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MIMMO IANNELLI

MATHEMATICAL THEORY OF AGE-STRUCTURED POPULATION DYNAMICS

APPLIED MATHEMATICS MONOGRAPHS
COMITATO NAZIONALE PER LE SCIENZE MATEMATICHE
CONSIGLIO NAZIONALE DELLE RICERCHE (C.N.R.)

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PREFACE

La prefazione occupa una posizione privilegiata rispetto al testo cui si riferisce; mi sia concesso quindi usare qui la mia lingua per poche parole in apertura di questa monografia. Non per dire dei suoi contenuti e obiettivi, cosa che faccio nell'introduzione, ma per aggiungere al contorno qualche osservazione di tono più personale.

Una prima introduzione agli argomenti svolti in questo libro faceva parte del programma del corso di Biomatematica, da me tenuto a Trento nel triennio 1988/91, ed è in quest'ambito che è nata l'esigenza di una monografia che sviluppasse tali argomenti oltre gli scopi di un corso di curriculum. Questa esigenza si coniugava anche al mio desiderio di sistemare ed esporre i contenuti di una teoria che si è andata sviluppando notevolmente negli ultimi venti anni e della quale mi sono occupato ampiamente nelle mie ricerche: ho quindi raccolto con piacere l'invito del Comitato a contribuire alla collana di monografie di Matematica Applicata.

Ora che ho finalmente posto il punto finale al lavoro (e già mi accorgo che desidererei andare oltre questa introduzione per trattare più specialisticamente alcuni degli argomenti discussi) mi rendo conto che, in questi venti anni, i metodi matematici che sottendono la teoria delle popolazioni con struttura di età si sono sviluppati sempre più nella direzione delle equazioni di evoluzione astratte confermando la necessità di una saldatura tra l'Analisi Funzionale e le Applicazioni. Ciò è per me di una certa soddisfazione perché di tale necessità sono sempre stato convinto e mi sono trovato spesso a difenderla di fronte a qualche attivo sostenitore delle spavalde tendenze dei nostri tempi.

Di venti anni di ricerche non restano soltanto i risultati (quelli compiuti e quelli lasciati a metà, per pigrizia o incapacità), ma anche i segni del quotidiano e del personale che accompagnano l'elaborazione del lavoro: non per nulla i libri che si scrivono sono dedicati alle mogli e ai figli per la loro presenza paziente e discreta. Nel mio caso, e standoo solo agli ultimi tre anni che mi hanno visto (irregolarmente) impegnato nella stesura di questa monografia, molti eventi personali, felici e infelici, sono inscritti tra le righe e al bordo di ogni pagina. Tra tutti questi la scomparsa di Stravros Busenberg, amico fraterno e collaboratore per tanti anni, segna particolarmente questo libro il cui contenuto ruota intorno alle ricerche che abbiamo svolto insieme: così questa prefazione si dipana nell'amarezza e nel rammarico di non poter commentare con lui il risultato del lavoro, festeggiandone la conclusione e facendo progetti per il futuro.

Introduction

The first step when modeling a population is to consider some significant variables that allow to divide the population into internally homogeneous subgroups, in order to describe the dynamics as the interaction of these groups, ruled by mechanisms that depend on those variables.

Thus, depending on the phenomenon that has to be modeled, the population is given a structure that is often responsible for special behaviors not occurring when the structure is absent, i.e. when the population can be considered homogeneous with respect to the parameters that determine the structure.

Age is one of the most natural and important parameters structuring a population. In fact many internal variables, at the level of the single individual, are strictly depending on the age because different ages mean different reproduction and survival capacities and, also, different behaviors. Then, though for a long time the interest for age structure has been restricted to demography, nowadays it plays a fundamental role in fields like ecology, epidemiology, cell growth etc.

Demographic documents reporting data that group individuals by their age can be found very far in the past (see [61] where examples concerning Stone Age, Bronze Age and Roman Age are quoted), but the very first population model that considers age structure seems to appear in the famous *Liber Abaci* by Leonardo Pisano called the Fibonacci (1228). Actually in the rabbit problem, generating the celebrated sequence of Fibonacci numbers, it is assumed that the rabbits start reproducing only two months after their birth, i.e. their fertility window opens at the age of two months.

We quote this example just to show that age structure occurs in a natural way within the context of population problems. However the theory we are going to present in this book is rather recent and stems from the basic models of population dynamics. These latter consider the different interacting species as internally homogeneous and should be our reference mark in order to show the effect of age structure. Thus, when needed, we will spend a few words to recall our comparison prototype.

Among all population models, the simplest one is entitled from T. R. Malthus who wrote a famous treatise [77] on the growth of the human popula-

Naturalmente devo ringraziare qualche persona che ha contribuito a far sì che arrivassi in fondo, in particolare Fabio Milner, Andrea Pugliese e Horst Thieme per aver letto e commentato più volte il manoscritto, e anche Luisa De Carli per il suo pronto soccorso di dattilografia.

Infine, ovviamente, dedico il tutto a Mariaconcetta, Marta, Jacopo e Giovanni, per la loro presenza paziente e discreta, appunto, anche se mi hanno più volte diffidato dal farlo.

M. IANNELLI
April 11, 1994

tion, predicting that it would be exponential in time with all the catastrophic consequences that one can imagine.

To introduce this model we consider a single homogeneous population; that is, we assume that all individuals of the population are identical so that the only variable that we have to deal with is the number of the individuals as a function of time $P(t)$ (*total population size*).

In addition we suppose that the population lives *isolated* in an *invariant habitat* with *no limit to resources*. Thus the population is subject to constant *fertility* and *mortality* rates that we respectively call β and μ (their difference $\alpha = \beta - \mu$ is usually called the *Malthusian parameter* of the population) and the growth is governed by the following equation

$$\text{(1.1)} \qquad \frac{d}{dt} P(t) = \beta P(t) - \mu P(t) = \alpha P(t).$$

Thus

$$\text{(1.2)} \qquad P(t) = P(0)e^{\alpha t}.$$

In Chapter I we will start our theory introducing a model that is strictly analogous to the Malthus model.

Our aim through this book is to present the basic theory of age structured populations and the related mathematical methods. Since our intention was to give an introduction to the subject we have not touched many topics that could be of some interest; on the other hand we have tried to give a complete introductive presentation. Actually we have tried to present the essentials of the theory with the purpose of enabling the reader to go further and work on current problems with more sophisticated mathematical methods.

Thus we have focused our exposition first on the modeling of a single species and then, in the last two chapter, on simple epidemics. Actually we have disregarded all that part of the theory that is concerned with interacting species or with multi-group dynamics (see for instance [11], [22], [25], [27], [45], [48], [79], [86], [87]) though it is a subject that has received some attention. Also we have not considered the direct extension of the theory to size-structured populations [79] nor have we considered diffusion of age-structured populations ([12], [44], [47], [50], [51], [71], [72]); moreover we have not mentioned the numerical analysis of the problems, that presents some specific features and is connected with discrete modeling ([28], [35], [58], [60], [81]).

Concerning the mathematics, we have limited ourselves to direct methods, that are based on the theory of Volterra integral equations, disregarding the functional analytic approach that in recent years has provided a natural and powerful framework to the theory ([18]-[20], [29], [64], [79], [99]-[101]). Doing so, it has not been possible to prove some of the most recent results that arise within that framework, and we have limited ourselves to quote the results. But, as we have already said, our purpose was to provide just an introduction, taking the reader to the more advanced theory, to motivate the latter both biologically and mathematically. We are in fact convinced that in order to be able to work with the tools provided by an abstract setting it is necessary to have full knowledge of the direct methods that can give answers to special problems.

In conclusion, if we have fulfilled our intentions, the outcome of this book is a systematic introduction to the classic theory, providing a first step in the field.

I
The basic linear theory

The linear theory we are going to develop in this chapter applies to the idealized situation which, when age structure is disregarded, corresponds to the Malthus model mentioned in the introduction. Actually we will deal with a strict analogue of the latter model: we consider a single population living *isolated*, in an *invariant habitat*, all of its individuals being perfectly equal but for their age, in particular we assume that there are *no sex differences*.

In accordance with this phenomenological setting, *fertility* and *mortality* are intrinsic parameters of the population growth and do not depend on time, nor on the population size: they are functions of age only.

This chapter is devoted to the introduction of the classical Lotka-McKendrick system for the description of such a population and to its analysis through the renewal equation: though extremely simple as a model, it provides a fundamental insight into age structured phenomena.

1 Introduction of the basic parameters

With the premises presented above, the evolution of the population is described by its *age density* function at time t:

$$p(a, t) \qquad a \in [0, a_\dagger], \ t \geq 0$$

where $a_\dagger$ denotes the *maximum age* which we assume to be finite (see section II.4 for some considerations about the case $a_\dagger = +\infty$). Thus the integral:

$$\int_{a_1}^{a_2} p(a, t)da$$

gives the number of individuals that, at time t, have age in the interval $[a_1, a_2]$; and

$$P(t) = \int_0^{a_\dagger} p(a, t)da \tag{1.1}$$

is the *total population* at time t.

Concerning *fertility* and *mortality* we first introduce:

$\beta(a) \equiv$ *age specific fertility,*

which can be defined as the number of newborn, in one time unit, coming from a single individual whose age is in the infinitesimal age interval $[a, a + da]$. Thus

$$\int_{a_1}^{a_2} \beta(a)p(a, t)da$$

gives the number of newborn in one time unit, coming from individuals with age in $[a_1, a_2]$. We also consider the *total birth rate*

$$B(t) = \int_0^{a_\dagger} \beta(a)p(a, t)da \tag{1.2}$$

which gives the total number of newborn in one time unit.

We also introduce

$\mu(a) \equiv$ *age specific mortality.*

It is the death rate of people having age in $[a, a + da]$; then the *total death rate* is:

$$D(t) = \int_0^{a_\dagger} \mu(a)p(a, t)da \tag{1.3}$$

and gives the total number of deaths occurring in one time unit.

The functions $\beta(\cdot)$ and $\mu(\cdot)$ are, of course, non negative: they are also called *vital rates* and are viewed as deterministic rates; in practice they are determined on a statistical basis. In figures 1.1 and 1.2 we show some classical examples of these functions, drawn from demography.

deterministic : approach admitting all thing is generated by a cause, and all thing is predicted in advance.

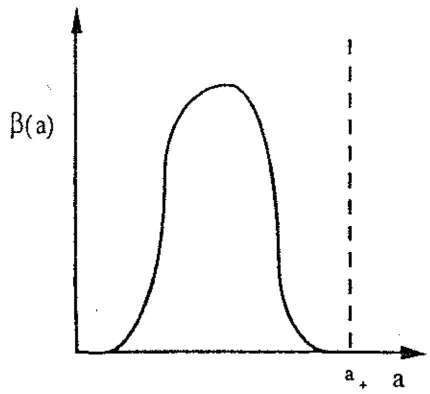

Figure 1.1

Other meaningful quantities are derived from $\beta(\cdot)$ and $\mu(\cdot)$; namely

(1.4) $$\Pi(a) = e^{-\int_0^a \mu(\sigma)d\sigma}, \qquad a \in [0, a_\dagger]$$

denotes the *survival probability*, i.e. the probability for an individual to survive to age a; thus it must be $\Pi(a_\dagger) = 0$; moreover the function

(1.5) $$K(a) = \beta(a)\Pi(a), \qquad a \in [0, a_\dagger]$$

is called the *maternity function* and syntesizes the dynamics of the population; it is related to the parameter

(1.6) $$R = \int_0^{a_\dagger} \beta(a)\Pi(a)\, da$$

which is called the *net reproduction rate* and gives the number of the newborn that an individual is expected to produce during his reproductive life. We will see that this parameter will play a role in the discussion of the asymptotic behavior of the population; in fact we expect the population to show an increasing trend when $R > 1$, decreasing if $R < 1$, stable when $R = 1$.

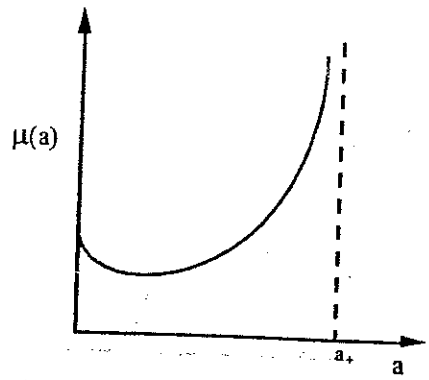

Figure 1.2

Finally we consider the *expected life*

(1.7) $$L = \int_0^{a_\dagger} \Pi(a)\, da\,.$$

This is the mean value of the life of an individual: actually (1.7) can be better understood if we note that $\mu(a)\Pi(a)da$ is the probability for an individual to survive to age a and then die in $[a, a + da]$; thus:

$$L = \int_0^{a_\dagger} a\mu(a)\Pi(a)\, da = \int_0^{a_\dagger} a \frac{d}{da}\left(e^{\int_0^a \mu(\sigma)d\sigma}\right) da = \int_0^{a_\dagger} a\left(\frac{d}{da}\int_0^a \mu(\sigma)d\sigma\right) e^{\int_0^a \mu(\sigma)d\sigma} da$$

$$= -\int_0^{a_\dagger} a \frac{d}{da}\Pi(a)\, da = -a\Pi(a)\,\big|_0^{a_\dagger} + \int_0^{a_\dagger} \Pi(a)\, da = \int_0^{a_\dagger} \Pi(a)\, da$$

where we have used $\Pi(a_\dagger) = 0$.

2 The Lotka-McKendrick equation

We now derive the basic equations which describe the evolution of the population under the phenomenological assumptions of the previous section. These equations arise as a consequence of the balance of births and deaths along time.

