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Selected Topics in Applied Functional Analysis

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Basic Facts from Functional Analysis and Banach Lattices

1.1 Spaces and Operators

1.1.1 General Notation

The symbol ':=' denotes 'equal by definition'. The sets of all natural (not including 0), integer, real, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, respectively. If $\lambda \in \mathbb{C}$, then we write $\Re\lambda$ for its real part, $\Im\lambda$ for its imaginary part, and $\bar{\lambda}$ for its complex conjugate. The symbols $[a, b]$, (a, b) denote closed and open intervals in $\mathbb{R}$. Moreover,

$$\begin{aligned} \mathbb{R}_+ &:= [0, \infty), \\ \mathbb{N}_0 &:= \{0, 1, 2, \ldots\}. \end{aligned}$$

If there is a need to emphasise that we deal with multidimensional quantities, we use boldface characters, for example $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Usually we use the Euclidean norm in $\mathbb{R}^n$, denoted by,

$$|\mathbf{x}| = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

If Ω is a subset of any topological space X, then by $\overline{\Omega}$ and $Int\,\Omega$ we denote, respectively, the closure and the interior of Ω with respect to X. If (X, d) is a metric space with metric d, we denote by

$$B_{x,r} := \{y \in X;\ d(x, y) \le r\}$$

the closed ball with centre x and radius r. If X is also a linear space, then the ball with radius r centred at the origin is denoted by B_r.

Let f be a function defined on a set Ω and $x \in \Omega$. We use one of the following symbols to denote this function: f, $x \to f(x)$, and $f(\cdot)$. The symbol $f(x)$ is in general reserved to denote the value of f at x, however, occasionally we abuse this convention and use it to denote the function itself.

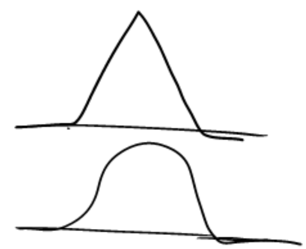
If $\{x_n\}_{n\in\mathbb{N}}$ is a family of elements of some set, then the sequence of these elements, that is, the function $n \to x_n$, is denoted by $(x_n)_{n\in\mathbb{N}}$. However, for simplicity, we often abuse this notation and use $(x_n)_{n\in\mathbb{N}}$ also to denote $\{x_n\}_{n\in\mathbb{N}}$.

The derivative operator is usually denoted by ∂. However, as we occasionally need to distinguish different types of derivatives of the same function, we use other commonly accepted symbols for differentiation. To indicate the variable with respect to which we differentiate we write $\partial_t, \partial_x, \partial^2_{tx} \ldots$. If $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then $\partial_{\mathbf{x}} := (\partial_{x_1}, \ldots, \partial_{x_n})$ is the gradient operator.

If $\beta := (\beta_1, \ldots, \beta_n)$, $\beta_i \geq 0$ is a multi-index with $|\beta| := \beta_1 + \cdots + \beta_n = k$, then symbol $\partial^\beta_{\mathbf{x}} f$ is any derivative of f of order k. Thus, $\sum_{|\beta|=0}^{k} \partial^\beta f$ means the sum of all derivatives of f of order less than or equal to k.

If $\Omega \subset \mathbb{R}^n$ is an open set, then for $k \in \mathbb{N}$ the symbol $C^k(\Omega)$ denotes the set of k times continuously differentiable functions in Ω. We denote by $C(\Omega) := C^0(\Omega)$ the set of all continuous functions in Ω and

$$C^\infty(\Omega) := \bigcap_{k=0}^{\infty} C^k(\Omega).$$



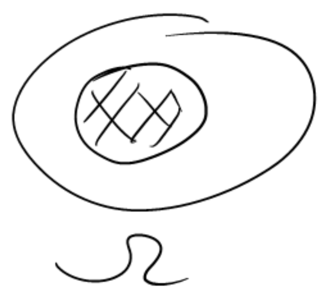
Functions from $C^k(\Omega)$ need not be bounded in Ω. If they are required to be bounded together with their derivatives up to the order k, then the corresponding set is denoted by $C^k(\overline{\Omega})$.

