$S(t)=e^{-\omega t}G(t)$

$S(t+s)=e^{-\omega(t+s)}G(t+s) = e^{-\omega t}e^{-\omega s}G(t)G(s)$

$S(t)S(s)$

$\|S(t)\|\le e^{-\omega t}e^{\omega t}=1$

$-\omega e^{-\omega t}G(t)x + e^{-\omega t}\frac{dG}{dt} = AG(t)x$

$=(\omega I + A)$

Corollary 2.12. *A linear operator A is the generator of a C_0 semigroup $(G(t))_{t\geq 0}$ satisfying $\|G(t)\| \leq e^{\omega t}$ if and only if*

(i) A is closed and $\overline{D(A)} = X$;
(ii) $\rho(A) \supset (\omega, \infty)$ and for such λ

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda - \omega}. \tag{2.33}$$

$\int_0^n e^{-\lambda t}G(t)x\,dt$

$\tilde A = -\omega I + A$

$R(\lambda, A) = ?$

$\lambda - A = f$

Proof. Follows from the contractive semigroup $S(t) = e^{-\omega t}G(t)$ being generated by $A - \omega I$.

$\lambda - A = f$

$(\lambda - A)^2 = 1$

$\lambda + \omega - \omega - A$

The full version of the Hille-Yosida theorem reads

Theorem 2.13. *$A \in \mathcal{G}(M, \omega)$ if and only if*

$= (\lambda-\omega) - A + \omega$

$= (\lambda-\omega) - \tilde A$

(a) A is closed and densely defined,
(b) there exist $M > 0, \omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and for all $n \geq 1, \lambda > \omega$,

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}. \tag{2.34}$$

$R(\lambda, A) = R(\lambda-\omega, \tilde A)$

$\lambda - \omega > 0$

$\|(\lambda I - A)^{-1}\| \le \frac{M}{\lambda-\omega}$

2.2.3 Dissipative operators and the Lumer-Phillips theorem

$\|R(\lambda-\omega, \tilde A)\| \le \frac{1}{\lambda-\omega}$

Let X be a Banach space (real or complex) and X^* be its dual. From the Hahn–Banach theorem, Theorem 1.7 for every $x \in X$ there exists $x^* \in X^*$ satisfying

$$<x^*, x>= \|x\|^2 = \|x^*\|^2.$$

$\langle J(x), x\rangle$

Therefore the *duality set*

$$\mathcal{J}(x) = \{x^* \in X^*;\ <x^*, x>= \|x\|^2 = \|x^*\|^2\} \tag{2.35}$$

$u_t = Au$

$u_t + Au = f$

is nonempty for every $x \in X$.

Definition 2.14. *We say that an operator $(A, D(A))$ is* dissipative *if for every $x \in D(A)$ there is $x^* \in \mathcal{J}(x)$ such that*

$$\Re <x^*, Ax>\leq 0. \tag{2.36}$$

If X is a real space, then the real part in the above definition can be dropped.

Theorem 2.15. *A linear operator A is dissipative if and only if for all $\lambda > 0$ and $x \in D(A)$,*

$$\|(\lambda I - A)x\| \geq \lambda\|x\|. \tag{2.37}$$

$\langle x^*, x\rangle \le \|x^*\|\,\|x\|$

Proof. Let A be dissipative, $\lambda > 0$ and $x \in D(A)$. If $x^* \in \mathcal{J}(x)$ and $\Re < Ax, x^* > \le 0$, then

$$\|\lambda x - Ax\|\|x\| \ge |<\lambda x - Ax, x^* >| \ge \Re < \lambda x - Ax, x^* > \ge \lambda\|x\|^2$$

$= \Re(\lambda\langle x, x^*\rangle) - \Re\langle Ax, x^*\rangle$

so that we get (2.65).