$\Pi(a) = e^{-\int_0^a \mu(\sigma)d\sigma}$

$\frac{d}{da}\Pi(a) = \frac{d}{da} e^{-\int_0^a \mu(\sigma)d\sigma}$

$\frac{d}{da}\Pi(a) = \left[-\frac{d}{da}\int_0^a \mu(\sigma)d\sigma\right] e^{-\int_0^a \mu(\sigma)d\sigma}$

$\frac{d}{da}\Pi(a) = -\mu(a)\Pi(a)$

$\frac{d}{da}\left[a\Pi(a)\right] = \Pi(a) + a\frac{d}{da}\Pi(a)$

$\Rightarrow a\frac{d}{da}\Pi(a) = \frac{d}{da}\left[a\Pi(a)\right] - \Pi(a)$

$\int a \frac{d}{da}\Pi(a)\,da = \int \frac{d}{da}\left[a\Pi(a)\right]da - \int \Pi(a)da$

$= a\Pi(a) - \int \Pi(a)da + c$

Consider first the function:

$$N(a, t) = \int_0^a p(\sigma, t)d\sigma$$

which represents the number of individuals that, at time t, have age less than or equal to a. Then we have, for $h > 0$

$$(2.1) \qquad N(a + h, t + h) = N(a, t) + \int_t^{t+h} B(s)ds - \int_0^h \int_0^{a+s} \mu(\sigma)p(\sigma, t + s)d\sigma\, ds$$

In fact, in (2.1), the second term on the right gives the input of all newborn in the time interval $[t, t + h]$: these have age less than or equal to h and, consequently have to be included in the number $N(a + h, t + h)$. Moreover, since:

$$\int_0^{a+s} \mu(\sigma)p(\sigma, t + s)d\sigma$$

is the number of individuals who die at the time $t + s$, having age less than or equal to $a + s$, the third terms on the right of (2.1) gives the loss from the initial group of $N(a, t)$ individuals and from the newborn, through the time interval $[t, t + h]$.

Now we differentiate (2.1) with respect to h, and set $h = 0$:

$$(2.2) \qquad p(a, t) + \int_0^a p_t(\sigma, t)d\sigma = B(t) - \int_0^a \mu(\sigma)p(\sigma, t)d\sigma$$

From this, putting $a = 0$, we get:

$$(2.3) \qquad p(0, t) = B(t)$$

and, differentiating (2.2) with respect to a:

$$(2.4) \qquad p_t(a, t) + p_a(a, t) + \mu(a)p(a, t) = 0$$

Thus (see also (1.2)) we arrive at the following system

$$(2.5)\qquad \begin{cases} i) & p_t(a,\,t) + p_a(a,\,t) + \mu(a)p(a,\,t) = 0 \\ ii) & p(0,\,t) = \displaystyle\int_0^{a_\dagger} \beta(\sigma)p(\sigma,\,t)d\sigma \\ iii) & p(a,\,0) = p_0(a) \end{cases}$$

where we have added the initial condition (iii).

The system (2.5) is the basic model which describes the evolution of a single population under the phenomenological conditions specified at the beginning of this chapter. Below we list the assumptions that the basic function $\beta(\cdot)$ and $\mu(\cdot)$ are supposed to fulfill in order to be biologically significant and to allow the mathematical treatment of (2.5).

(2.6) $\beta(\cdot)$ *is non-negative and belongs to* $L^\infty(0,\,a_\dagger)$

(2.7) $\mu(\cdot)$ *is non-negative and belongs to* $L^1_{loc}\,([0,\,a_\dagger))$,

(2.8) $\displaystyle\int_0^{a_\dagger} \mu(\sigma)d\sigma = +\infty$

(2.9) $p_0 \in L^1(0,\,a_\dagger)$, $\quad p_0(a) \geqslant 0$ *a.e. in* $[0,\,a_\dagger]$

Here $a_\dagger$ is the maximum age an individual of the population may reach and, as already noted, we assume $a_\dagger < +\infty$. Condition (2.8) is necessary for the survival probability $\Pi(a)$ to vanish at the age $a_\dagger$.

The treatment of problem (2.5), under these assumptions, will be developed through the subsequent sections; actually, instead of treating (2.5) directly, it will be transformed into a Volterra integral equation which is derived in the next section.

3 The renewal equation

We now derive a different formulation of problem (2.5). To this purpose we set:

(3.1) $\qquad q(a,\,t) = e^{\int_0^a \mu(\sigma)d\sigma}\,p(a,t)$

This new variable satisfies:

$$
(3.2)\qquad \begin{cases} i) & q_t(a, t) + q_a(a, t) = 0 \\ ii) & q(0, t) = B(t) \\ iii) & q(a, 0) = e^{\int_0^a \mu(\sigma)d\sigma} p_0(a) = q_0(a) \end{cases}
$$

If we assume $B(t)$ is given, q can be viewed as the solution of the first order partial differential equation (3.2, i) in the strip $\{a \in [0, a_\dagger],\ t \geqslant 0\}$ with the boundary conditions (3.2, ii) and (3.2, iii) on the halfline $\{a = 0, t > 0\}$ and on $\{a \in [0, a_\dagger],\ t = 0\}$, respectively. Thus q has the form:

$$
q(a, t) = \phi(a - t)
$$

where ϕ is determined by the boundary conditions; actually we have

$$
q(a, t) = \begin{cases} q_0(a - t) & \text{if } a \geqslant t \\ B(t - a) & \text{if } a < t \end{cases}
$$

which in turn, via (3.1), provides the following formula for $p(a, t)$:

$$
(3.3)\qquad p(a, t) = \begin{cases} p_0(a - t)\ \dfrac{\Pi(a)}{\Pi(a - t)} & \text{if } a \geqslant t \\ B(t - a)\ \Pi(a) & \text{if } a < t \end{cases}
$$

Now, formula (3.3) allows us to get an equation for the birth rate $B(t)$. In fact, plugging (3.3) into (2.5, ii) (see (2.3)) we have, for $t \leqslant a_\dagger$:

$$
B(t) = \int_0^{a_\dagger} \beta(a)p(a, t)da = \int_0^t \beta(a)\Pi(a)B(t - a)da + \\ + \int_t^{a_\dagger} \beta(a)\ \frac{\Pi(a)}{\Pi(a - t)}\ p_0(a - t)da
$$

and, for $t > a_\dagger$:

$$
B(t) = \int_0^{a_\dagger} \beta(a)\Pi(a)B(t - a)da
$$

Thus $B(t)$ satisfies the following Volterra integral equation of the second kind:

$$B(t) = F(t) + \int_0^t K(t-s)B(s)ds \tag{3.4}$$

with:

$$F(t) = \int_t^\infty \beta(a) \frac{\Pi(a)}{\Pi(a-t)} p_0(a-t)da = \tag{3.5}$$

$$= \int_0^\infty \beta(a+t) \frac{\Pi(a+t)}{\Pi(a)} p_0(a)da$$

$$K(t) = \beta(t)\Pi(t) \tag{3.6}$$

where $t \geqslant 0$, and the functions β, Π, p_0 are extended by zero outside the interval $[0, a_\dagger]$.

Equation (3.4) is known as the *renewal equation* and also as the *Lotka equation*; we see that the kernel $K(t)$ is the maternity function defined in (1.5). Our procedure above shows that, albeit only formally, (3.4) is equivalent to problem (2.5); actually (3.4) is the main tool to investigate this problem, the connection being provided by (3.5) and (3.6) together with formula (3.3). The following proposition states some properties of (3.4) on the basis of the assumptions (2.6)-(2.9).

Proposition 3.1. *Let (2.6)-(2.9) be satisfied, then:*

$$K(t) \geqslant 0 \text{ a.e.}, \quad K(t) = 0 \text{ for } t > a_\dagger, \quad K \in L^1(\mathbf{R}_+) \cap L^\infty(\mathbf{R}_+) \tag{3.7}$$

$$F(t) \geqslant 0, \quad F(t) = 0 \text{ for } t > a_\dagger, \quad F \in C(\mathbf{R}_+) \tag{3.8}$$

If moreover

$$p_0 \in W^{1,1}(0, a_\dagger) \text{ and } \mu(\cdot)p_0(\cdot) \in L^1(0, a_\dagger) \tag{3.9}$$

then $F \in W^{1,\infty}(\mathbf{R}_+)$.

Proof:

(3.7) and the first part of (3.8) are obvious. To prove that $F \in C(\mathbf{R}_+)$ take $t_0 \geqslant 0$ then we have

$$F(t) = \int_t^{\infty} \beta(a) \frac{\Pi(a)}{\Pi(a-t)} (p_0(a-t) - p_0(a-t_0))\, da +$$

$$+ \int_t^{\infty} \beta(a) \frac{\Pi(a)}{\Pi(a-t)} p_0(a-t_0) da$$

where, since $p_0 \in L^1(\mathbf{R})$:

$$\left| \int_t^{\infty} \beta(a) \frac{\Pi(a)}{\Pi(a-t)} (p_0(a-t) - p_0(a-t_0))\, da \right| \leqslant$$
$$\leqslant |\beta|_{L^\infty} \int_0^{\infty} |p_0(a-t) - p_0(a-t_0)|\, da \to 0$$

as $t \to t_0$, so that:

$$\lim_{t \to t_0} F(t) = \int_{t_0}^{\infty} \beta(a) \frac{\Pi(a)}{\Pi(a-t_0)} p_0(a-t_0) da = F(t_0)$$

Finally, in a similar way we can prove that the conditions in (3.9) imply $F \in W^{1,\infty}(\mathbf{R}_+)$. ■

4 Analysis of the Lotka-McKendrick equation

Now we study problem (2.5) by considering the renewal equation (3.4) with (3.5) and (3.6). First we have the following theorem which is actually part of the standard theory of Volterra equations (see Appendix II): here we give the proof of the theorem for the sake of completeness:

Theorem 4.1. *Let (2.6)-(2.9) be satisfied, then equation (3.4), with (3.5) and (3.6), has a unique solution $B \in C(\mathbf{R}_+)$ such that $B(t) \geqslant 0$ for all t. If in addition p_0 satisfies (3.9), then $B \in W^{1,\infty}_{loc}(\mathbf{R}_+)$ and:*

(4.1) $$B'(t) = F'(t) + K(t)B(0) + \int_0^t K(t-s)B'(s)ds$$

Proof:

Assume first that

(4.2) $$|K|_{L^1(R_+)} = \int_0^{\infty} K(s)ds < 1$$

Then the solution of (3.4) is obtained via the standard iteration procedure

$$(4.3)\qquad \begin{cases} B^0(t) = F(t) \\ B^{k+1}(t) = F(t) + \int_0^t K(t-s)B^k(s)ds. \end{cases}$$

In fact, take any $T > 0$, then by (3.7) and (3.8) we have $B^k \in C([0, T])$ and $B^k(t) \geqslant 0$; moreover

$$|B^{k+1}(t) - B^k(t)| \leqslant \int_0^t K(t-s)|\, B^k(s) - B^{k-1}(s)|\, ds$$

and

$$|B^{k+1} - B^k|_{C([0,T])} \leqslant |K|_{L^1(R_+)}\, |B^k - B^{k-1}|_{C([0,T])}.$$

Thus by (4.2) the sequence $B^k(t)$ converges, uniformly on $[0, T]$, to a solution $B(t)$ of (3.4), such that $B \in C([0, T])$ and $B(t) \geqslant 0$.

Concerning uniqueness of this solution we see that if $B(t)$ and $\bar{B}(t)$ are two solutions of (3.4) we must have

$$|B - \bar{B}|_{C([0,T])} \leqslant |K|_{L^1(R_+)}\, |B - \bar{B}|_{C([0,T])}$$

so that, by (4.2), $B(t) = \bar{B}(t)$.

If in addition p_0 satisfies (3.9) then, by Proposition 3.1 and (4.3), we have $B^k \in W^{1,\infty}(\mathbf{R}_+)$, and, setting:

$$V^k(t) = \frac{d}{dt} B^k(t) \ a.e.,$$

we have $V^k \in L^\infty(\mathbf{R}_+)$ and

$$(4.4)\qquad V^{k+1}(t) = F'(t) + K(t)F(0) + \int_0^t K(t-s)V^k(s)ds,$$

which yields

$$|V^{k+1} - V^k|_{L^\infty(R_+)} \leqslant |K|_{L^1(R_+)}\, |V^k - V^{k-1}|_{L^\infty(R_+)}.$$

Thus, again by (4.2), the sequence V^k converges in $L^\infty(\mathbf{R}_+)$ to $V(t) = \frac{d}{dt}B(t)$ *a.e.* Of course (4.1) follows from (4.4).

Finally, if (4.2) is not fulfilled, take α such that

$$\int_0^\infty e^{-\alpha t}K(t)dt < 1$$

Setting $\tilde{B} = e^{-\alpha t}B(t)$, $\tilde{F}(t) = e^{-\alpha t}F(t)$, $\tilde{K}(t) = e^{-\alpha t}K(t)$ equation (3.4) is transformed into the equivalent one:

$$\tilde{B}(t) = \tilde{F}(t) + \int_0^t \tilde{K}(t-s)\tilde{B}(s)ds$$

which, because $\tilde{K}(t)$ satisfies (4.2), can be solved with the previous argument. ■

The preceding theorem allows us to state results for problem (2.5) via formula (3.3). In fact we have:

Theorem 4.2. *Let (2.6)-(2.9) and (3.9) be satisfied. Assume also that:*

(4.5) $$p_0(0) = \int_0^\infty \beta(a)p_0(a)da$$

and let $p(a, t)$ be defined by (3.3) where $B(t)$ is the solution of (3.4)-(3.6). Then:

(4.6) $$p \in C([0, a_\dagger] \times \mathbf{R}_+), \quad p(a,t) \geq 0, \quad \mu(\cdot)p(\cdot,t) \in L^1(0, a_\dagger) \quad \forall\, t > 0$$

(4.7) $$\frac{\partial p}{\partial t}(a,t), \ \frac{\partial p}{\partial a}(a,t) \text{ exist a.e. in } [0, a_\dagger] \times [0, +\infty]$$

and problem (2.5) is satisfied. Moreover $p(a, t)$ is the only solution in the sense of (4.6), (4.7).