For a continuous function f, defined on Ω, we define the *support* of f as

$$\mathrm{supp} f = \overline{\{\mathbf{x} \in \Omega;\ f(x) \neq 0\}}.$$

The set of all functions with compact support in Ω which have continuous derivatives of order smaller than or equal to k is denoted by $C^k_0(\Omega)$. As above, $C_0(\Omega) := C^0_0(\Omega)$ is the set of all continuous functions with compact support in Ω and

$$C^\infty_0(\Omega) := \bigcap_{k=0}^{\infty} C^k_0(\Omega).$$



Another important standard class of spaces are the spaces $L_p(\Omega)$, $1 \leq p \leq \infty$ of functions integrable with power p. To define them, let us establish some general notation and terminology. We begin with a *measure space* (Ω, Σ, μ), where Ω is a set, Σ is a σ-algebra of subsets of Ω, and μ is a σ-additive measure on Σ. We say that μ is *σ-finite* if Ω is a countable union of sets of finite measure.

In most applications in this book, $\Omega \subset \mathbb{R}^n$ and Σ is the σ-algebra of Lebesgue measurable sets. However, occasionally we need the family of *Borel sets* which, by definition, is the smallest σ-algebra which contains all open sets. The measure μ in the former case is called the Lebesgue measure and in the latter the Borel measure. Such measures are σ-finite.

A function $f : \Omega \to \mathbb{R}$ is said to be *measurable* (with respect to Σ, or with respect to μ) if $f^{-1}(B) \in \Sigma$ for any Borel subset B of $\mathbb{R}$. Because Σ is a

σ-algebra, f is measurable if (and only if) preimages of semi-infinite intervals are in Σ.

We identify two functions which differ from each other on a set of μ-measure zero, therefore, when speaking of a function in the context of measure spaces, we usually mean a class of equivalence of functions. For most applications the distinction between a function and a class of functions is irrelevant. Sometimes, however, some care is necessary, as explained in Example 1.25 and Subsection 1.2.6.

The space of equivalence classes of all measurable real functions on Ω is denoted by $L_0(\Omega, d\mu)$ or simply $L_0(\Omega)$.

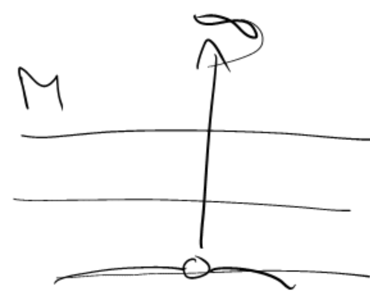
The integral of a measurable function f with respect to measure μ over a set Ω is written as

$$\int_\Omega f d\mu = \int_\Omega f(\mathbf{x}) d\mu_\mathbf{x},$$

where the second version is used if there is a need to indicate the variable of integration. If μ is the Lebesgue measure, we abbreviate $d\mu_\mathbf{x} = d\mathbf{x}$.

For $1 \le p < \infty$ the spaces $L_p(\Omega)$ are defined as subspaces of $L_0(\Omega)$ consisting of functions for which

$$\|f\|_p := \|f\|_{L_p(\Omega)} = \left(\int_\Omega |f(\mathbf{x})|^p d\mathbf{x} \right)^{1/p} < \infty. \tag{1.1}$$



The space $L_p(\Omega)$ with the above norm is a Banach space. It is customary to complete the scale of L_p spaces by the space $L_\infty(\Omega)$ defined to be the space of all Lebesgue measurable functions which are bounded almost everywhere in Ω, that is, bounded everywhere except possibly on a set of measure zero. The corresponding norm is defined by

$$\|f\|_\infty := \|f\|_{L_\infty(\Omega)} := \inf\{M;\ \mu(\{\mathbf{x} \in \Omega;\ |f(\mathbf{x})| > M\}) = 0\}. \tag{1.2}$$

The expression on the right-hand side of (1.2) is frequently referred to as the *essential supremum* of f over Ω and denoted $\operatorname{ess\,sup}_{\mathbf{x}\in\Omega} |f(\mathbf{x})|$.