Conversely, let $x \in D(A)$ and $\lambda\|x\| \le \|\lambda x - Ax\|$ for $\lambda > 0$. Consider $y^*_\lambda \in \mathcal{J}(\lambda x - Ax)$ and $z^*_\lambda = y^*_\lambda/\|y^*_\lambda\|$.

$$\begin{aligned}\lambda\|x\| \le \|\lambda x - Ax\| = \|\lambda x - Ax\|\|z^*_\lambda\| &= \|y^*_\lambda\|^{-1}\|\lambda x - Ax\|\|y^*_\lambda\| = \|y^*_\lambda\|^{-1} < \lambda x - Ax, y^*_\lambda > \\ &= < \lambda x - Ax, z^*_\lambda > = \lambda\Re < x, z^*_\lambda > - \Re < Ax, z^*_\lambda > \\ &\le \lambda\|x\| - \Re < Ax, z^*_\lambda >\end{aligned}$$

for every $\lambda > 0$. From this estimate we obtain that $\Re < Ax, z^*_\lambda > \le 0$ and, by $|\alpha| \ge \Re\alpha$,

$\Re\langle Ax, z^*_\lambda\rangle \le 0$

$$\lambda\Re < x, z^*_\lambda > = \lambda\|x\| + \Re < Ax, z^*_\lambda > \ge \lambda\|x\| - |\Re < Ax, z^*_\lambda >| \ge \lambda\|x\| - \|Ax\|$$

$\Re\langle x, z^*_\lambda\rangle \ge \|x\| - \frac{1}{\lambda}\|Ax\|$

or $\Re < x, z^*_\lambda > \ge \|x\| - \lambda^{-1}\|Ax\|$. Now, the unit ball in X^* is weakly-$*$ compact and thus there is a sequence $(z^*_{\lambda_n})_{n\in\mathbb{N}}$ converging to z^* with $\|z^*\| = 1$. From the above estimates, we get

$|\langle x, x^*\rangle| \ge \Re$

$$\Re < Ax, z^* > \le 0$$

$|\alpha| = \beta$
$\Re\alpha = \beta$
$\alpha = \beta$

and $\Re < x, z^* > \ge \|x\|$. Hence, also, $| < x, z^* > | \ge \|x\|$ On the other hand, $\Re < x, z^* > \le | < x, z^* > | \le \|x\|$ and hence $< x, z^* > = \|x\|$. Taking $x^* = z^*\|x\|$ we see that $x^* \in \mathcal{J}(x)$ and $\Re < Ax, x^* > \le 0$ and thus A is dissipative.

Theorem 2.16. *Let A be a linear operator with dense domain $D(A)$ in X.*

(a) If A is dissipative and there is $\lambda_0 > 0$ such that the range $Im(\lambda_0 I - A) = X$, then A is the generator of a C_0-semigroup of contractions in X.
(b) If A is the generator of a C_0 semigroup of contractions on X, then $Im(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, for every $x \in D(A)$ and every $x^* \in \mathcal{J}(x)$ we have $\Re < Ax, x^* > \le 0$.

$(\lambda_0 I - A)x = f$

$\Lambda = (0,\infty)$

Proof. Let $\lambda \ge 0$, then dissipativeness of A implies $\|\lambda x - Ax\| \ge \lambda\|x\|$ for $x \in D(A)$, $\lambda > 0$. This gives injectivity and, since by assumption, the $Im(\lambda_0 I - A)D(A) = X$, $(\lambda_0 I - A)^{-1}$ is a bounded everywhere defined operator and thus closed. But then $\lambda_0 I - A$, and hence A, are closed. We have to prove that $Im(\lambda I - A)D(A) = X$ for all $\lambda > 0$. Consider the set $\Lambda = \{\lambda > 0;\ Im(\lambda I - A)D(A) = X\}$. Let $\lambda \in \Lambda$. This means that $\lambda \in \rho(A)$ and, since $\rho(A)$ is open, Λ is open in the induced topology. We have to prove that Λ is closed in the induced topology. Assume $\lambda_n \to \lambda$, $\lambda > 0$. For every $y \in X$ there is $x_n \in D(A)$ such that

$x_n \to x$
$(A+B)x_n \to y$
$Bx_n \to z$
Ax_n

$$\lambda_n x_n - Ax_n = y.$$

$(\lambda_0 I - A)x = f$

$\|(\lambda_0 I - A)^{-1} f\| \le \frac{1}{\lambda_0}\|f\|$

$\lambda_0 I - A$ closed
A - [illegible]