Proof:

The proof of (4.6)-(4.7) is quite straightforward and follows from the properties of $B(t)$, stated in theorem 4.1. We only note the following inequality concerning the last part of (4.6):

$$\int_0^{a_\dagger} \mu(a)p(a, t)da =$$

$$\int_0^{t\wedge a_\dagger} \mu(a)B(t-a)\Pi(a)\, da + \int_{t\wedge a_\dagger}^{a_\dagger} \mu(a)p_0(a-t)\ \frac{\Pi(a)}{\Pi(a-t)}\ da \leqslant$$

$$\leqslant \max_{s\in[0,t]} |B(s)| \int_0^{t\wedge a_\dagger} \mu(a)\Pi(a)\, da +$$

$$+ e^{\int_0^{(t\vee a_\dagger)-t}\mu(\sigma)d\sigma} |p_0|_{C([0,a_\dagger])} \int_{t\wedge a_\dagger}^{a_\dagger} \mu(a)\Pi(a)\, da \leqslant$$

$$\leqslant \max_{s\in[0,t]} |B(s)| + e^{\int_0^{(t\vee a_\dagger)-t}\mu(\sigma)d\sigma} |p_0|_{C([0,a_\dagger])}$$

and stress the fact that (4.5) is intended to guarantee the continuity of $p(a, t)$ through the line $a = t$; in fact

$$B(0) = \int_0^\infty \beta(a)p_0(a)da = p_0(0).$$

As far as uniqueness is concerned, we have already seen (formally, but now the procedure can be repeated rigorously) that a solution of (2.5) must be of the form (3.3) with $B(t)$ satisfying (3.4)-(3.6): the uniqueness of the solution of this latter problem yields uniqueness for (2.5). ■

We have seen that, with the assumptions (3.9) and (4.5), formula (3.3) provides us a solution which we can call classical; actually, this formula is meaningful even if these conditions are not satisfied; in fact (3.9) is enough to provide a solution in the sense stated in the following:

Theorem 4.3. *Let (2.6)-(2.9) be satisfied, then $p(a, t)$, defined by (3.3), has the following properties:*

(4.8) $\quad p(\cdot, t) \in C([0, T]; L^1(0, a_\dagger)),\ p(a, t) \geqslant 0$ *a.e. in* $[0, a_\dagger] \times \mathbf{R}_+$,

(4.9) $\quad |p(\cdot, t)|_{L^1} \leqslant e^{t|\beta|_{L^\infty}} |p_0|_{L^1}$,

(4.10) $\quad p(a, t)$ *is continuous for* $a < t$ *and satisfies (2.5, ii) for* $t > 0$,

(4.11) $\quad \lim_{h\to 0} \frac{1}{h} [p(a+h,t+h)-p(a,t)] = -\mu(a)p(a,t)$ *a.e. in* $[0, a_\dagger]\times\mathbf{R}_+$

Proof:

Let us prove (4.9) first. From (3.5) we have:

$$F(t) \leqslant |\beta|_{L^\infty} |p_0|_{L^1}, \quad K(t) \leqslant |\beta|_{L^\infty};$$

then, from (3.4):

$$B(t) \leqslant |\beta|_{L^\infty} |p_0|_{L^1} + |\beta|_{L^\infty} \int_0^t B(s)ds.$$

Thus, by Gronwall's inequality

$$B(t) \leqslant |\beta|_{L^\infty} e^{t|\beta|_{L^\infty}} |p_0|_{L^1}. \tag{4.12}$$

From this estimate, formula (3.3) yields:

$$|p(\cdot, t)|_{L^1} = \int_0^t B(t-a)\Pi(a)da + \int_0^\infty \frac{\Pi(a+t)}{\Pi(a)} p_0(a)da \leqslant$$

$$\leqslant \left(|\beta|_{L^\infty} \int_0^t e^{(t-a)|\beta|_{L^\infty}} da + 1 \right) |p_0|_{L^1} =$$

$$= e^{t|\beta|_{L^\infty}} |p_0|_{L^1}$$

Now (4.8) follows easily from (4.9); in fact, for a given $p_0 \in L^1(0, a_\dagger)$, let p_0^n be a sequence such that:

$$p_0^n \textit{ satisfy } (3.9) \textit{ and } (4.5), \quad \lim_{n \to +\infty} |p_0^n - p_0|_{L^1} = 0,$$

and let p^n be the solution of (2.5) corresponding to p_0^n. Thus $p^n \in C([0, T]; L^1(0, a_\dagger))$ and, by (4.9) and linearity, we have:

$$|p^n(\cdot, t) - p(\cdot, t)|_{L^1} \leqslant e^{t|\beta|_{L^\infty}} |p_0^n - p_0|_{L^1}$$

so that p is the limit of the sequence p^n in the space $C([0, T]; L^1(0, a_\dagger))$ i.e. (4.8) is true.

Finally, (4.10) and (4.11) are straightforward. ■

The previous theorem shows that even when the initial datum p_0 is not regular the solution $p(a, t)$ still has some regularity. We also note that the estimate

(4.9) provides continuity of the solution p with respect to the initial datum p_0 (hence wellposedness), in the norm of the space $L^1(0, a_\dagger)$: this is a main feature of the problem and it is in agreement with the biological meaning of the population density $p(a, t)$.

5 The asymptotic behaviour

Here we investigate the asymptotic behaviour of the birth rate $B(t)$, i.e. we discuss the asymptotic behaviour of the solution of the renewal equation (3.4)-(3.6): again any result on $B(t)$ can be transfered to $p(a, t)$, via formula (3.3).

First we note that, by (4.12), $B(t)$ is absolutely Laplace transformable and

(5.1) $$\hat{B}(\lambda) = \frac{\hat{F}(\lambda)}{1 - \hat{K}(\lambda)} = \hat{F}(\lambda) + \frac{\hat{F}(\lambda)\hat{K}(\lambda)}{1 - \hat{K}(\lambda)}$$

were $\hat{f}(\lambda)$ denotes the Laplace transform of $f(t)$.

Thus we can use classical Laplace transform techniques which relate the asymptotic behaviour of $B(t)$ to the singularities of $\hat{B}(\lambda)$ (see Appendix I). Since $F(t)$ and $K(t)$ vanish for $t > a_\dagger$, their transforms $\hat{F}(\lambda)$ and $\hat{K}(\lambda)$ are entire analytical functions of λ; then by (5.1) $\hat{B}(\lambda)$ can only have poles which have to be found among the roots of the equation:

(5.2) $$\hat{K}(\lambda) = 1$$

With respect to this latter equation we have:

Theorem 5.1. *Equation (5.2) has one and only one real solution α^* which is a simple root. $\alpha^* < 0$ if and only if $\int_0^\infty K(t)dt < 1$. Any other solution α of (5.2) is such that $\Re\alpha < \alpha^*$. Within any strip $\sigma_1 < \Re\lambda < \sigma_2$ there is at most a finite number of roots.*

Proof:

Consider the real function:

(5.3) $$x \rightarrow \hat{K}(x) = \int_0^\infty e^{-xt}K(t)dt, \quad x \in \mathbf{R}$$

which, since $K(t) \geqslant 0$, is strictly decreasing and such that:

$$\lim_{x \to -\infty} \hat{K}(x) = +\infty\,, \quad \lim_{x \to +\infty} \hat{K}(x) = 0.$$

Then there is one and only one real solution α^* of (5.2) and, since

$$\frac{d}{dx}\hat{K}(x)|_{x=\alpha^*} = -\int_0^\infty te^{-\alpha^* t}K(t)dt > 0\,,$$

α^* is simple. Of course $\alpha^* < 0$ if and only if $\hat{K}(0) = \int_0^\infty K(t)dt < 1$.

Let α be a solution different from α^*, then:

$$\int_0^\infty e^{-\alpha^* t}K(t)dt = 1 = \Re\left(\int_0^\infty e^{-\alpha t}K(t)dt\right) =$$

$$= \int_0^\infty e^{-\Re\alpha t}\cos(\Im\alpha\; t)K(t)dt < \int_0^\infty e^{-\Re\alpha t}K(t)dt$$

so that, since (5.3) is strictly decreasing, it follows that $\Re\alpha < \alpha^*$.

Finally, since $\hat{K}(\lambda) \to 0$ as $|\lambda| \to +\infty$, all the roots within the strip $\sigma_1 < \Re\lambda < \sigma_2$ must lie in some bounded subset and be finite in number because otherwise $\hat{K}(\lambda)$ would vanish identically. ■

Now we are ready to state:

Theorem 5.2. *Let p_0 satisfy (2.9) and let α^* be defined in the previous theorem. Then*

$$B(t) = b_0 e^{\alpha^* t}\,(1 + \Omega(t)) \tag{5.4}$$

where:

$$b_0 \geqslant 0 \quad \text{and} \quad \lim_{t \to +\infty} \Omega(t) = 0$$

Proof:

We first consider the last term in (5.1). We have:

$$\lim_{\substack{|\lambda| \to +\infty \\ \Re\lambda > \delta}} \frac{\hat{F}(\lambda)\hat{K}(\lambda)}{1 - \hat{K}(\lambda)} = 0 \tag{5.5}$$

$$(5.6) \qquad \int_{-\infty}^{+\infty} \left| \frac{\hat{F}(\sigma + iy)\hat{K}(\sigma + iy)}{1 - \hat{K}(\sigma + iy)} \right| dy < +\infty$$

where $\delta \in \mathbf{R}$ is arbitrarily chosen and $\sigma \in \mathbf{R}$ is such that the line $\Re\lambda = \sigma$ does not meet any root of (5.2). Condition (5.5) holds because:

$$(5.7) \qquad \lim_{|\lambda| \to +\infty} \hat{K}(\lambda) = \lim_{|\lambda| \to +\infty} \hat{F}(\lambda) = 0$$

in any half plane $\Re\lambda > \delta$. Concerning (5.6), we first note that (5.7) also implies:

$$m_\sigma = \inf_{y \in \mathbf{R}} |1 - \hat{K}(\sigma + iy)| > 0$$

Furthermore, defining the functions

$$f_\sigma(t) = e^{-\sigma t} \overset{\circ}{F}(t) \quad \forall\, t \geq 0; \quad f_\sigma(t) = 0 \quad \forall\, t < 0,$$

and

$$g_\sigma(t) = e^{-\sigma t} K(t) \quad \forall\, t \geq 0; \quad g_\sigma(t) = 0 \quad \forall\, t < 0,$$

we see that, since they vanish outside of $[0, a_+]$, their Fourier transforms $f_\sigma^*(y)$, $g_\sigma^*(y)$ belong to $L^2(\mathbf{R})$ and also

$$f_\sigma^*(y) = \hat{F}(\sigma + iy), \; g_\sigma^*(y) = \hat{K}(\sigma + iy).$$

Thus:

$$(5.8) \qquad \left| \frac{\hat{F}(\sigma + iy)\hat{K}(\sigma + iy)}{1 - \hat{K}(\sigma + iy)} \right| \leqslant \frac{1}{m_\sigma} |f_\sigma^*(y) g_\sigma^*(y)|$$

and (5.6) is satisfied.

Now we take $\sigma > \alpha^*$ and consider the function:

$$(5.9) \qquad H(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\hat{F}(\lambda)\hat{K}(\lambda)}{1 - \hat{K}(\lambda)} e^{\lambda t}\, d\lambda$$

which, by (5.5) and (5.6) is well defined and has Laplace transform $\hat{H}(\lambda) = \dfrac{\hat{F}(\lambda)\hat{K}(\lambda)}{1 - \hat{K}(\lambda)}$; consequently, by (5.1), we have

(5.10) $$B(t) = F(t) + H(t)\ .$$

Finally, we consider $\sigma_1 < \alpha^*$ such that any root of (5.2), other than α^*, lies strictly on the left of the line $\Re\lambda = \sigma_1$. By (5.5) and (5.6) we shift the integration abscissa in (5.9), from σ to σ_1 (see Appendix I); this yields:

(5.11) $$H(t) = e^{\alpha^* t}\,(b_0 + \Omega_0(t)),$$

where

(5.12) $$b_0 = Res\left[\frac{\hat{F}(\lambda)\hat{K}(\lambda)}{1 - \hat{K}(\lambda)}\right]_{\lambda=\alpha^*} = \frac{\int_0^\infty e^{-\alpha^* t}F(t)dt}{\int_0^\infty te^{-\alpha^* t}K(t)dt}$$

and

(5.13) $$|\Omega_0(t)| = \frac{e^{-\alpha^* t}}{2\pi}\left|\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\frac{\hat{F}(\lambda)\hat{K}(\lambda)}{1 - \hat{K}(\lambda)}\,e^{\lambda t}\,d\lambda\right| \leqslant$$
$$\leqslant \frac{e^{-(\alpha^*-\sigma_1)t}}{m_{\sigma_1}}\,|f^*_{\sigma_1}|_{L^2(R)}\,|g^*_{\sigma_1}|_{L^2(R)}$$

We see that $b_0 = 0$ if and only if $F(t) = 0$ for all $t \geqslant 0$, but in this case the only solution to (3.4) is the trivial one $B(t) \equiv 0$. On the other hand if $b_0 > 0$, by (5.10) and (5.11):

$$B(t) = b_0 e^{\alpha^* t}\left(1 + \frac{e^{-\alpha^* t}F(t)}{b_0} + \frac{1}{b_0}\,\Omega_0(t)\right)$$

and (5.4) is proved. ■

A few comments on b_0 are now in order. First we want to interpret the case $b_0 = 0$: we have seen, in the proof of the theorem, that this case occurs if and only if $F(t) \equiv 0$, that is, if and only if

$$\int_0^\infty \beta(a + t)p_0(a)\,\frac{\Pi(a + t)}{\Pi(a)}\,da = 0 \quad \forall\, t \geqslant 0$$

and, consequently, if and only if, for all $t \geqslant 0$,

(5.14) $$\beta(a + t)p_0(a) = 0 \quad a.e.\ on\ [0, a_\dagger].$$

Now (5.14) occurs if and only if the support of $\beta(\cdot)$ lies to the left of the support of p_0, that is when all of the initial individuals are too old to be able to become fertile. In this case we have

$$(5.15) \qquad p(a,t) = \begin{cases} p_0(a-t)\dfrac{\Pi(a)}{\Pi(a-t)} & \text{if } a \geq t \\ 0 & \text{if } a < t \end{cases}$$

We see that the behaviour of $p(a, t)$ can be trivial even if the initial datum p_0 is not identically vanishing. An initial datum that does not satisfy condition (5.14) is called a *non-trivial datum*.