If $\mu(\Omega) < \infty$, then for $1 \le p \le p' \le \infty$ we have

$$L_{p'}(\Omega) \subset L_p(\Omega) \tag{1.3}$$

and for $f \in L_\infty(\Omega)$

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p, \tag{1.4}$$

which justifies the notation. However,

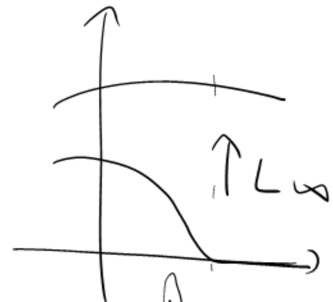
$$\bigcap_{1\le p<\infty} L_p(\Omega) \ne L_\infty(\Omega),$$

as demonstrated by the function $f(x) = \ln x,\ x \in (0, 1]$. If $\mu(\Omega) = \infty$, then neither (1.3) nor (1.4) hold.

Occasionally we need functions from $L_0(\Omega)$ which are L_p only on compact subsets of $\mathbb{R}^n$. Spaces of such functions are denoted by $L_{p,loc}(\Omega)$. A function $f \in L_{1,loc}(\Omega)$ is called *locally integrable* (in Ω).

Let $\Omega \subset \mathbb{R}^n$ be an open set. It is clear that

$$C_0^\infty(\Omega) \subset L_p(\Omega)$$



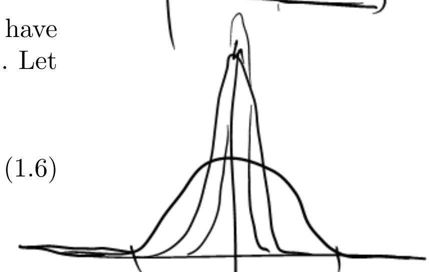
for $1 \le p \le \infty$. If $p \in [1,\infty)$, then we have even more: $C_0^\infty(\Omega)$ is dense in $L_p(\Omega)$.

$$\overline{C_0^\infty(\Omega)} = L_p(\Omega), \tag{1.5}$$

where the closure is taken in the L_p-norm.

Example 1.1. Having in mind further applications, it is worthwhile to have some understanding of the structure of this result; see [4, Lemma 2.18]. Let us define the function

$$\omega(\mathbf{x}) = \begin{cases} \exp\left(\frac{1}{|\mathbf{x}|^2-1}\right) & \text{for } |\mathbf{x}| < 1, \\ 0 & \text{for } |\mathbf{x}| \ge 1. \end{cases} \tag{1.6}$$



This is a $C_0^\infty(\mathbb{R}^n)$ function with support B_1.

Using this function we construct the family

$$\omega_\epsilon(\mathbf{x}) = C_\epsilon \omega(\mathbf{x}/\epsilon),$$

$\operatorname{supp} \omega_\varepsilon$ to $\overline{B(0,\varepsilon)}$

where C_ϵ are constants chosen so that $\int_{\mathbb{R}^n} \omega_\epsilon(\mathbf{x}) d\mathbf{x} = 1$; these are also $C_0^\infty(\mathbb{R}^n)$ functions with support B_ϵ, often referred to as *mollifiers*. Using them, we define the *regularisation* (or *mollification*) of f by taking the convolution

$\varphi(x) = e^{-\frac{1}{x}}$ $x > 0$

$\varphi(x) = 0$ $x < 0$

$$(J_\epsilon * f)(\mathbf{x}) := \int_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y})\omega_\epsilon(\mathbf{y})d\mathbf{y} = \int_{\mathbb{R}^n} f(\mathbf{y})\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y}. \tag{1.7}$$

$\left[a\left(\frac{1}{x}\right)^n + b\left(\frac{1}{x}\right)^{n-1} \dots\right] e^{-\frac{1}{x}}$

Precisely speaking, if $\Omega \ne \mathbb{R}^n$, we integrate outside the domain of definition of f. Thus, in such cases below, we consider f to be extended by 0 outside Ω.

