences of this theorem is the Bounded Inverse Theorem.

Corollary 1.26. *If $A \in \mathcal{L}(X, Y)$ is such that $KerA = \{0\}$ and $ImA = Y$, then $A^{-1} \in \mathcal{L}(Y, X)$.*

The corollary follows as the assumptions on the kernel and the image ensure the existence of a linear operator A^{-1} defined on the whole Y. The operator A^{-1} is continuous by the Open Mapping Theorem, as the preimage of any open set in X through A^{-1}, that is, the image of this set through A, is open.

Throughout the book we are faced with invertibility of unbounded operators. An operator $(A, D(A))$ is said to be *invertible* if there is a bounded operator $A^{-1} \in \mathcal{L}(Y, X)$ such that $A^{-1}Ax = x$ for all $x \in D(A)$ and $A^{-1}y \in D(A)$ with $AA^{-1}y = y$ for any $y \in Y$. We have the following useful conditions for invertibility of A.

Proposition 1.27. *Let X, Y be Banach spaces and $A \in L(X, Y)$. The following assertions are equivalent.*

$\|Ax\| \geq m\|x\| \Rightarrow Ker A = \{0\}$

(i) A is invertible;
(ii) $ImA = Y$ and there is $m > 0$ such that $\|Ax\| \geq m\|x\|$ for all $x \in D(A)$;
(iii) A is closed, $\overline{ImA} = Y$ and there is $m > 0$ such that $\|Ax\| \geq m\|x\|$ for all $x \in D(A)$;
(iv) A is closed, $ImA = Y$, and $KerA = \{0\}$.

$\|A^{-1}y\| \leq \frac{1}{m}\|y\|$

Proof. The equivalence of (i) and (ii) follows directly from the definition of invertibility. By Theorem 1.28, the graph of any bounded operator is closed and because the graph of the inverse is given by

$$G(A) = \{(x, y);\ (y, x) \in G(A^{-1})\},$$

A jest domknięty to $A : D(A) \to Y$

$A^{-1} : Y \to D(A)$

$D(A)$, $\|x\|_A = \|x\| + \|Ax\|$ $A^{-1} \in \mathcal{L}(Y, X)$

$(D(A), \|x\|_A)$ jest przestrzenią Banacha

$(x_n) \subset D(A)$ $\|x_n - x_m\| \to 0$ $x_n \to x$ w X $x \in D(A)$

$\|Ax_n - Ax_m\| \xrightarrow[n,m\to\infty]{} 0$ $Ax_n \to y$ w Y $Ax = y$

we see that the graph of any invertible operator is closed and thus any such an operator is closed. Hence, (i) and (ii) imply (iii) and (iv). Assume now that (iii) holds. $G(A)$ is a closed subspace of $X \times Y$, therefore it is a Banach space itself. The inequality $\|Ax\| \geq m\|x\|$ implies that the mapping $G(A) \ni (x, Ax) \to Ax \in ImA$ is an isomorphism onto ImA and hence ImA is also closed. Thus $ImA = Y$ and (ii) follows. Finally, if (iv) holds, then Corollary 1.26 can be applied to A from $D(A)$ (with the graph norm) to Y to show that $A^{-1} \in \mathcal{L}(Y, D(A)) \subset \mathcal{L}(Y, X)$. □

Norm equivalence. An important result is that if X is a Banach space with respect to two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ and there is C such that $\|x\|_1 \leq C\|x\|_2$, then both norms are equivalent.

The Closed Graph Theorem

It is easy to see that a bounded operator defined on the whole Banach space X is closed. That the inverse also is true follows from the Closed Graph Theorem.

Theorem 1.28. *Let X, Y be Banach spaces. An operator $A \in L(X, Y)$ with $D(A) = X$ is bounded if and only if its graph is closed.*

We can rephrase this result by saying that an everywhere defined closed operator in a Banach space must be bounded.

Proof. Indeed, consider on X two norms, the original norm $\|\cdot\|$ and the graph norm

$$\|x\|_{D(A)} = \|x\| + \|Ax\|.$$

$\|x\| \leq \|x\|_A$

By closedness, X is a Banach space with respect to $D(A)$ and A is continuous in the norm $\|\cdot\|_{D(A)}$. Hence, the norms are equivalent and A is continuous in the norm $\|\cdot\|$.

$\|x\| + \|Ax\| \leq C\|x\|$

$\|Ax\| \leq \|x\|$

To give a nice and useful example of an application of the Closed Graph Theorem, we discuss a frequently used notion of relatively bounded operators. Let two operators $(A, D(A))$ and $(B, D(B))$ be given. We say that B is A-*bounded* if $D(A) \subset D(B)$ and there exist constants $a, b \geq 0$ such that for any $x \in D(A)$,

$$\|Bx\| \leq a\|Ax\| + b\|x\|. \tag{1.31}$$

Note that the right-hand side defines a norm on the space $D(A)$, which is equivalent to the graph norm (1.15).