Another remark follows from (5.13); in fact, since

$$F(t) = \int_0^\infty \beta(a+t)p_0(a)\,\frac{\Pi(a+t)}{\Pi(a)}\,da \leq |\beta|_{L^\infty}\,|p_0|_{L^1}\,,$$

it follows that:

$$(5.16) \qquad b_0 \leq M_0\,|p_0|_{L^1}\,,\ |f^*_{0}|_{L^2(R)} \leq M_0\,|p_0|_{L^1}\,,$$

where M_0 is a constant independent of p_0. Thus the estimate (4.12) can be improved to:

$$(5.17) \qquad |p(\cdot,t)|_{L^1} \leq M\,e^{\alpha^* t}\,|p_0|_{L^1}\,,$$

where M is a constant independent of p_0. Actually (5.17) follows from (5.16) because, for $t > a_\dagger$

$$p(a,t) = e^{\alpha^*(t-a)}\,(b_0 + \Omega_0(t-a))\,\Pi(a)\,.$$

We also note that formula (5.4) implies:

(5.18) *$B(t)$ is either identically vanishing or eventually positive.*

Thus we can state:

Proposition 5.3. *Let $p(a, t)$ be the solution of (2.5) under the assumption of Theorem 4.2 and let $b_0 > 0$ in (5.4), then*

$$P(t) = \int_0^{a_\dagger} p(a,t)\,da > 0 \quad \textit{for all} \quad t \geq 0$$

Proof:

We first note that if we set $q(a, t) = p(a, t + t_0)$, $t \geq 0$, then $q(a, t)$ is the solution of the same problem (2.5) with the initial condition:

$$q(a, 0) = p(a, t_0)$$

Thus we can argue by contradiction. In fact if $P(t_0) = 0$ for some t_0 then

$$p(a, t_0) = 0 \qquad a.e. \ on \quad [0, a_\dagger]$$

and consequently, for $t \geq t_0$

$$p(a, t) = 0 \qquad a.e. \ on \quad [0, a_\dagger]$$

and this implies

$$P(t) = 0 \qquad for \ t \geq t_0 \ .$$

Now, if $b_0 > 0$, by (5.18) and formula (5.4) we have that $P(t)$ is eventually positive, so that it is impossible that $P(t)$ vanishes at t_0. ■

Before we end this section we go back to equation (5.2) to discuss the meaning of α^*. Actually, this equation is called the *Lotka characteristic equation* and α^* the *intrinsic Malthusian parameter*: they determine the growth of the population through the birth rate $B(t)$, whose behavior is given in Theorem 5.2, and are related to the net reproduction rate defined in (1.5) by the following statement:

$$\begin{array}{lll} R > 1 & \text{if and only if} & \alpha^* > 0 \,, \\ R = 1 & \text{if and only if} & \alpha^* = 0 \,, \\ R < 1 & \text{if and only if} & \alpha^* < 0 \,. \end{array} \tag{5.19}$$

This is part of Theorem 5.1 because $R = \hat{K}(0)$: thus the natural connection between the two parameters R and α^* is made precise.

6 Comments and references

Though it is nice to go far into the past, looking for vestiges of the concepts and ideas which we are interested in, the origin of the theory we have presented in this chapter has to be dated many centuries after the rabbits of Fibonacci (see also [6] and [36]). In fact this theory starts in 1911 with the work of Alfred Lotka ([87], [74]-[76]) and with the paper by McKendrick [78] where the system is formulated in the form that we have presented and its connection with the renewal equation is stressed. Actually some important aspects of the theory are part of the theory of Volterra integral equations and have been clarified somewhat later by Feller [37].

This linear description of the growth of an age structured population is a basic mathematical tool in demography and has been largely used since 1911. Monographs like Coale [21], Impagliazzo [61], Keyfitz [68], contain the fundamentals of the theory, also formulated in a discrete age-time setting: in these texts the reader can find a rich documentation of the application of the theory to demographic data.

More attention to the theory has been paid starting in 1974, when age structure was recognized as a fundamental aspect in the context of population ecology (see [46], [57], [89]). Since then the mathematical tools have been also developed and the results are nowadays set in a functional analytic framework within which nonlinear problems can also be approached throught the methods of abstract evolution equations ([18]-[20], [29], [99]-[101]).

II
Further developments of the linear theory

This chapter is devoted to some developments of the linear theory; namely we discuss a few aspects of the model introduced in the previous chapter so as to treat some basic questions that are relevant in the description of changing populations.

In Section 1 we are concerned with the relation between the overall growth of the population and the evolution of how individuals are distributed through the age classes: this allows a comparison with the simple Malthus model which disregards age structure. Later, in Section 2 we modify the model in order to allow the vital rates to change in time and we examine the asymptotic behavior in the new situation so that we can deal with the special question of ergodicity (Section 3). Finally in Section 4 we treat the case of inifinite maximum age $a_{\dagger} = +\infty$.

1 The age profile

In this section we point out some of the features of problem (I.2.5) in order to give a complementary interpretation of the asymptotic behaviour of the solution. We consider the following variables by which we will describe the evolution of the population:

$$\text{(1.1)} \qquad \textit{age profile: } \omega(a, t) = \frac{p(a, t)}{P(t)} ,$$

$$\text{(1.2)} \qquad \textit{total population: } P(t) = \int_0^{a_{\dagger}} p(a, t)da .$$

The previous description is recovered by the formula

$$(1.3) \qquad p(a,\ t) = P(t)\omega(a,\ t)\ .$$

Proceeding formally, by (1.1) and (I.2.5) we get

$$(1.4) \qquad \omega_t(a,\ t) + \omega_a(a,\ t) + \mu(a)\omega(a,\ t) + \omega(a,\ t)\ \frac{1}{P(t)}\ \frac{d}{dt}P(t) = 0$$

and also, by (I.2.5)

$$\frac{d}{dt}P(t) = \int_0^{a_\dagger} p_t(a,\ t)da = -\int_0^{a_\dagger} p_a(a,\ t)da - \int_0^{a_\dagger} \mu(a)p(a,\ t)da =$$

$$(1.5) \qquad = -p(a_\dagger,\ t) + p(0,\ t) - \int_0^{a_\dagger} \mu(a)p(a,\ t)da =$$

$$= \int_0^{a_\dagger} [\beta(a)-\mu(a)]p(a,t)da = P(t)\int_0^{a_\dagger} [\beta(a)-\mu(a)]\omega(a,t)da$$

where we have used $p(a_\dagger,\ t) = 0$. Thus putting (1.4) and (1.5) together, and by the definition of $P(t)$ and $\omega(a,\ t)$ itself, we arrive at the following two sets of equations:

$$(1.6) \qquad \begin{cases} \omega_t(a,\ t) + \omega_a(a,\ t) + \mu(a)\omega(a,\ t) + \\ \quad + \omega(a,\ t)\displaystyle\int_0^{a_\dagger} [\beta(\sigma) - \mu(\sigma)]\omega(\sigma,\ t)d\sigma = 0 \\ \omega(0,\ t) = \displaystyle\int_0^{a_\dagger} \beta(a)\omega(a,\ t)da; \quad \int_0^{a_\dagger} \omega(a,\ t)da = 1 \\ \omega(a,\ 0) = \omega_0(a) \end{cases}$$

$$(1.7) \qquad \begin{cases} \dfrac{d}{dt}P(t) = \alpha(t)P(t) \\ P(0) = P_0 \end{cases}$$

where:

$$\omega_0(a) = \frac{p_0(a)}{\int_0^{a_\dagger} p_0(\sigma)d\sigma}, \quad P_0 = \int_0^{a_\dagger} p_0(\sigma)d\sigma$$

and

$$\alpha(t) = \int_0^{a_\dagger} [\beta(\sigma) - \mu(\sigma)]\omega(\sigma, t)d\sigma .$$

We see that the age profile $\omega(a, t)$ satisfies an equation of its own which is not coupled with the other variable $P(t)$, so its evolution is determined only by the initial age profile $\omega_0(a)$. Once that the evolution of the age profile is known, we can find the behaviour of the total population $P(t)$ which is influenced through the coefficient $\alpha(t)$ in equation (1.7); $\alpha(t)$ can be viewed as a transient Malthusian coefficient.

It is interesting to treat problem (1.6) in itself, also in view of some developments that we will treat in Chapter V; clearly any result on problem (1.6) is strictly dependent on the previous theorems on problem (I.2.5). First we have to rule out *trivial initial profiles*, that is those ω_0 that, having support beyond the maximum reproductive age, satisfy the following condition (see I.5.14)

$$\beta(a + t)\omega_0(a) = 0 \quad \text{a.e. for } a \in [0, a_\dagger] \quad \text{and for all} \quad t \geq 0 .$$

Then we have

Theorem 1.1 *Let ω_0 be non-trivial and such that:*

$$(1.8) \qquad \begin{cases} \omega_0 \in W^{1,1}(0, a_\dagger), \quad \mu(\cdot)\omega_0(\cdot) \in L^1(0, a_\dagger) \\ \omega_0(a) \geq 0, \ \omega_0(0) = \int_0^{a_\dagger} \beta(a)\omega_0(a)da, \ \int_0^{a_\dagger} \omega_0(a)da = 1 \end{cases}$$

Then there exists one and only one $\omega \in C([0, a_\dagger] \times \mathbf{R}_+)$ such that

$$(1.9) \qquad \begin{cases} \omega(a, t) \geq 0, \ \int_0^{a_\dagger} \mu(a)\omega(a, t)da < +\infty, \ \int_0^{a_\dagger} \omega(a, t)da = 1 \\ \dfrac{\partial\omega}{\partial t}(a, t), \dfrac{\partial\omega}{\partial a}(a, t) \text{ exist a.e. in } [0, a_\dagger] \times \mathbf{R}_+ \end{cases}$$

and problem (1.6) is satisfied.

Proof:

Consider problem (I.2.5) with $p_0 = \omega_0$ and let $q(a, t)$ be its solution given by Theorem I.4.2, then set

$$(1.10) \qquad \omega(a, t) = \frac{q(a, t)}{\int_0^{a_\dagger} q(\sigma, t)d\sigma}$$

where the denominator does not vanish by Proposition I.5.3: it is easy to check that this is the solution we are looking for. On the other hand, let ω be a solution of (1.6) (in the sense specified in (1.9)) and set

$$(1.11) \qquad \begin{cases} \alpha(t) = \int_0^{a_\dagger} [\beta(\sigma) - \mu(\sigma)]\omega(\sigma, t)d\sigma \\ q(a, t) = \omega(a, t)e^{\int_0^t \alpha(s)ds} \end{cases}$$

where $\alpha(t)$ is meaningful by (1.9). It is easy to show that q solves problem (I.2.5) with $p_0 = \omega_0$ and consequently it is uniquely determined because the solution of this problem is unique. Moreover, since by (1.11):

$$\int_0^{a_\dagger} q(\sigma, t)d\sigma = e^{\int_0^t \alpha(s)ds},$$

then also $e^{\int_0^t \alpha(s)ds}$ is uniquely determined, consequently $\omega(a, t)$ is uniquely determined by:

$$\omega(a, t) = q(a, t)e^{-\int_0^t \alpha(s)ds}$$

and the proof is complete.

■

Since (I.2.5) and (1.6) are strictly related we expect that the asymtpotic behaviour of ω can be obtained from the previous result on p; however, we start by considering the stationary problem relative to (1.6):

$$
(1.12) \quad \begin{cases} i) & \omega_a(a)+\mu(a)\omega(a)+\omega(a)\displaystyle\int_0^{a_\dagger}[\beta(\sigma)-\mu(\sigma)]\omega(\sigma)d\sigma = 0 \\ ii) & \omega(0) = \displaystyle\int_0^{a_\dagger}\beta(\sigma)\omega(\sigma)d\sigma \\ iii) & \displaystyle\int_0^{a_\dagger}\omega(\sigma)d\sigma = 1 . \end{cases}
$$

This problem has a unique non trivial solution whose form can be determined as follows. Let $\omega^*(a)$ be a solution of (1.12) and set

$$
\lambda = \int_0^{a_\dagger}[\beta(\sigma)-\mu(\sigma)]\omega^*(\sigma)d\sigma
$$

then by (1.12, i) and (1.12, iii)

$$
\omega^*(a) = \frac{e^{-\lambda a}\Pi(a)}{\displaystyle\int_0^{a_\dagger} e^{-\lambda\sigma}\Pi(\sigma)d\sigma.}
$$

and, since $\omega^*(a)$ must satisfy (1.12, ii) we get the following condition on λ:

$$
1 = \int_0^{a_\dagger} e^{-\lambda\sigma}\beta(\sigma)\Pi(\sigma)d\sigma
$$

This is exactly the Lotka equation (I.5.2) that we have already dealt with; thus it must be $\lambda = \alpha^*$ and a possible solution of (1.12) must be:

$$
(1.13) \qquad \omega^*(a) = \frac{e^{-\alpha^* a}\Pi(a)}{\displaystyle\int_0^{a_\dagger} e^{-\alpha^*\sigma}\Pi(\sigma)d\sigma}
$$

On the other hand, $\omega^*(a)$, as defined in (1.13), is a solution of (1.12): this fact is easy to check, we merely need to note that

$$\int_0^{a_\dagger} [\beta(a) - \mu(a)]\omega^*(a)da =$$

$$= \left[\int_0^{a_\dagger} \beta(a)e^{-\alpha^* a}\Pi(a)da - \int_0^{a_\dagger} \mu(\sigma)e^{-\alpha^* a}\Pi(\alpha)da \right] / \int_0^{a_\dagger} e^{-\alpha^* \sigma}\Pi(\sigma)d\sigma =$$

$$(1.14) \quad = \left[1 + \int_0^{a_\dagger} e^{-\alpha^* a} \frac{d}{da} \Pi(a)da \right] / \int_0^{a_\dagger} e^{-\alpha^* \sigma}\Pi(\sigma)d\sigma =$$

$$= \left[1 + [e^{-\alpha^* a}\Pi(a)]_0^{a_\dagger} + \alpha^* \int_0^{a_\dagger} e^{-\alpha^* a}\Pi(a)da \right] / \int_0^{a_\dagger} e^{-\alpha^* \sigma}\Pi(\sigma)d\sigma =$$

$$= \alpha^* .$$

Thus we have proved the following:

Theorem 1.2. *Problem (1.12) has one and only one non trivial solution given by (1.13)* ■

The stationary solution $\omega^*(a)$ is the asymptotic age profile of $\omega(a, t)$ as $t \to +\infty$, unless the initial profile ω_0 is trivial. In fact we have:

Theorem 1.3. *Let ω_0 be non trivial, then:*

$$(1.15) \quad \lim_{t \to \infty} \int_0^{a_\dagger} |\omega(a, t) - \omega^*(a)| da = 0$$

Proof:

We first recall that, by the proof of theorem 1.1., $\omega(a, t)$ is given by(1.10). Moreover, by Theorem I.5.2:

$$q(a, t) = q_0 e^{\alpha^*(t-a)}\Pi(a)(1 + \Omega(t - a)) \quad for\ t > a_\dagger$$

where $\lim_{t \to \infty} \Omega(t) = 0$ and $q_0 > 0$, since ω_0 is non trivial. Then:

$$(1.16) \qquad \omega(a, t) = \frac{e^{-\alpha^* a}\Pi(a)(1 + \Omega(t - a))}{\int_0^{a_\dagger} e^{-\alpha^* a}\Pi(a)(1 + \Omega(t - a))da} \qquad for\ t > a_\dagger$$

and (1.15) follows easily. ■

Turning to equation (1.7) we first note that in correspondence with the stationary profile $\omega^*(a)$ we have $\alpha(t) \equiv \alpha^*$ (see (1.14)) and that, again by (1.16), if ω_0 is not trivial, it is easy to prove that:

$$\lim_{t \to +\infty} \alpha(t) = \alpha^* \tag{1.17}$$

Thus, when the age-profile stays at the stationary profile $\omega^*(a)$, equation (1.7) becomes

$$\frac{d}{dt} P(t) = \alpha^* P(t); \; P(0) = P_0 \tag{1.18}$$

and the total population undergoes pure exponential growth:

$$P(t) = e^{\alpha^* t} P_0$$

Moreover, in the general case, when ω_0 is not trivial, because of (1.17) equation (1.18) plays the role of limiting equation of (1.7). Finally we note that, when the age profile is stationary, by (1.3), we have the so called *persistent solution* of (I.2.5):

$$p^*(a, t) = p_0 e^{\alpha^*(t-a)} \Pi(a) = P_0 e^{\alpha^* t} \omega^*(a) \tag{1.19}$$

2 Time dependent rates

The model that has been treated in chapter I assumes that the fertility and the mortality rates $\beta(\cdot)$ and $\mu(\cdot)$ do not vary explicitly with the time, here we want to consider an extension of the model by considering time dependent rates $\beta(a, t)$ and $\mu(a, t)$, so that we can account for environmental variations such as permanent changes in the life conditions or periodic changes due to seasonal fluctuations. We will also introduce possible migration by a function $m(a, t)$ which is assumed to be known.

With these changes in the model, (I.2.5) is modified into the following non autonomous system:

$$\begin{cases} i) \; p_t(a, t) + p_a(a, t) + \mu(a, t) p(a, t) = m(a, t) \\ ii) \; p(0, t) = \int_0^{a_+} \beta(\sigma, t) \omega(\sigma, t) p(\sigma, t) d\sigma \\ iii) \; p(a, 0) = p_0(a) \end{cases} \tag{2.1}$$

For this problem we will at least assume that for each fixed t, the functions $\beta(\cdot,t)$ and $\mu(\cdot,t)$ satisfy (I.2.6), (I.2.7), (I.2.8); further assumptions will be specified when needed. Proceeding formally, we can treat (2.1) via the same procedure followed in section I.3; this leads to the following integrated form of (2.1):

$$(2.2)\quad p(a,t) = \begin{cases} p_0(a-t)\Pi(a,t,t) + \int_0^t \Pi(a,t,\sigma)m(a-\sigma,t-\sigma)d\sigma & \text{if } a \geq t \\ p(0,t-a)\Pi(a,t,a,) + \int_0^a \Pi(a,t,\sigma)m(a-\sigma,t-\sigma)d\sigma & \text{if } a < t \end{cases}$$

where

$$(2.3)\qquad \Pi(a,\ t,\ x) = e^{-\int_0^x \mu(a-\sigma,t-\sigma)d\sigma}$$

is defined for $x \in [0,\ a \wedge t]$. We note that (2.3) can be interpreted as the probability that an individual of age $(a - x)$ at the time $(t - x)$ will survive up to time t (with age a).

Moreover, via (2.2) we get the following integral equation for the birth rate $B(t) = p(0,\ t)$

$$(2.4)\qquad B(t) = F(t) + \int_0^t K(t,\ t - s)B(s)ds$$

with:

$$(2.5)\qquad K(t,\ s) = \begin{cases} \beta(s,\ t)\Pi(s,t,s) & \text{if } 0 < s \leq t \wedge a_\dagger \\ 0 & \text{elsewhere} \end{cases}$$

and

$$(2.6)\qquad \begin{aligned} F(t) = &\int_0^\infty \beta(a + t,t)p_0(a)\Pi(a + t,\ t,\ t)da + \\ &\int_0^\infty \beta(a,\ t) \int_0^{t\wedge a} \Pi(a,\ t,\ \sigma)m(a - \sigma,t - \sigma)d\sigma da \end{aligned}$$

where the functions β, p_0 and Π are extended by zero.

Thus the study of (2.1) depends on the analysis of the non-convolution equation (2.4). We will make the following main assumptions:

$$(2.7)\qquad \begin{aligned} &\beta \in C(\mathbf{R}_+,\ L^\infty(0,\ a_\dagger)), \quad \mu \in C(\mathbf{R}_+,\ L^\infty(0,\ A)) \quad \forall A \in [0,\ a_\dagger) \\ &m \in C(\mathbf{R}_+,\ L^1(0,\ a_\dagger)), \quad p_0 \in L^1(0,\ a_\dagger) \end{aligned}$$

which guarantee existence and uniqueness of a solution. Namely we have

Theorem 2.1. *Let (2.7) be satisfied, then equation (2.4) has a unique continuous solution B(t).* ■

We do not go through the proof of this theorem which is quite similar to that given in section I.4 for equation (I.3.4), and we also omit all the comments on the solution of (2.1) which is obtained via formula (2.2). We focus instead on the problem of the asymptotic behavior of $B(t)$ which actually depend on how the vital rates vary with t: we will examine two particular cases of main interest, namely the case of vital rates presenting a converging trend as time goes on, and the case of vital rates which are periodic with respect to time. In both cases we have to exclude trivial initial data p_0 such that

(2.8) $$\beta(a + t, t)p_0(a) = 0 \quad \text{a.e. for } a \in [0, a_\dagger] \quad \text{and for all } t \geq 0 \,.$$

In fact this condition would give $F(t) = 0 \ \forall t \geq 0$ and, consequently, $B(t) = 0$ $\forall t \geq 0$.

With respect to the case of converging rates we have the following results:

Theorem 2.2. *Let $B(t)$ be the solution of (2.4), (2.5), (2.6). Assume $m(a, t) \equiv 0$ and let $K^* \in L^\infty(\mathbf{R}_+)$ be such that $K^*(t) = 0$ for $t > a_\dagger$ and*

(2.9) $$\lim_{t \to +\infty} |K(t, \cdot) - K^*(\cdot)|_{L^\infty(0,a_\dagger)} = 0$$

(2.10) $$\int_0^\infty |K(t, \cdot) - K^*(\cdot)|_{L^\infty(0,a_\dagger)}\, dt < +\infty$$

Then $B(t)$ can be written as

(2.11) $$B(t) = B^*(t)\,(b_0 + \Omega(t))$$

where $b_0 \geq 0$, $\lim_{t \to +\infty} \Omega(t) = 0$ and $B^*(t)$ is the solution of the limiting equation:

(2.12) $$B^*(t) = F(t) + \int_0^t K^*(t - s)B^*(s)ds$$

Proof:

If the initial datum p_0 satisfies (2.8) then (2.11) is trivial, thus we assume that (2.8) is not true.

Let $R^*(t)$ be the resolvent kernel of equation (2.12). We recall that (see Appendix II)

$$(2.13) \qquad B^*(t) = F(t) - \int_0^t R^*(t-s)F(s)ds$$

and that, by the results of Theorem I.5.2 (also applied to the resolvent equation (1.3) of Appendix II), we have

$$(2.14) \qquad B^*(t) = b_0^* e^{\alpha^* t} (1 + \Omega_1(t)), \; R^*(t) = r_0^* e^{\alpha^* t} (1 + \Omega_2(t))$$

where $b_0^* > 0$, $r_0^* > 0$, $\lim_{t\to+\infty} \Omega_1(t) = \lim_{t\to+\infty} \Omega_2(t) = 0$ and α^* is the (unique) real solution of the equation

$$\int_0^{a_\dagger} e^{-\alpha t} K^*(t)dt = 1$$

Besides, we write (2.4) as:

$$B(t) = F(t) + \int_0^t K^*(s)B(t-s)ds + \int_0^t \varepsilon(t,s)B(t-s)ds$$

where $\varepsilon(t, s) = K(t, s) - K^*(s)$ and, using the resolvent kernel $R^*(t)$, we get:

$$(2.15) \qquad \begin{aligned} B(t) &= F(t) + \int_0^t \varepsilon(t,s)B(t-s)ds + \\ &\quad - \int_0^t R^*(t-s)F(s)ds - \int_0^t R^*(t-s) \int_0^s \varepsilon(s,\sigma)B(s-\sigma)d\sigma ds \\ &= B^*(t) + \int_0^t \varepsilon(t,s)B(t-s)\,ds + \\ &\quad - \int_0^t R^*(t-s) \int_0^s \varepsilon(s,\sigma)B(s-\sigma)d\sigma ds \end{aligned}$$

We first prove that

$$(2.16) \qquad |B(t)| \leq M e^{\alpha^* t}$$

In fact, by (2.14), let C_∞ be a constant such that $C_\infty > e^{-\alpha^* t} B^*(t) + e^{-\alpha^* t} R^*(t)$ for all $t \geq 0$, and let $T > a_\dagger$ be sufficiently large so that (see (2.9) and (2.10))

$$\int_0^{a_t} |\varepsilon(t, s)|\, ds < \frac{e^{-|\alpha^*|a_t}}{4} \qquad for\ t \geq T\,,$$

$$\int_T^{\infty} \int_0^{a_t} |\varepsilon(s, \sigma)|\, d\sigma ds < \frac{e^{-|\alpha^*|a_t}}{4C_\infty}$$

Then, setting $M_{a,b} = \max_{t\in[a,b]} |e^{-\alpha^* t} B(t)|$, we have for $t \in [T, \tau]$:

$$|e^{-\alpha^* t} B(t)| \leq |e^{-\alpha^* t} B^*(t)| + e^{|\alpha^*|a_t} (M_{0,T}+M_{T,\tau}) \int_0^{a_t} |\varepsilon(t,s)| ds +$$

$$+ C_\infty e^{|\alpha^*|a_t} M_{0,T} \int_0^T \int_0^{a_t} |\varepsilon(s,\sigma)|\, d\sigma ds +$$

$$+ C_\infty e^{|\alpha^*|a_t} (M_{0,T}+M_{T,\tau}) \int_T^{\infty} \int_0^{a_t} |\varepsilon(s,\sigma)|\, d\sigma ds$$

so that, for all $\tau > T$,

$$M_{T,\tau} \leq 2C_\infty + \left(1+2C_\infty e^{|\alpha^*|a_t} \int_0^T \int_0^{a_t} |\varepsilon(s,\sigma)| d\sigma ds\right) M_{0,T}$$

which implies that $e^{-\alpha^* t} B(t)$ is bounded on $[0, +\infty]$, that is (2.16).

Now, by (2.9) and (2.10) we have that:

$$\left| e^{-\alpha^* t} \int_0^t \varepsilon(t, s) B(t-s) ds \right| \leq e^{|\alpha^*|a_t} \int_0^{a_t} |\varepsilon(t, s)|\, ds \overset{t\to\infty}{\longrightarrow} 0 \tag{2.17}$$

and also

$$e^{-\alpha^* t} \int_0^t R^*(t-s) \int_0^s \varepsilon(s,\sigma) B(s-\sigma) d\sigma ds = \int_0^t r_0^* (1+\Omega_2(t-s))\, g(s) ds$$

where the function:

$$g(s) = e^{-\alpha^* s} \int_0^s \varepsilon(s,\sigma) B(s-\sigma) d\sigma$$

belongs to $L^1(0, +\infty)$ because:

$$|g(s)| \leq M\, e^{|\alpha^*|a_t} \int_0^{a_t} |\varepsilon(s, \sigma)|\, d\sigma$$

Thus we have:

$$(2.18) \qquad e^{-\alpha^* t} \int_0^t R^*(t-s) \int_0^s \varepsilon(s,\sigma) B(s-\sigma) d\sigma ds \overset{t\mapsto\infty}{\to} r_0^* \int_0^\infty g(s) ds$$

Taking (2.17) and (2.18) into (2.15) we have (2.11) with

$$b_0 = \left(1 - \frac{r_0^* \int_0^\infty g(s)ds}{b_0^*}\right).$$

■

The previous result rests upon assumptions (2.9)-(2.10) which state that the kernel $K(t, s)$ "rapidly converges" to $K^*(s)$ as $t\mapsto+\infty$: that is a condition on how the rates $\beta(a, t)$ and $\mu(a, t)$ evolve converging to a limiting vital dynamics. We note that (2.11) is not quite satisfactory because it is not clear whether b_0 is strictly positive. Actually (2.11) is not really significant if $b_0 = 0$, but our proof does not give informations on the value of b_0.

Concerning the case of periodic rates, we have the following result due to H. Thieme ([91], [92]).

Theorem 2.3. *Let $B(t)$ be the solution of (2.4), (2.5), (2.6). Assume $m(a, t) \equiv 0$ and suppose that there exists a period $T > 0$ such that:*

$$(2.19) \qquad K(t + T, s) = K(t, s), \ \forall\, t \geq 0 \ s \in [0, a_\dagger]$$

Then there exist a unique $\alpha^* \in \mathbf{R}$ and a unique T-periodic function $b^*(\cdot) \in C(\mathbf{R}_+)$ that solve the functional equation

$$(2.20) \qquad b(t) = \int_0^\infty e^{\alpha s} K(t, s) b(t-s)\, ds$$

and such that $B(t)$ can be written as

$$(2.21) \qquad B(t) = e^{\alpha^* t} b^*(t)\,(1 + \Omega(t))$$

where $\lim_{t\to+\infty} \Omega(t) = 0.$

■

The proof of this theorem requires concepts and tools that go beyond the purposes of this presentation, thus it must be omitted; we have nevertheless presented this result because the statements of Theorem 2.2 and 2.3 are important examples for the concept of ergodicity that we will discuss in the next section.