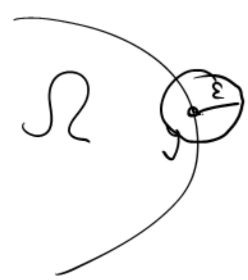
Then, we have

Theorem 1.2. *With the notation above,*

1. *Let $p \in [1,\infty)$. If $f \in L_p(\Omega)$, then*

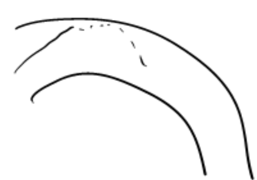
$$\lim_{\epsilon \to 0^+} \|J_\epsilon * f - f\|_p = 0.$$

2. *If $f \in C(\Omega)$, then $J_\epsilon * f \to f$ uniformly on any $\bar{G} \subset \Omega$.*
3. *If $\bar{\Omega}$ is compact and $f \in C(\bar{\Omega})$, then $J_\epsilon * f \to f$ uniformly on $\bar{\Omega}$.*
4. *If $f \in L_\infty(\mathbb{R}^n)$ and support of f is bounded, then $J_\epsilon * f \to f$ almost everywhere on $\mathbb{R}^n$.*



$K \subset\subset \Omega$ omecne, i.e. $\overline{K} \subset \Omega$

ostroty



R. Adams Sobolev spaces

Proof. For 1., even if $\mu(\Omega) = \infty$, then any $f \in L_p(\Omega)$ can be approximated by (essentially) bounded (simple) functions with compact supports. By Luzin's theorem, such functions can be approximated in $L_p(\Omega)$ (and almost everywhere) by continuous functions with compact support. Thus, it is enough to prove the result for continuous compactly supported functions.

Because the effective domain of integration in the second integral is $B_{\mathbf{x},\epsilon}$, $J_\epsilon * f$ is well defined whenever f is locally integrable and, similarly, if the support of f is bounded, then suppf_ϵ is also bounded and it is contained in the ϵ-neighbourhood of supp$f$. The functions f_ϵ are infinitely differentiable with

$J^*_\varepsilon f(x) \doteq \int_{\mathbb{R}^n} f(y)\omega_\varepsilon(x-y)dy$

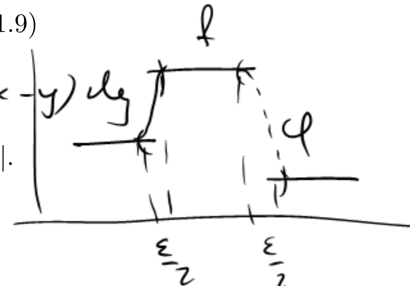
$$\partial_x^\beta (J_\varepsilon * f)(x) \doteq \partial^\beta_{\mathbf{x}} f_\varepsilon(\mathbf{x}) = \int\limits_{\mathbb{R}^n} f(\mathbf{y})\partial^\beta_{\mathbf{x}}\omega_\epsilon(\mathbf{x}-\mathbf{y})d\mathbf{y} \tag{1.8}$$

for any β. By Hölder inequality, if $f \in L_p(\mathbb{R}^n)$, then $f_\epsilon \in L_p(\mathbb{R}^n)$ with

$$\|f_\epsilon\|_p \le \|f\|_p \tag{1.9}$$

for any $\epsilon > 0$ and

$f(x) = f(x)\cdot 1 = \int_\Omega f(x)\omega_\varepsilon(x-y)dy$



$$|(J_\epsilon * f)(\mathbf{x}) - f(\mathbf{x})| \le \int\limits_{\mathbb{R}^n} |f(\mathbf{y}) - f(\mathbf{x})|\omega_\epsilon(\mathbf{x}-\mathbf{y}) \le \sup_{\|\mathbf{x}-\mathbf{y}\|\le\epsilon} |f(\mathbf{x}) - f(\mathbf{y})|.$$

By compactness of support and of f we obtain

$$f = \lim_{\epsilon\to 0^+} f_\epsilon, \qquad \text{in } L_p(\mathbb{R}^n) \tag{1.10}$$

$\forall_\varepsilon \exists \varphi_\varepsilon$

as well as in $C(\bar{\Omega})$, where in the latter case we extend f outside Ω by a continuous function (e.g. Urysohn theorem). For 4., the statement follows directly from Luzin's theorem.