$x = (\lambda_0 I - A)^{-1} f$

$y \in D(A^{-1})$, $z = A^{-1}y$

$\Leftarrow$ A^{-1} closed. $\Rightarrow$ A closed

$y_n \to y$, $A^{-1}y_n \to z$ $\Leftrightarrow$ $Ax_n \to y$, $y_n \to z$

$y_n = Ax_n$

$z \in D(A)$

$\to \lambda \quad \to x$

$\lambda_n x_n - A x_n = y$

$\| \lambda_n x_n - A x_n \| \geq \lambda_n \|x_n\|$

$\|y\| \geq \lambda_n \|x_n\|$

From (~~??~~), $\|x_n\| \leq \frac{1}{\lambda_n}\|y\| \leq C$ for some $C > 0$. Now

$$\begin{aligned}\lambda_m\|x_n - x_m\| &\leq \|\lambda_m(x_n - x_m) - A(x_n - x_m)\| \\ &= \|\lambda_m x_n - \lambda_m x_m - \lambda_n x_n + \lambda_n x_n - Ax_n + Ax_m\| \\ &= |\lambda_n - \lambda_m|\|x_n\| \leq C|\lambda_n - \lambda_m|\end{aligned}$$

$= y \qquad - y$

Thus, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $x_n \to x$, then $Ax_n \to \lambda x - y$. Since A is closed, $x \in D(A)$ and $\lambda x - Ax = y$. Thus, for this λ, $Im(\lambda I - A)D(A) = X$ and $\lambda \in \Lambda$. Thus Λ is also closed in $(0,\infty)$ and since $\lambda_0 \in \Lambda$, $\Lambda \neq \emptyset$ and thus $\Lambda = (0,\infty)$ (as the latter is connected). Thus, the thesis follows from the Hille-Yosida theorem.

On the other hand, if A is the generator of a semigroup of contractions $(G(t))_{t\geq 0}$, then $(0,\infty) \subset \rho(A)$ and $Im(\lambda I - A)D(A) = X$ for all $\lambda > 0$. Furthermore, if $x \in D(A), x^* \in \mathcal{J}(x)$, then

$\frac{G(t)x - x}{t} \to Ax$

$$| < G(t)x, x^* > | \leq \|G(t)x\|\|x^*\| \leq \|x\|^2 \quad = \|x$$

and therefore $\quad - \langle x, x^* \rangle$

$$\Re < G(t)x - x, x^* >= \cancel{\Re < G(t)x, x^* >} -\|x\|^2 \leq 0$$

and, dividing the left hand side by t and passing with $t \to \infty$, we obtain

$$\mathrm{Re} < Ax, x^* >\leq 0.$$

Since this holds for every $x^* \in \mathcal{J}(x)$, the proof is complete.

Adjoint operators

Before we move to an important corollary, let as recall the concept of the adjoint operator. If $A \in \mathcal{L}(X,Y)$, then the adjoint operator A^* is defined as

$$<y^*, Ax> = <A^*y^*, x> \tag{2.38}$$

and it can be proved that it belongs to $\mathcal{L}(Y^*, X^*)$ with $\|A^*\| = \|A\|$. If A is an unbounded operator, then the situation is more complicated. In general, A^* may not exist as a single-valued operator. In other words, there may be many operators B satisfying φ^*

$$<y^*, Ax> = <By^*, x>, \qquad x \in D(A),\ y^* \in D(B). \tag{2.39}$$

Operators A and B satisfying (2.39) are called *adjoint to each other*.