3 Strong and weak ergogidicity

A main principle concerning the demographic evolution of a population is the statement that any population *eventually forgets its initial age distribution*. This phenomenological claim, that should be valid independently of the particular dynamics which is responsible for the population growth, is known as the *ergodic* behavior of the population and is precisely formulated by the two concepts of *strong ergodicity* and *weak ergodicity*.

Actually, the term "strong ergodicity" has been traditionally intended as concerning the case of fixed vital rates and was just a different enunciation of Theorem I.5.2. with its consequences exposed in section 1. In fact in Theorem 1.3 and in (1.17) we have seen that the age profile $\omega(a, t)$ and the Malthusian rate $\alpha(t)$ respectively attain the asymptotic profile $\omega^*(a)$ and the intrinsic rate α^*, which are both independent of the initial distribution $p_0(a)$. Presently we need to extend the concept to any population, whenever a similar situation occurs: namely we adopt the following

Definition 3.1. *An age structured population is said to be strongly ergodic if the age-profile $\omega(a, t)$ and the malthusian coefficient $\alpha(t) = \int_0^{a_\dagger} [\beta(\sigma, t) - \mu(\sigma, t)]\omega(\sigma, t)d\sigma$ have asymptotic limits that are independent of the initial datum $p_0(a)$.* ∎

This definition is still somewhat vague because it does not specify what kind of asymptotic limits occur, but this point must be specified case by case. Actually this definition applies to the time dependent cases studied in section 2; in fact we have the following two theorems that are just corollaries of Theorem 2.2 and Theorem 2.3:

Theorem 3.2. *Consider problem (2.1) with $m(a, t) \equiv 0$ and let the vital rates satisfy:*

$$\lim_{t \to +\infty} \beta(\cdot, t) = \beta^*(\cdot) \quad in \quad L^\infty(0, a_\dagger)$$

$$\lim_{t \to +\infty} \mu(\cdot, t) = \mu^*(\cdot) \quad in \quad L^1_{loc}([0, a_\dagger)) .$$

Suppose also that the maternity function $K(t, a)$ satisfies (2.10) and that in (2.11) $b_0 > 0$. Then the population is strongly ergodic. ∎

Theorem 3.3. *Consider problem (2.1) with $m(a, t) \equiv 0$ and let the maternity function satisfy (2.19), then the population is strongly ergodic.* ∎

The proof of these results is already contained in the previous section. Namely for Theorem 3.2 we are under the assumptions (2.9), (2.10) and also we have

$$\lim_{t \to +\infty} \Pi(a, t, a) = \Pi^*(a) = e^{-\int_0^a \mu^*(\sigma) d\sigma}$$

so that, by (2.11) ,

(3.1) $$\lim_{t \to +\infty} \omega(a, t) = \omega^*(a) = \frac{e^{-\alpha^* a}\Pi^*(a)}{\int_0^{a_\dagger} e^{-\alpha^* \sigma}\Pi^*(\sigma)d\sigma} \quad in \quad L^1(0, a_\dagger)$$

and

$$P(t) = P_0 e^{\int_0^t \alpha(s)ds} = e^{\alpha^* t}(c_0 + \Omega(t))$$

where $c_0 > 0$ and $\lim_{t \to +\infty} \Omega(t) = 0$. This yields

(3.2) $$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \alpha(s)ds = \alpha^*$$

Concerning Theorem 3.3, using (2.20) and (2.21) it is easy to see that

$$\lim_{t \to +\infty} \sup_{s \in [0,T]} |\omega(\cdot, t + s) - \omega^*(\cdot, s)|_{L^1} = 0$$

where

$$\omega^*(a, t) = \frac{e^{-\alpha^* a} b^*(t - a)\Pi(a, t, a)}{\int_0^{a_\dagger} e^{-\alpha^* \sigma} b^*(t - \sigma)\Pi(\sigma, t - \sigma, \sigma)d\sigma}$$

and that (3.2) is true also in this case.

Passing to the concept of weak ergodicity we have:

Definition 3.4. *An age structured population is said to be weakly ergodic if, letting $\omega^1(a, t)$ and $\omega^2(a, t)$ be the age-profiles corresponding to initial data p_0^1 and p_0^2 respectively, we have:*

$$\lim_{t \to +\infty} |\omega^1(\cdot, t) - \omega^2(\cdot, t)|_{L^1} = 0$$ ■

Of course, strong ergodicity implies weak ergodicity. However, the latter concept is enough to interprete the idea of a population that forgets its initial age distribution.

We have seen that in the strongly ergodic cases considered in theorems 3.2 and 3.3, in order to identify the limit distribution of $p(a, t)$, we must assume some specific limit behavior of the rates; weak ergodicity, in turn, can be stated under fairly general assumptions on system (2.1). This is done by H. Inaba [62] with methods that we cannot introduce here, thus we limit ourselves to present the following sufficient conditions:

Theorem 3.4. *Consider problem (2.1) with $m(a, t) \equiv 0$ and suppose that*

$$K(a, t) \geqslant b > 0, \quad (a, t) \in [a_1, a_2] \times [0, +\infty]$$

for some interval $[a_1, a_2]$, and

$$\int_0^{a_\dagger} \beta(a + s, t + s)\Pi(a + s, t + s, s)\, ds > 0 \quad \text{a.e. in } [0, a_\dagger] \times [0, +\infty)$$

Then the population is weakly ergodic. ■

4 Infinite maximum age

The theory presented throughout the previous sections always assumes that the maximum age $a_\dagger$ is finite and that, consequently, condition (I.2.8) is satisfied. This assumption, which is rather realistic, can however be disregarded if we consider time ranges that are comparable with the life span of the population: actually the early models considering age structure did not even pose the problem, letting age assume any non-negative value, but implicitly assuming some hypotheses (such as finite fertility windows) which allow a mathematical treatment analogous to the one that we have given. As a matter of fact, if we let $a_\dagger = +\infty$ we have to be careful with the behavior of the rates at inifinity in order to perform the asymptotic analysis of section I.5. In this section we will not go through this point but will rather show that with $a_{\dagger} = +\infty$ we can use some specific form for the rates $\beta(a)$ and $\mu(a)$ such that it is possible to transform the renewal equation into a system of ordinary differential equations.

Let us assume the following constitutive equations

$$\beta(a) = \beta_0 a e^{-\varrho a}, \quad \mu(a) = \mu_0 \tag{4.1}$$

where β_0, ϱ, μ_0 are prescribed positive parameters. Then the renewal equation (I.3.4) has the form:

$$B(t) = \beta_0 \int_0^t (t-s)e^{-\gamma(t-s)}B(s)ds + \beta_0 e^{-\gamma t}\left(t\int_0^\infty e^{-\varrho a}p_0(a)da + \int_0^\infty a e^{-\varrho a}p_0(a)da\right) \tag{4.2}$$

where $\gamma = \varrho + \mu_0$. Now we introduce the auxiliary variable:

$$Q(t) = \beta_0 \int_0^t e^{-\gamma(t-s)}B(s)ds + \beta_0 e^{-\gamma t}\int_0^\infty e^{-\varrho a}p_0(a)da \tag{4.3}$$

so that, by a straighforward calculation, (4.2) is tranformed into the following system for the couple $(B(t), Q(t))$:

$$\begin{cases} \dfrac{d}{dt}B(t) = -\gamma B(t) + Q(t), \quad B(0) = \beta_0 \displaystyle\int_0^\infty a e^{-\varrho a}p_0(a)da \\ \dfrac{d}{dt}Q(t) = \beta_0 B(t) - \gamma Q(t), \quad Q(0) = \beta_0 \displaystyle\int_0^\infty e^{-\varrho a}p_0(a)da \end{cases} \tag{4.4}$$

and we get the following explicit form for $B(t)$:

$$B(t) = b_0 e^{\alpha^* t}\left(1 + b_1 e^{-2\sqrt{\beta_0}\,t}\right) \tag{4.5}$$

$$\alpha^* = -\gamma + \sqrt{\beta_0} \tag{4.6}$$

and b_0, b_1 are constants depending on $B(0)$ and $Q(0)$.

We note that $B(t)$, as given by (4.5), has the same form as in (I.5.4) though the rates (4.1) do not fit into the theory of the previous sections. We also note the following formula for $p(a, t)$

$$(4.7) \qquad p(a,t) = \begin{cases} e^{-\mu_0 t} p_0(a-t) & \text{if } a \geq t \\ b_0 e^{\alpha^* t} (1 + b_1 e^{-2\sqrt{\beta_0}(t-a)}) \, e^{(\varrho - \sqrt{\beta_0})a} & \text{if } a < t \end{cases}$$

where $p_0 \in L^1(0, +\infty)$. Using this formula, it is possible to show the following result concerning the age profile

$$(4.8) \qquad \text{If } (\sqrt{\beta_0} - \varrho) < 0 \quad \text{then } \lim_{t \to +\infty} \omega(a,t) = \omega^*(a) = (\sqrt{\beta_0} - \varrho) \, e^{(\varrho - \sqrt{\beta_0})a}$$

$$(4.9) \qquad \text{If } (\sqrt{\beta_0} - \varrho) > 0 \quad \text{then } \lim_{t \to +\infty} \omega(a, t) = 0$$

The limit in (4.8) is in $L^1(0, +\infty)$ while that in (4.9) occurs pointwise. We remark that this latter result is anomalous with respect to the case with $a_\dagger < +\infty$.

5 Comments and references

The results that we have shown in this chapter continue the basic linear theory of Chapter I. The description of the evolution through the age profile and the total population is just a presentation of the same results from a different point of view, but gives some further understanding which is of some use when considering erogidicity and also when treating some special class of non linear models that we will consider later.

The time dependent case is not of secondary importance though it has not been studied very much. First attention to this case was given by Langhaar in [70], but mathematical results are quite recent, also in connection with the question of ergodicity. The latter is of great interest in demography and has attracted much attention in the context of discrete models ([21], [68], [73]): an extended discussion can be found in [62] where also some historical remarks can be found. Actually, a general mathematical definition of ergodicity, which can be applied in all circumstances, does not seem to exist, thus we have adopted those given in Definitions 3.1 and 3.4 and shown some significant examples. Theorems 2.2 and 3.4 are due to Inaba ([62], [64]).

Finally, concerning the case of $a_\dagger = +\infty$, we have mentioned some difficulties arising in the asymptotic analysis, first remarked by Feller in [37] where some examples are given. The problem can be easily understood in the context of a functional analytic approach to the Lotka-McKendrick system [100]. However, while such pathologies can be disregarded because they do not add anything biologically significant to the models, the example of Section 4 shows that with $a_\dagger = +\infty$ we can take advantage of reduction to ordinary differential equations, in correspondence with some special constitutive assumptions on the vital rates. This reduction has been systematically used ([45], [48]) to investigate nonlinear models: we will go back to this point in Chapter V.

III

Nonlinear models

The linear model we have been dealing with in the previous chapters has been presented as the age structured version of the so called Malthus model; thus the criticisms of the latter also apply to the former. In fact, the simple Malthus model is not realistic, unless we want to follow the growth of the population for a limited time until the rough assumptions that we have made are satisfied. Actually, even if we disregard the external habitat variations, we have to consider that the population itself causes modifications of its own condition of life, thus we must assume that fertility and mortality depend on the population size and the linear equation of the Malthus model must be replaced by a non-linear one

$$\frac{d}{dt} P(t) = \alpha(P(t))\, P(t)$$

where the function $\alpha(x) : [0, +\infty) \to \mathbf{R}$ must describe the effect of the population size on fertility and mortality. Usually $\alpha(x)$ is supposed to satisfy the following assumptions:

$$\begin{cases} i) \quad \alpha'(x) > 0 \qquad if \quad 0 < x < x_0 \\ ii) \quad \alpha'(x) < 0 \qquad if \quad x > x_0 \\ iii) \quad \lim_{x \to +\infty} \alpha(x) < 0 \end{cases}$$

where $x_0 \geqslant 0$ (if $x_0 = 0$ only *ii)* and *iii)* are meaningful). These assumptions contain the main phenomenology of the single population growth, in fact *i)* shapes the so called *Allee effect* stating that, at a low population density, an increase of the population size has a positive effect on the population growth; assumptions *ii)* and *iii)*, conversely, introduce the *logistic effect* claiming that at high population densities an increase of the size has a negative influence on the population growth. A special case is the following *Verhulst model* [98],

$$\alpha(x) = \alpha_0 \left(1 - \frac{x}{K} \right)$$

which is *purely logistic*.

Such models prevent the population to growth to infinity; in fact $P(t)$ always converges monotonically to an equilibrium size. The solution of the special Verhulst case is reported in Figure 0.1. In this case $P(t)$ always tends to the so called *carrying capacity K*.

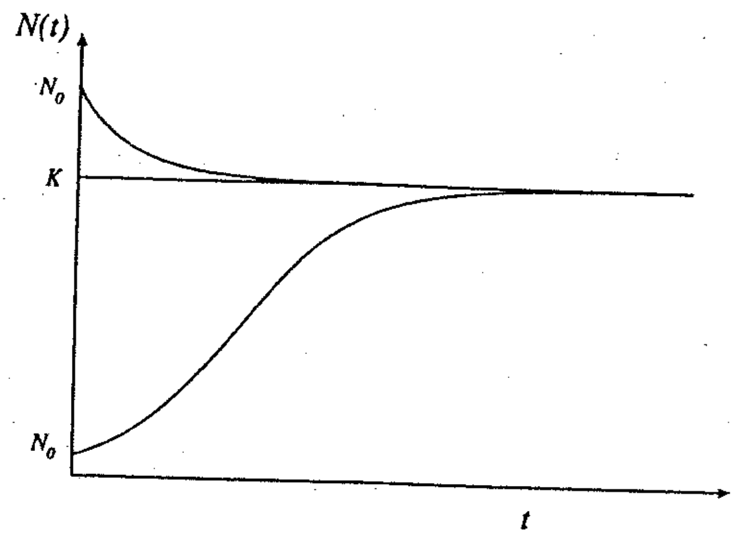

Figure 0.1

Thus, going back to our age-structured model, to describe the effect of crowding on the population growth we should take into account a posible dependence of the age specific vital rates on the population itself, considering mechanisms such as the logistic effect and the Allee effect, but here, because of age structure, a large variety of ways by which these mechanisms are realized can be envisaged.