$f_n \rightrightarrows f$

$\int_\Omega |f_n - f|^p dx \le \varepsilon^p \int_\Omega dx$

$A_\varepsilon = \mu\{x : \varphi \ne f\} < \varepsilon$

We observe that, if f is nonnegative, then f_ϵ are also nonnegative by (1.7) and hence any non-negative $f \in L_p(\mathbb{R}^n)$ can be approximated by nonnegative, infinitely differentiable, functions with compact support.

Remark 1.3. Spaces $L_p(\Omega)$ often are defined as a completion of $C_0(\Omega)$ in the $L_p(\Omega)$ norm, thus avoiding introduction of measure theory. The theorem above shows that these two definitions are equivalent.

$\varepsilon = \frac{1}{n^2}$

$\varphi_n \to f$

wszędzie poza $\bigcap_{k=1}^{\infty}\bigcup_{n=k} A_{\frac{1}{n^2}}$

1.1.2 Operators

Let X, Y be real or complex Banach spaces with the norm denoted by $\|\cdot\|$ or $\|\cdot\|_X$.

An *operator* from X to Y is a linear rule $A : D(A) \to Y$, where $D(A)$ is a linear subspace of X, called the *domain* of A. The set of operators from X to Y is denoted by $L(X,Y)$. Operators taking their values in the space of scalars are called *functionals*. We use the notation $(A, D(A))$ to denote the operator

Additions f - measurable with bounded support

Luzin theorem => convergence a.e

$$\forall_{\varepsilon} \exists_{\varphi_\varepsilon \in C_0(\mathbb{R}^n)} \mu(\{x ; \varphi_\varepsilon(x) \neq f(x)\}) \leq \varepsilon \qquad |\varphi_\varepsilon| \leq |f|$$

$$A_n = \{x : \varphi_n(x) \neq f(x)\}$$

$$\mu(A_n) \leq \frac{1}{n^2}$$

$$A = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$$

Claim $\forall_{x \notin A} \varphi_n(x) \to f(x)$

$$x \notin A \Rightarrow \exists_k \forall_{n \geq k} x \notin A_n \Rightarrow \varphi_n(x) = f(x)$$

$$\varphi_n(x) \to f(x)$$

$$\mu(A) \leq \lim_{k \to \infty} \sum_{n=k}^{\infty} \frac{1}{n^2} \to 0$$

$$\mu(A) = 0$$

$J_\varepsilon * \varphi \to \varphi$ dla $\varphi \in C_0(\mathbb{R}^n)$

$f \in L_p(\Omega)$ $J_\varepsilon * f \to f$

$\|A_n\| \leq M$ $D \subset X$ $\overline{D} = X$ $\forall_{x \in D} A_n x \to f_x$

$\forall_{x \in X} A_n x \to f_x$

$$\| J_\varepsilon * f - f \| \leq \| J_\varepsilon * f - J_\varepsilon * \varphi \| + \| J_\varepsilon * \varphi - \varphi \| + \| \varphi - f \|$$

$$\leq \underbrace{\underbrace{\| f - \varphi \|}_{<\,\varepsilon} + \underbrace{\varepsilon}_{\text{z dowodu}} + \underbrace{\| f - \varphi \|}_{<\,\varepsilon}}_{\text{z epsiloni}}$$

A with domain $D(A)$. If $A \in L(X, X)$, then we say that A (or $(A, D(A))$) is an operator in X.

By $\mathcal{L}(X, Y)$, we denote the space of all bounded operators between X and Y; $\mathcal{L}(X, X)$ is abbreviated as $\mathcal{L}(X)$. The space $\mathcal{L}(X, Y)$ can be made a Banach space by introducing the norm of an operator X by

$$\|A\| = \sup_{\|x\| \leq 1} \|Ax\| = \sup_{\|x\|=1} \|Ax\|. \tag{1.11}$$

If $(A, D(A))$ is an operator in X and $Y \subset X$, then the *part* of the operator A in Y is defined as

$$A_Y y = Ay \tag{1.12}$$

on the domain

$$D(A_Y) = \{x \in D(A) \cap Y;\ Ax \in Y\}.$$

A *restriction* of $(A, D(A))$ to $D \subset D(A)$ is denoted by $A|_D$. For $A, B \in L(X, Y)$, we write $A \subset B$ if $D(A) \subset D(B)$ and $B|_{D(A)} = A$.