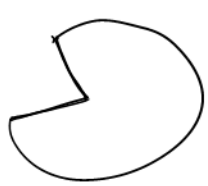
However, if $D(A)$ is dense in X, then there is a unique maximal operator A^* adjoint to A; that is, any other B such that A and B are adjoint to each other, must satisfy $B \subset A^*$. This A^* is called the *adjoint operator* to A. It can be constructed in the following way. The domain $D(A^*)$ consists of all elements y^* of Y^* for which there exists $f^* \in X^*$ with the property

$Au = \Delta u \qquad W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega)$

$\forall\, v \in \mathring{W}_2^1(\Omega) \cap W_2^2(\Omega) \quad \int \Delta u\, v = \int \nabla u \nabla v = \int \Delta v \cdot u$

$\{\mathring{W}_2^1(\Omega) \ni u,\ \Delta u \in L_2\}$

$\{u \in L_2(\Omega),\ \Delta u \in L_2\}$



$\forall\, x \in D(A)$ $\langle y^*, Ax\rangle = \langle f^*, x\rangle$
$\langle y^*, Ax\rangle = \langle g^*, x\rangle$

g^* f^*

$0 = \langle f^* - g^*, x\rangle$ $\forall\, x \in D(A)$

$$<y^*, Ax> = <f^*, x> \qquad x \in D(A) \tag{2.40}$$

for any $x \in D(A)$. Because $D(A)$ is dense, such element f^* can be proved to be unique and therefore we can define $A^*y^* = f^*$. Moreover, the assumption $\overline{D(A)} = X$ ensures that A^* is a closed operator though not necessarily densely defined. In reflexive spaces the situation is better: if both X and Y are reflexive, then A^* is closed and densely defined with

$x_n \to x$ $x \in D(A)$ $y \in Im($

$$\overline{A} = (A^*)^*; \tag{2.41}$$

$Ax_n = y_n + x_n$

see [105, Theorems III.5.28, III.5.29].

Ax_n then $Im(I - A)$ and closedness

Corollary 2.17. *Let A be a densely defined closed linear operator. If both A and A^* are dissipative, then A is the generator of a C_0-semigroup of contractions on X.*

$y_n = (I - A)x_n$

$y_n \to y$

$\|x_n - x_m\| \le\ \le\ \|(x_n - x_m) - A(x_n - x_n)\| = \|y_n - y_m\|$

Proof. It suffices to prove that, e.g., $Im(I - A) = X$. Since A is dissipative and closed, $Im(I - A)$ is a closed subspace of X. Indeed, if $y_n \to y$, $y_n \in Im(I - A)$, then, by dissipativity, $\|x_n - x_m\| \le \|(x_n - x_m) - (Ax_n - Ax_m)\| = \|y_n - y_m\|$ and $(x_n)_{n\in\mathbb{N}}$ converges. But then $(Ax_n)_{n\in\mathbb{N}}$ converges and, by closedness, $x \in D(A)$ and $x - Ax = y \in Im(I - A)$. Assume $Im(I - A) \neq X$, then by H-B theorem, there is $0 \neq x^* \in X^*$ such that $< x^*, x - Ax >= 0$ for all $x \in D(A)$. But then $x^* \in D(A^*)$ and, by density of $D(A)$, $x^* - A^*x^* = 0$ but dissipativeness of A^* gives $x^* = 0$.

$\langle 0, x\rangle$

$\langle x^*, x\rangle - \langle A^*x^*, x\rangle = \langle x^* - A^*x^*, x\rangle = 0$ $\forall\, x \in D(A)$

The Cauchy problem for the heat equation

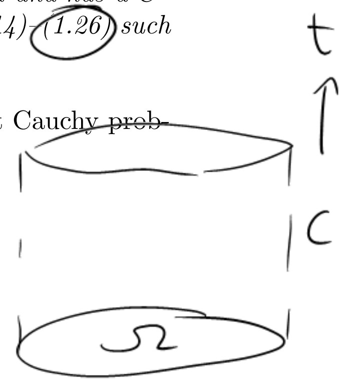
Let $C = \Omega \times (0, \infty), \Sigma = \partial\Omega \times (0, \infty)$ where Ω is an open set in $\mathbb{R}^n$. We consider the problem

$$\partial_t u = \Delta u, \quad \text{in}\Omega \times [0, \cancel{T}\infty], \tag{2.42}$$

$$u = 0, \quad \text{on}\Sigma, \tag{2.43}$$

$$u = u_0, \quad \text{on}\Omega. \tag{2.44}$$

Theorem 2.18. *Assume that $u_0 \in L_2(\Omega)$ where Ω is bounded and has a C^2 boundary. Then there exists a unique function u satisfying (2.44)–(1.26) such that $u \in C([0, \infty); L_2(\Omega)) \cap C((0, \infty); W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega))$,*