In this chapter we will consider a fairly general model and some special cases of it, which are mathematically tractable, though we are aware of the fact that it is not exhaustive of the many possible mechanisms.

1 A general nonlinear model

We consider a single population and assume that fertility and mortality depend on a set of n significant variables (*sizes*) which represent different ways of weighing the age distribution:

$$(1.1) \qquad S_i(t) = \int_0^{a_\dagger} \gamma_i(a)p(a, t)da, \quad i = 1, \ldots, n$$

thus $\beta(a)$ and $\mu(a)$ are replaced by

$$\beta(a, S_1(t), \ldots, S_n(t)), \quad \mu(a, S_1(t), \ldots, S_n(t))$$

and the linear model of the previous chapter is modified into the following one:

$$(1.2) \qquad \begin{cases} p_t(a, t) + p_a(a, t) + \mu(a, S_1(t), \ldots, S_n(t))p(a, t) = 0 \\ p(0, t) = \int_0^{a_\dagger} \beta(\sigma, S_1(t), \ldots, S_n(t))p(\sigma, t)d\sigma \\ p(a, 0) = p_0(a) \\ S_i(t) = \int_0^{a_\dagger} \gamma_i(\sigma)p(\sigma, t)d\sigma, \quad i = 1, \ldots, n \end{cases}$$

In the next section we will provide a general theorem stating existence and uniqueness of solutions to this problem, here we introduce the assumptions that we will make throughout this chapter on β, μ and γ_i.

(1.3) $\forall\ (x_1,\ldots,x_n) \in \mathbf{R}^n$ *the functions* $\beta(\cdot,x_1,\ldots,x_n)$ *and* $\mu(\cdot,x_1,\ldots,x_n)$ *belong to* $L^1(0, a_\dagger)$ *and* $L^1_{loc}([0, a_\dagger))$, *respectively.*

(1.4) $\forall\ (x_1,\ldots,x_n) \in \mathbf{R}^n$ *it is* $0 \leq \beta(a,x_1,\ldots,x_n) \leq \beta_+$ *a.e. in* $[0, a_\dagger]$

(1.5) $\forall\ (x_1,\ldots,x_n) \in \mathbf{R}^n$ *it is* $\mu(a,x_1,\ldots,x_n) \geq 0$ *a.e. in* $[0, a_\dagger]$ *and*

$$\int_0^{a_\dagger} \mu(\sigma,x_1,\ldots,x_n)d\sigma = +\infty$$

(1.6) *$\forall M > 0$ there exists a constant $H(M) > 0$ such that, if $|x_i| \leqslant M$ and $|\bar{x}_i| \leqslant M$, for $i = 1, 2, \ldots, n$, then:*

$$|\beta(a,x_1,\ldots,x_n) - \beta(a,\bar{x}_1,\ldots,\bar{x}_n)| \leqslant H(M) \sum_{i=1}^{n} |x_i - \bar{x}_i|$$

$$|\mu(a,x_1,\ldots,x_n) - \mu(a,\bar{x}_1,\ldots,\bar{x}_n)| \leqslant H(M) \sum_{i=1}^{n} |x_i - \bar{x}_i|$$

(1.7) *The functions $\gamma_i(\cdot)$ belong to $L^\infty(0, a_\dagger)$ and $\gamma(a) \geqslant 0$ a.e. in $[0, a_\dagger]$*

We note that conditions (1.3), (1.4), (1.5) mean that, for a fixed set of the variables x_i, the rates β and μ satisfy the assumptions of the linear case; moreover, by (1.6) they are Lipschitz continuous with respect to the x_i's, uniformly in $a \in [0, a_\dagger]$.

2 Existence and uniqueness

Existence and uniqueness of the solution to (1.2) will be proved via a fixed point argument which is based on an integrated form of the problem. To establish this formula we consider the variables $S_i(t)$ as a given functions of t so that problem (1.2) can be viewed as a non autonomous linear problem (see section II.2); then integration along characteristics yields:

$$(2.1) \qquad p(a, t) = \begin{cases} p_0(a - t)\Pi(a, t, t; S) & \text{if } a \geqslant t \\ b(t - a; S)\Pi(a, t, a; S) & \text{if } a < t \end{cases}$$

where S stays for $(S_1(t), S_2(t), \ldots, S_n(t))$ and belongs to the space $C([0, T]; \mathbf{R}^n)$; moreover we have set (see II.2.3):

$$(2.2) \qquad \Pi(a, t, x; S) = \exp\left[- \int_0^x \mu(a - \sigma, S_1(t - \sigma), \ldots, S_n(t - \sigma))d\sigma \right]$$

and $b(t; S)$ is the solution of the equation:

$$(2.3) \qquad u(t) = F(t; S) + \int_0^t K(t, t - \sigma; S)u(\sigma)d\sigma$$

where:

$$F(t; S) = \int_t^\infty \beta(a, S_1(t),\ldots, S_n(t))\Pi(a, t, t; S)p_0(a-t)da$$
(2.4)
$$= \int_0^\infty \beta(a+t, S_1(t),\ldots, S_n(t))\Pi(a+t, t, t; S)p_0(a)da$$

and

(2.5) $$K(t, \sigma; S) = \beta(\sigma, S_1(t),\ldots, S_n(t))\Pi(\sigma, t, \sigma; S)$$

Note that, for a given $S \in C([0, T]; \mathbf{R}^n)$ the function $F(\cdot, S)$ is continuous and, by the results of section II.2, also $b(\cdot, S)$ is continuous. Finally we note that the function $t \rightarrow p(\cdot, t)$, defined by (2.1) belongs to $C([0, T]; L^1(0, a_\dagger))$.

Before going through the proof of existence and uniqueness of a solution to problem (1.2) we need to state some estimates. First we note that for $S, \bar{S} \in C([0, T]); \mathbf{R}^n)$ such that $|S_i(t)| \leqslant M$, $|\bar{S}_i(t)| \leqslant M$, for $i = 1, \ldots, n$ and $t \in [0, T]$, we have

(2.6) $$|\beta(a,S_1(t),\ldots,S_n(t))-\beta(a,\bar{S}_1(t),\ldots,\bar{S}_n(t))| \leqslant H(M) \sum_{i=1}^n |S_i(t)-\bar{S}_i(t)|$$

(2.7) $$|\Pi(a,t,x;S)-\Pi(a,t,x;\bar{S})| \leqslant H(M) \sum_{i=1}^n \int_{t-x}^t |S_i(\sigma)-\bar{S}_i(\sigma)|d\sigma$$

We then obtain

Lemma 2.1. *Let $S, \bar{S} \in C([0, T]; \mathbf{R}^n)$ such that $|S_i(t)| \leqslant M$, $|\bar{S}_i(t)| \leqslant M$, for $i = 1, \ldots, n$ and $t \in [0, T]$, then:*

(2.8) $$b(t; S) \leqslant \beta_+ e^{\beta_+ t}|p_0|_{L^1}$$

and there exists $L(M) > 0$ such that:

(2.9)
$$|b(t;S)-b(t;\bar{S})| \leqslant L(M)\, |p_0|_{L^1} \sum_{i=1}^n \left[|S_i(t)-\bar{S}_i(t)| + \int_0^t |S_i(\sigma)-\bar{S}_i(\sigma)|d\sigma \right]$$

Proof:

To prove (2.8) one can proceed as in the proof of Theorem I.4.3, in fact we only need to note that $F(t; S) \leqslant \beta_+|p_0|_{L^1}$ and $K(t, \sigma; S) \leqslant \beta_+$.

Concerning (2.9) we have from (2.3)-(2.7):

$$|b(t;\, S)-b(t;\, \bar{S})| \leqslant$$

$$\leqslant \int_t^{\infty} |\beta(a,S_1(t),\dots,S_n(t))-\beta(a,\bar{S}_1(t),\dots\bar{S}_n(t))|p_0(a-t)da+$$

$$+\ \beta_+ \int_0^{\infty} |\Pi(a+t,t,t;S)-\Pi(a+t,t,t;\bar{S})|p_0(a)da+$$

$$+ \int_0^t |\beta(\sigma;S_1(t),\dots,S_n(t))-\beta(\sigma,\bar{S}_1(t),\dots,\bar{S}_n(t))|b(t-\sigma;S)d\sigma+$$

$$+\ \beta_+ \int_0^t |\Pi(\sigma,t,\sigma;S) - \Pi(\sigma,t,\sigma;\bar{S})|b(t-\sigma;S)d\sigma+$$

$$+\ \beta_+ \int_0^t |b(\sigma;S) - b(\sigma;\bar{S})|d\sigma$$

$$\leqslant H(M)|p_0|_{L^1} \sum_{i=1}^{n} |S_i(t) - \bar{S}_i(t)| +$$

$$+\ \beta_+ H(M)|p_0|_{L^1} \sum_{i=1}^{n} \int_0^t |S_i(\sigma) - \bar{S}_i(\sigma)|d\sigma+$$

$$+\ \beta_+ H(M) \int_0^t e^{\beta_+ \sigma}d\sigma|p_0|_{L^1} \sum_{i=1}^{n} |S_i(t) - \bar{S}_i(t)| +$$

$$+\ \beta_+^2 H(M)|p_0|_{L^1} \sum_{i=1}^{n} \int_0^t e^{\beta_+ \sigma} \int_0^t |S_i(r)-\bar{S}_i(r)|dr\, d\sigma+$$

$$+\ \beta_+ \int_0^t |b(\sigma;\, S) - b(\sigma;\, \bar{S})|d\sigma$$

$$\leqslant 2H(M)(1+\beta_+)e^{\beta_+ T}|p_0|_{L^1} \sum_{i=1}^{n} \left[\, |S_i(t)-\bar{S}_i(t)|+ \int_0^t |S_i(\sigma)-\bar{S}_i(\sigma)|d\sigma\right]$$

$$+\ \beta_+ \int_0^t |b(\sigma;\, S) - b(\sigma;\, \bar{S})|d\sigma$$

so that, by the Gronwall inequality, (2.9) holds with $L(M) = 2(1 + \beta_+)H(M)e^{2\beta_+ T}$.

■

We now consider the space $E = C([0, T]; L^1(0, a_\dagger))$ and the set

(2.10) $$\mathcal{K} \equiv \{q \in E \mid q(a, t) \geqslant 0, \ |q(\cdot, t)|_{L^1} \leqslant M\}$$

which is a closed set in E. Then, for $q \in \mathcal{K}$ we set:

(2.11) $$Q \equiv (Q_1(t),\dots, Q_n(t)), \quad Q_i(t) = \int_0^{a_\dagger} \gamma_i(a)q(a, t)da$$

and define the mapping $\mathcal{T} : \mathcal{K} \subset E \to E$ (see (2.1)) by:

(2.12) $$(\mathcal{T}q)(a, t) = \begin{cases} p_0(a - t)\Pi(a, t, t; Q) & \text{if } a \geqslant t \\ b(t - a; Q)\Pi(a, t, a; Q) & \text{if } a < t \end{cases}$$

where $p_0 \in L^1(0, a_\dagger)$ is fixed.

We look for a fixed point of this mapping in order to provide an existence and uniqueness proof for problem (1.2). To this aim, for a fixed initial datum $p_0 \in L^1(0, a_\dagger)$ we take M such that:

(2.13) $$M > e^{\beta_+ T}|p_0|_{L^1}$$

Then we have:

Lemma 2.2. *Let $\mathcal{K}$ be defined in (2.10) with M satisying (2.13), then the mapping $\mathcal{T}$, defined in (2.11)-(2.12), maps $\mathcal{K}$ into itself and for $q, \bar{q} \in \mathcal{K}$, $t \in [0, T]$ we have*

(2.14) $$|(\mathcal{T} q)(\cdot, t) - (\mathcal{T} \bar{q})(\cdot, t)|_{L^1} \leqslant C(M,T) \int_0^t |q(\cdot, \sigma)-\bar{q}(\cdot, \sigma)|_{L^1}d\sigma$$

where $C(M, T)$ is a constant (depending on M and T)

Proof:

Let $q \in \mathcal{K}$, then by the remarks on formula (2.1), $(\mathcal{T} q)(a, t) \geqslant 0$, and by (2.8):

$$\int_0^{a_\dagger} (\mathcal{T} q)(a,t)da= \int_0^t b(t-a;Q)\, \Pi\,(a,t,a;Q)da+ \int_t^{a_\dagger} p_0(a-t)\, \Pi\,(a,t,t;Q)da \leqslant$$

$$\leqslant \int_0^t b(a;Q)da+ \int_t^{a_\dagger} p_0(a-t)da \leqslant e^{\beta_+ t}|p_0|_{L^1}<M$$

Thus $\mathcal{T}(\mathcal{H}) \subset \mathcal{H}$. Let now $q, \bar{q} \in \mathcal{H}$, then, since:

$$Q_i(t) \leqslant \gamma_+ \int_0^{a_\dagger} q(a,t)da = \gamma_+ |q(\cdot,t)|_{L^1} < \gamma_+ M$$

where

$$\gamma_+ = \max_{i=1,\ldots,n} |\gamma_i|_{L^\infty}$$

we have:

$$\int_0^{a_\dagger} |(\mathcal{T})q(a,t) - (\mathcal{T})\bar{q}(a,t)|\, da \leqslant$$

$$\leqslant \int_0^t |b(t-a;Q) - b(t-a;\bar{Q})|\, da +$$

$$+ \int_0^t b(t-a;\bar{Q})|\Pi(a,t,a;Q) - \Pi(a,t,a;\bar{Q})|\, da +$$

$$+ \int_t^\infty p_0(a-t)|\Pi(a,t,t;Q) - \Pi(a,t,t;\bar{Q})|\, da$$

$$\leqslant L(\gamma_+ M)|p_0|_{L^1} \sum_{i=1}^n \int_0^t \left[|Q_i(a)-\bar{Q}_i(a)| + \int_0^a |Q_i(\sigma)-\bar{Q}_i(\sigma)|d\sigma\right] da$$

$$+ \beta_+ H(\gamma_+ M)|p_0|_{L^1} \sum_{i=1}^n \int_0^t e^{a\beta^+} \int_a^t |Q_i(\sigma)-\bar{Q}_i(\sigma)|d\sigma\, da$$

$$+ H(\gamma_+ M)|p_0|_{L^1} \sum_{i=1}^n \int_0^t |Q_i(\sigma)-\bar{Q}_i(\sigma)|d\sigma$$

$$\leqslant [(1+T)L(\gamma_+ M)+e^{\beta_+ T}H(\gamma_+ M)]\, |p_0|_{L^1} \sum_{i=1}^n \int_0^t |Q_i(\sigma)-\bar{Q}_i(\sigma)|d\sigma$$

$$\leqslant n\gamma_+ [(1+T)L(\gamma_+ M)+e^{\beta_+ T}H(\gamma_+ M)]\, |p_0|_{L^1} \int_0^t |q(\cdot,\sigma)-\bar{q}(\cdot,\sigma)|_{L^1} d\sigma$$

and (2.14) follows.