Two operators $A, B \in \mathcal{L}(X)$ are said to commute if $AB = BA$. It is not easy to extend this definition to unbounded operators due to the difficulties with defining the domains of the composition. The extension is usually done to the case when one of the operators is bounded. Thus, an operator $A \in L(X)$ is said to *commute* with $B \in \mathcal{L}(X)$ if

$$BA \subset AB. \tag{1.13}$$

This means that for any $x \in D(A)$, $Bx \in D(A)$ and $BAx = ABx$.

We define the *image* of A by

$$ImA = \{y \in Y;\ y = Ax \text{ for some } x \in D(A)\}$$

and the *kernel* of A by

$$KerA = \{x \in D(A);\ Ax = 0\}.$$

We note a simple result which is frequently used throughout the book.

Proposition 1.4. *Suppose that $A, B \in L(X, Y)$ satisfy: $A \subset B, KerB = \{0\}$, and $ImA = Y$. Then $A = B$.*

Proof. If $D(A) \neq D(B)$, we take $x \in D(B) \setminus D(A)$ and let $y = Bx$. Because A is onto, there is $x' \in D(A)$ such that $y = Ax'$. Because $x' \in D(A) \subset D(B)$ and $A \subset B$, we have $y = Ax' = Bx'$ and $Bx' = Bx$. Because $KerB = \{0\}$, we obtain $x = x'$ which is a contradiction with $x \notin D(A)$. $\square$

Furthermore, the *graph* of A is defined as

$$G(A) = \{(x, y) \in X \times Y;\ x \in D(A), y = Ax\}. \tag{1.14}$$

Przez operator ograniczony rozumiemy operator taki, że $D(A) = X$

$\|A\| < +\infty$

We say that the operator A is *closed* if $G(A)$ is a closed subspace of $X \times Y$. Equivalently, A is closed if and only if for any sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$, if $\lim_{n\to\infty} x_n = x$ in X and $\lim_{n\to\infty} Ax_n = y$ in Y, then $x \in D(A)$ and $y = Ax$.

An operator A in X is *closable* if the closure of its graph $\overline{G(A)}$ is itself a graph of an operator, that is, if $(0, y) \in \overline{G(A)}$ implies $y = 0$. Equivalently, A is closable if and only if for any sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$, if $\lim_{n\to\infty} x_n = 0$ in X and $\lim_{n\to\infty} Ax_n = y$ in Y, then $y = 0$. In such a case the operator whose graph is $\overline{G(A)}$ is called the *closure* of A and denoted by $\overline{A}$.

By definition, when A is closable, then

$$\begin{aligned} D(\overline{A}) &= \{x \in X;\ \text{there is } (x_n)_{n\in\mathbb{N}} \subset D(A) \text{ and } y \in X \text{ such that} \\ &\qquad \|x_n - x\| \to 0 \text{ and } \|Ax_n - y\| \to 0\}, \\ \overline{A}x &= y. \end{aligned}$$

For any operator A, its domain $D(A)$ is a normed space under the *graph norm*

$$\|x\|_{D(A)} := \|x\|_X + \|Ax\|_Y. \tag{1.15}$$

The operator $A : D(A) \to Y$ is always bounded with respect to the graph norm, and A is closed if and only if $D(A)$ is a Banach space under (1.15).

One of the simplest and most often used unbounded, but closed or closable, operators is the operator of differentiation. If X is any of the spaces $C([0,1])$ or $L_p([0,1])$, then considering $f_n(x) := C_n x^n$, where $C_n = 1$ in the former case and $C_n = (np+1)^{1/p}$ in the latter, we see that in all cases $\|f_n\| = 1$. However,

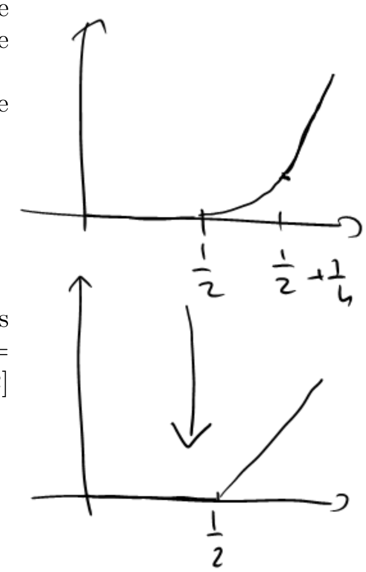
$$\|f_n'\| = n\left(\frac{np+1}{np+1-p}\right)^{1/p}$$

in $L_p([0,1])$ and $\|f_n'\| = n$ in $C([0,1])$, so that the operator of differentiation is unbounded.