Proof. The strategy is to consider (2.44–$\cancel{(1.26)}$ as the abstract Cauchy problem

$$u' = Au, \quad u(0) = u_0$$

in $X = L_2(\Omega)$ where A is the unbounded operator defined by

$$Au = \Delta u$$

for

$$u \in D(A) = \{u \in \overset{\circ}{W}{}_2^1(\Omega); \Delta u \in L_2(\Omega)\} = W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)).$$

First we observe that A is densely defined as $C_0^\infty(\Omega) \subset \overset{\circ}{W}{}_2^1(\Omega)$ and $\Delta C_0^\infty(\Omega) \subset L_2(\Omega)$. Next, A is dissipative. For $u \in L_2(\Omega)$, $\mathcal{J}u = u$ and

$$(Au, u) = -\int\limits_\Omega |\nabla u|^2 d\mathbf{x} \leq 0$$

$u - \Delta u = f \in L_2$

$u \in D(A)$

Further, we consider the variational problem associated with $I - A$, that is, to find $u \in \overset{\circ}{W}{}_2^1(\Omega)$ to

$\lambda > 0$

$$a(u, v) = \int\limits_\Omega \nabla u \nabla v d\mathbf{x} + \int\limits_\Omega uv d\mathbf{x} = \int\limits_\Omega fv d\mathbf{x}, \quad v \in \overset{\circ}{W}{}_2^1(\Omega)$$

$\lambda < 0$

where $f \in L_2(\Omega)$ is given. Clearly, $a(u, u) = \|u\|_{1,\Omega}^2$ and thus is coercive. Hence there is a unique solution $u \in \overset{\circ}{W}{}_2^1$ (Ω) which, by writing

$$\int\limits_\Omega \nabla u \nabla v d\mathbf{x} = \int\limits_\Omega fv d\mathbf{x} - \int\limits_\Omega uv d\mathbf{x} = \int\limits_\Omega (f - u)v d\mathbf{x},$$

can be shown to be in $W_2^2(\Omega)$. This ends the proof of generation.

If we wanted to use the Hille-Yosida theorem instead, then to find the resolvent, we would have to solve

$$a(u, v) = \int\limits_\Omega \nabla u \nabla v d\mathbf{x} + \lambda \int\limits_\Omega uv d\mathbf{x} = \int\limits_\Omega fv d\mathbf{x}, \quad v \in \overset{\circ}{W}{}_2^1(\Omega)$$

$\lambda > 0$

$u_\lambda = R(\lambda, A) f$

for $\lambda > 0$. The procedure is the same and we get in particular for ~~the~~ solution

$$\|\nabla u_\lambda\|_{0,\Omega}^2 + \lambda \|u_\lambda\|_{0,\Omega}^2 \leq \|f\|_{0,\Omega} \|u_\lambda\|_{0,\Omega}.$$

$R(\lambda, A) : L_2(\Omega) \to W_2^2 \cap \overset{\circ}{W}{}_2^1$

Since $u_\lambda = R(\lambda, A)f$ we obtain

$$\|R(\lambda, A)f\|_{0,\Omega}^2 \leq \lambda^{-1} \|f\|_{0,\Omega}.$$

$L_2(\Omega) \to L_2(\Omega)$

Closedness follows from continuous invertibility.

$j \circ R(\lambda, A) : L_2(\Omega) \to L_2(\Omega)$

$u_0 \in L_2(\Omega)$

$u(t) = G(t) u_0 \in C([0, \infty), L_2(\Omega))$

$u_0 \in D(A) = W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$

$G(t) u_0 \in C^1([0, \infty), L_2(\Omega)) \cap C([0, \infty), W_2^2(\Omega))$