Corollary 1.29. *If A is closed and B closable, then $D(A) \subset D(B)$ implies that B is A-bounded.*

Proof. If A is a closed operator, then $D(A)$ equipped with the graph norm is a Banach space. If we assume that $D(A) \subset D(B)$ and $(B, D(B))$ is closable, then $D(A) \subset D(\overline{B})$. Because the graph norm on $D(A)$ is stronger than the norm induced from X, the operator $\overline{B}$, considered as an operator from $D(A)$ to X is everywhere defined and closed. On the other hand, $\overline{B}|_{D(A)} = B$; hence $B : D(A) \to X$ is bounded by the Closed Graph Theorem and thus B is A-bounded. □

Methods of Hilbert space

1. Radon-Nikodym theorem

Ω, σ-algebra on Ω, Σ and

two measures ν and μ (w.r.t

the same σ-algebra).

$0 < \mu(\Omega) < +\infty$, $0 < \nu(\Omega) < +\infty$

If from $\mu(E) = 0$ it follows that $\nu(E) = 0$, then we say that ν is absolutely continuous w.r.t μ

Th. RN. If ν is a.c. w.r.t to μ, then there exists a unique $g \in L_1(\mu)$ such that for any $E \in \Sigma$:

$$\nu(E) = \int_E g \, d\mu$$

Proof. Let us consider Hilbert space

$$H = L_2(\Omega, d\mu + d\nu) \text{ on } \mathbb{R}$$

$$\|x\|^2 = \int_\Omega |x|^2 (d\mu + d\nu) \qquad x \in H$$

The functional

$$\langle x^*, f \rangle = \int_\Omega f \, d\mu$$

is bounded on H.

Indeed

$$|\langle x^*, f \rangle| \leq \int_\Omega 1 \cdot f \, d\mu \leq$$

$$\leq \mu(\Omega) \cdot \int_\Omega f^2 d\mu \leq \sqrt{\mu(\Omega)} \sqrt{\int_\Omega f^2 (d\mu + d\nu)}$$

$$\leq C \|f\|_H$$

Thus, by the Riesz theorem, there exists $y \in H$ such that

$$\forall_f \quad \int_\Omega f \, d\mu = \int_\Omega f y \, d(\mu + \nu)$$

Thus we obtain

$$\int_\Omega f(1-y) \, d\mu = \int_\Omega f y \, d\nu$$

We claim that $0 < y \leq 1$ a.e μ (and so for ν)

Indeed, if $F = \{x \in \Omega;\ y \leq 0\}$

we take $\chi_F = f$ so that

$$\underbrace{\int_F (1-y) \, d\mu}_{\geq 0} = \underbrace{\int_F y \, d\nu}_{\leq 0}$$

Contradiction $\Rightarrow$ $\mu(F) = 0$ $\Rightarrow$ $\nu(F) = 0$

If $E = \{x \in \Omega;\ y > 1\}$, $f = \chi_E$

$$\underbrace{\int_E (1-y) \, d\mu}_{< 0} = \underbrace{\int_E y \, d\nu}_{\geq \nu(E)}$$

$\Rightarrow \mu(E) = 0$

So, we have

$$\int_\Omega \overset{u}{y}\overset{\varphi}{f}\,\underbrace{\left(\frac{1-y}{y}\right)}\,d\mu = \int_\Omega \overbrace{f y}^{u}\,d\nu$$

$$0 < y \le 1 \quad \mu \text{ a.e.}$$

we can modify y to satisfy these inequalities everywhere without changing either integral.

To complete, we let $u = fy$ and $g = \frac{1-y}{y}$ - it is well defined finite nonnegative function

Let $E_\varepsilon = \{x \in \Omega;\ g(x) \le \frac{1}{\varepsilon}\}$ and consider

$$u_\varepsilon = \chi_{E_\varepsilon}$$

$$\int_\Omega ug\,d\mu = \int_\Omega u\,d\nu$$

$$\int_\Omega \underbrace{u_\varepsilon g}\,d\mu = \int_\Omega u_\varepsilon\,d\nu$$

$$u_\varepsilon \nearrow 1 \text{ on } \Omega \qquad \mu(\Omega) < \infty$$

From monotone convergence

$$u_\varepsilon g \nearrow g$$

$$\sup_\varepsilon \int_\Omega u_\varepsilon g\,d\mu < +\infty$$

$\Rightarrow$ g integrable

For any $E \in \Sigma$, we take $u = \chi_E$ and so

$$\int_\Omega u\,d\nu = \int_\Omega \chi_E\,d\nu = \nu(E) = \int_E g\,d\mu$$