Let us define $Tf = f'$ as an unbounded operator on $D(T) = \{f \in X;\ Tf \in X\}$, where X is any of the above spaces. We can easily see that in $X = C([0,1])$ the operator T is closed. Indeed, let us take $(f_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} Tf_n = g$ in X. This means that $(f_n)_{n\in\mathbb{N}}$ and $(f_n')_{n\in\mathbb{N}}$ converge uniformly to, respectively, f and g, and from basic calculus f is differentiable and $f' = g$.

The picture changes, however, in L_p spaces. To simplify the notation, we take $p = 1$ and consider the sequence of functions

$$f_n(x) = \begin{cases} 0 & \text{for } 0 \le x \le \frac{1}{2}, \\ \frac{n}{2}\left(x - \frac{1}{2}\right)^2 & \text{for } \frac{1}{2} < x \le \frac{1}{2} + \frac{1}{n}, \\ x - \frac{1}{2} - \frac{1}{2n} & \text{for } \frac{1}{2} + \frac{1}{n} < x \le 1. \end{cases}$$



These are differentiable functions and it is easy to see that $(f_n)_{n\in\mathbb{N}}$ converges in $L_1([0,1])$ to the function f given by $f(x) = 0$ for $x \in [0, 1/2]$ and $f(x) = x - 1/2$ for $x \in (1/2, 1]$ and the derivatives converge to $g(x) = 0$ if $x \in [0, 1/2]$

and to $g(x) = 1$ otherwise. The function f, however, is not differentiable and so T is not closed. On the other hand, g seems to be a good candidate for the derivative of f in some more general sense. Let us develop this idea further. First, we show that T is closable. Let $(f_n)_{n\in\mathbb{N}}$ and $(f_n')_{n\in\mathbb{N}}$ converge in X to f and g, respectively. Then, for any $\phi \in C_0^\infty((0,1))$, we have, integrating by parts,

$$\int_0^1 f_n'(x)\phi(x)dx = -\int_0^1 f_n(x)\phi'(x)dx$$

and because we can pass to the limit on both sides, we obtain

$$\int_0^1 g(x)\phi(x)dx = -\int_0^1 f(x)\phi'(x)dx. \tag{1.16}$$

Using the equivalent characterization of closability, we put $f = 0$, so that

$$\int_0^1 g(x)\phi(x)dx = 0$$

for any $\phi \in C_0^\infty((0,1))$ which yields $g(x) = 0$ almost everywhere on $[0,1]$. Hence $g = 0$ in $L_1([0,1])$ and consequently T is closable.

The domain of $\overline{T}$ in $L_1([0,1])$ is called the Sobolev space $W_1^1([0,1])$ which is discussed in more detail in Subsection 2.2.1.

These considerations can be extended to hold in any $\Omega \subset \mathbb{R}^n$. In particular, we can use (1.16) to generalize the operation of differentiation in the following way: we say that a function $g \in L_{1,loc}(\Omega)$ is the *generalised (or distributional) derivative* of $f \in L_{1,loc}(\Omega)$ of order α, denoted by $\partial_{\mathbf{x}}^\alpha f$, if

$$\int_\Omega g(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = (-1)^{|\beta|}\int_\Omega f(\mathbf{x})\partial_{\mathbf{x}}^\beta \phi(\mathbf{x})d\mathbf{x} \tag{1.17}$$

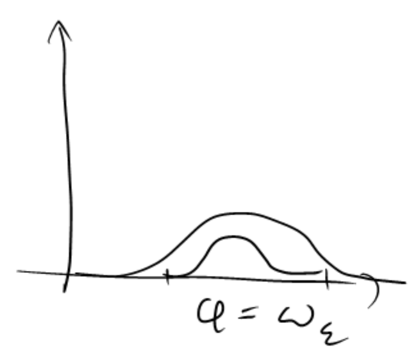
for any $\phi \in C_0^\infty(\Omega)$.

This operation is well defined. This follows from the Du Bois Reymond lemma.

Lemma 1.5. *If $f \in L_{1,loc}(\Omega)$ satisfies*

$$\int_\Omega f\phi d\mathbf{x} = 0 \tag{1.18}$$

for any $\phi \in C_0^\infty(\Omega)$, then $f = 0$ a.e. in Ω.



Proof. We can assume f real. Consider any $K \Subset \Omega$ and $\Phi = \operatorname{sign} f$ on K and zero elsewhere. There is a sequence of $\phi_n \in C_0^\infty(\Omega)$ with $|\phi_n(x)| \leq 1$ and $\phi_n \to \Phi$ a.e. on K. Thus, by Lebesgue dominated convergence theorem

$$\int\limits_K |f| d\mathbf{x} = \int\limits_K f \lim \phi_n d\mathbf{x} = \lim \int\limits_K f\phi_n d\mathbf{x} = 0.$$

From the considerations above it is clear that $\partial_{\mathbf{x}}^\beta$ is a closed operator extending the classical differentiation operator (from $C^{|\beta|}(\Omega)$). One can also prove that $\partial_{\mathbf{x}}^\beta$ is the closure of the classical differentiation operator. If $\Omega = \mathbb{R}^n$, then this statement follows directly from (1.7) and (1.8). Indeed, let $f \in L_p(\mathbb{R}^n)$ and $g = D^\alpha f \in L_p(\mathbb{R}^n)$. We consider $f_\epsilon = J_\epsilon * f \to f$ in L_p. By Fubini theorem, we prove

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} (J_\epsilon * f)(\mathbf{x}) D^\alpha \phi(\mathbf{x}) d\mathbf{x} &= \int\limits_{\mathbb{R}^n} \omega_\epsilon(y) \int\limits_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y}) D^\alpha \phi(\mathbf{x}) d\mathbf{x} d\mathbf{y} \\
&= (-1)^{|\alpha|} \int\limits_{\mathbb{R}^n} \omega_\epsilon(y) \int\limits_{\mathbb{R}^n} g(\mathbf{x}-\mathbf{y}) \phi(\mathbf{x}) d\mathbf{x} d\mathbf{y} \\
&= (-1)^{|\alpha|} \int\limits_{\mathbb{R}^n} (J_\epsilon * g) \phi(\mathbf{x}) d\mathbf{x}
\end{aligned}$$

so that $D^\alpha f_\epsilon = J_\epsilon * D^\alpha f = J_\epsilon * g \to g$ as $\epsilon \to 0$ in L_p. This shows that action of the distributional derivative can be obtained as the closure of the classical derivation.

Otherwise the proof is more complicated (see, e.g., [4, Theorem 3.16]) since we do not know whether we can extend f outside Ω in such a way that the extension still will have the generalized derivative.

In one-dimensional spaces the concept of the generalised derivative is closely related to a classical notion of absolutely continuous function. Let $I = [a, b] \subset \mathbb{R}^1$ be a bounded interval. We say that $f : I \to \mathbb{C}$ is *absolutely continuous* if, for any $\epsilon > 0$, there is $\delta > 0$ such that for any finite collection $\{(a_i, b_i)\}_i$ of disjoint intervals in $[a, b]$ satisfying $\sum_i (b_i - a_i) < \delta$, we have $\sum_i |f(b_i) - f(a_i)| < \epsilon$. The fundamental theorem of calculus, [150, Theorem 8.18], states that any absolutely continuous function f is differentiable almost everywhere, its derivative f' is Lebesgue integrable on $[a, b]$, and $f(t) - f(a) = \int_a^t f'(s) ds$. It can be proved (e.g., [61, Theorem VIII.2]) that absolutely continuous functions on $[a, b]$ are exactly integrable functions having integrable generalised derivatives and the generalised derivative of f coincides with the classical derivative of f almost everywhere.