■

Thus we are ready to prove the following:

Theorem 2.3. *Let $p_0 \in L^1(0, a_\dagger)$, then there is one and only one $p \in \mathcal{K}$ such that:*

$$(2.15) \qquad p(a, t) = \begin{cases} p_0(a-t)\Pi(a, t, t; S) & \text{if } a \geqslant t \\ b(t-a; S)\Pi(a, t, a; S) & \text{if } a < t \end{cases}$$

$$(2.16) \qquad S_i(t) = \int_0^{a_\dagger} \gamma_i(a)p(a, t)da, \quad i = 1, \ldots, n$$

Moreover, $p(a, t)$ has the following properties:

$$(2.17) \qquad \lim_{h\to 0} \frac{1}{h} [p(a+h,t+h)-p(a,t)] = -\mu(a)p(a,t) \text{ a.e. in } [0, a_\dagger] \times \mathbf{R}_+$$

$$(2.18) \qquad |p(\cdot, t)|_{L^1} \leqslant e^{\beta_+ t}|p_0|_{L^1}$$

$$(2.19) \qquad |p(\cdot, t) - \bar{p}(\cdot, t)|_{L^1} \leqslant e^{C(M,T)t}|p_0 - \bar{p}_0|_{L^1}$$

where $\bar{p}(\cdot, t)$ is the solution relative to the initial datum $\bar{p}_0$.

Proof:

A straightforward consequence of (2.14) is that for any integer $N > 0$:

$$(2.20) \qquad |\mathcal{T}^N q - \mathcal{T}^N \bar{q}|_E \leqslant \frac{C(M, T)^N T^N}{N!} |q - \bar{q}|_E$$

so that, if N is sufficiently large, $\mathcal{T}^N$ is a contraction and, consequently, $\mathcal{T}$ has a unique fixed point in $\mathcal{K}$. Thus, the first part of the thesis is proven and, also, (2.18) is a consequence of the choice (2.13). Finally, to prove (2.19), we call $\mathcal{T}_{p_0}$ and $\mathcal{T}_{\bar{p}_0}$ the mappings defined by (2.12),relative to p_0 and $\bar{p}_0$, respectively; then:

$$p(\cdot, t) = (\mathcal{T}_{p_0}p)(\cdot, t), \quad \bar{p}(\cdot, t) = (\mathcal{T}_{\bar{p}_0}\bar{p})(\cdot, t)$$

and, by (2.14):

$$
\begin{aligned}
&|p(\cdot, t) - \bar{p}(\cdot, t)|_{L^1} \leq \\
&\leq |(\mathcal{T}_{p_0}p)(\cdot, t) - (\mathcal{T}_{\bar{p}_0}p)(\cdot, t)|_{L^1} + |(\mathcal{T}_{\bar{p}_0}p)(\cdot, t) - (\mathcal{T}_{\bar{p}_0}\bar{p})(\cdot, t)|_{L^1} \leq \\
&\leq \int_t^\infty |p_0(a-t) - \bar{p}_0(a-t)|da + C(M, T) \int_0^t |p(\cdot, \sigma) - \bar{p}(\cdot, \sigma)|_{L^1} d\sigma
\end{aligned}
$$

so that (2.19) follows from the Gronwall inequality. ∎

We omit here the discussion of the regularity of the solution to problem (1.2); this depends on the regularity of p_0 and on how the rates depend on the variables x_i: in the following, unless explicitely noted, we will always consider the generalized solution given by Theorem 2.3.

3 The search for equilibria

We now consider problem (1.2) with the assumptions (1.3)-(1.7) and look for equilibria, i.e. for stationary solutions of the form $p(a, t) = v(a)$. Namely, such solutions must satisfy the system:

$$
(3.1) \qquad \begin{cases} v_a(a) + \mu(a, V_1, \ldots, V_n)v(a) = 0 \\ v(0) = \int_0^{a_\dagger} \beta(a, V_1, \ldots, V_n)v(a)da \\ V_i = \int_0^{a_\dagger} \gamma_i(a)v(a)da \end{cases}
$$

that at least has the trivial solution $v(a) \equiv 0$. Nontrivial solutions of this problem can be found as follows: the first of the equations in (3.1) yields:

$$
(3.2) \qquad v(a) = v(0)\, e^{-\int_0^a \mu(\sigma, V_1, \ldots, V_n)d\sigma} = v(0)\Pi(a, 0, a; V)
$$

where $V \equiv (V_1, \ldots, V_n)$; then, setting $\Pi(a; V) = \Pi(a, 0, a; V)$, we plug (3.2) into the other equations:

$$v(0) = v(0) \int_0^{a_\dagger} \beta(\sigma, V_1,\ldots, V_n)\, \Pi(\sigma; V) d\sigma$$

$$V_i = v(0) \int_0^{a_\dagger} \gamma_i(\sigma)\, \Pi(\sigma; V) da$$

and get the following system on the V_i's:

$$\int_0^{a_\dagger} \beta(\sigma, V_1,\ldots, V_n)\, \Pi(\sigma; V) d\sigma = 1 \tag{3.3}$$

$$\frac{V_1}{\int_0^{a_\dagger} \gamma_1(\sigma)\Pi(\sigma; V)d\sigma} = \frac{V_2}{\int_0^{a_\dagger} \gamma_2(\sigma)\Pi(\sigma; V)d\sigma} = \cdots = \frac{V_n}{\int_0^{a_\dagger} \gamma_n(\sigma)\Pi(\sigma; V)d\sigma} \tag{3.4}$$

Thus we see that any set $(V_1,\ldots, V_n)$ which solves (3.3)-(3.4) determines a solution of (3.1) via formula (3.2) where $v(0) > 0$ is given by

$$v(0) = \frac{V_i}{\int_0^{a_\dagger} \gamma_i(\sigma)\Pi(\sigma; V)d\sigma} \tag{3.5}$$

As a consequence we can state the following:

Theorem 3.1. *Let (1.3)-(1.7) be satisfied, then $v(a)$ is a nontrivial stationary solution of problem (1.2) if and only if it has the form (3.2), where $V \equiv (V_1,\ldots, V_n)$ satisfies (3.3), (3.4) and $v(0)$ is given by (3.5).* ■

Equations (3.3)-(3.4) are the main tool to investigate existence of equilibria, we can understand that they may provide a large variety of situations such that existence of multiple equilibria occurs. In the following sections we will consider some simple examples that show how this can happen, depending on the vital parameters of the population.

Concerning condition (3.3) we note that

$$(3.6)\qquad R(V_1,\ldots,V_n) = \int_0^{a_\dagger} \beta(\sigma, V_1,\ldots,V_n)\,\Pi(\sigma; V)d\sigma$$

is the net growth rate at the constant sizes $V_1,\ldots, V_n$; thus, condition (3.3) means that at the equilibrium, the net rate must be equal to 1 (see I.5.17).

4 The Allee-logistic model with a single size

First we consider the modeling of a population with vital rates depending on the single variable

$$(4.1)\qquad S(t) = \int_0^{a_\dagger} \gamma(\sigma)p(\sigma, t)d\sigma$$

In this case we have the system:.

$$(4.2)\qquad \begin{cases} p_t(a, t) + p_a(a,t) + \mu(a, S(t))p(a, t) = 0 \\ p(0, t) = \int_0^{a_\dagger} \beta(\sigma, S(t))p(\sigma, t)d\sigma \\ p(a, 0) = p_0(a) \end{cases}$$

and the search for equilibria leads to analyze only equation (3.3) for the net reproduction rate:

$$(4.3)\qquad R(V) = \int_0^{a_\dagger} \beta(a, V)e^{-\int_0^a \mu(\sigma, V)d\sigma}da$$

whose behavior as a function of V depends on the mechanism of growth. As a matter of fact, in order to model the growth we can introduce constitutive assumptions, directly on $R(V)$; namely, a realistic behaviour of the net rate versus V is described by the following assumptions:

$$(4.4)\qquad \begin{cases} i) \quad R'(V) > 0 & \text{if } 0 < V < V_0 \\ ii) \quad R'(V) < 0 & \text{if } V > V_0 \\ iii) \quad \lim_{V\to+\infty} R(V) = 0 & \end{cases}$$

where $V_0 \geqslant 0$ (if $V_0 = 0$, only *ii)* is meaningful). These are the standard assumptions on the single population growth, including both the *Allee effect* and the *logistic effect* that we have mentioned at the beginning of this chapter.

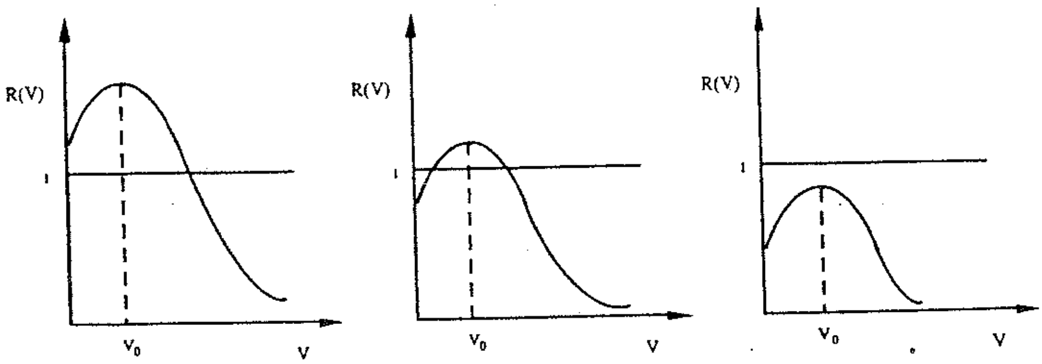

Figure 4.1

With respect to equation (3.3), the assumption (4.4) leads to different possible conclusions depending on the values $R(0)$ and $R(V_0)$, in fact, when $V_0 > 0$, we have (see figure 4.1):

(4.5)	*Exactly one nontrivial solution*	*if* $R(0) > 1$.
	Exactly two nontrivial solutions	*if* $R(0) < 1$ *and* $R(V_0) > 1$.
	No nontrivial solution	*if* $R(V_0) < 1$.

If, instead, $V_0 = 0$ we are in the *purely logistic* case and we have:

(4.6)	*Exactly one nontrivial solution*	*if* $R(0) > 1$.
	No nontrivial solution	*if* $R(0) \leqslant 1$.

We note that the behaviour of $R(V)$, described in (4.4) is accomplished, for instance, if we assume that $\beta(a, V)$ and $\mu(a, V)$ are, respectively, increasing and decreasing for $0 < V < V_0$ and, conversely, respectively decreasing and increasing for $V > V_0$.

As a concrete example we consider the following form for the vital rates:

$$(4.7) \qquad \beta(a, S) = \beta_0(a)\, e^{\varepsilon S\left(1 - \frac{S}{K}\right)}, \quad \mu(a, S) = \mu_0(a)$$

where K is a parameter and $\beta_0(\cdot)$, $\mu_0(\cdot)$ play the role of *intrinsic* vital rates satisfying the usual conditions (I.2.6)-(I.2.8). Assuming (4.7) we have:

$$R(V) = R_0 \, e^{\varepsilon V\left(1-\frac{V}{K}\right)}$$

with

$$(4.8) \qquad R_0 = \int_0^{a_\dagger} \beta_0(a)\, \Pi_0(a) da$$

where we have set

$$\Pi_0(a) = e^{-\int_0^a \mu_0(\sigma) d\sigma}$$

Thus $R(V)$ satisfies (4.4) with $V_0 = \dfrac{K}{2}$ and a solution V^* of (3.3) must satisfy

$$\varepsilon V^* \left(1 - \frac{V^*}{K}\right) = -ln(R_0)$$

so that we have:

$$(4.9) \qquad \begin{array}{ll} \textit{One nontrivial solution} & \textit{if } R_0 > 1 \\ \textit{Two nontrivial solutions} & \textit{if } e^{-\frac{\varepsilon K}{4}} < R_0 < 1 \\ \textit{No nontrivial solution} & \textit{if } 0 < R_0 < e^{-\frac{\varepsilon K}{4}} \end{array}$$

The relative bifurcation graph is shown in figure 4.2 where the size V^* of the equilibrium is plotted versus R_0.

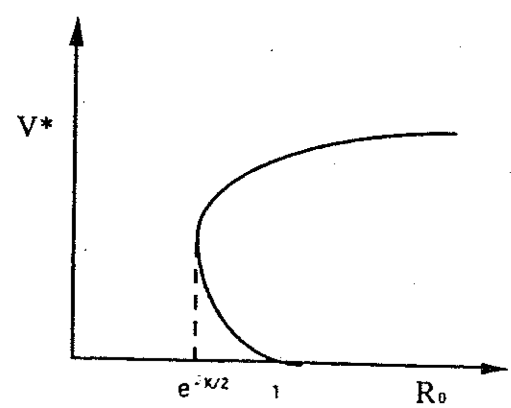

Figure 4.2

In correspondence with each equilibrium size V^* we have the stationary solution

$$p^*(a) = V^* \frac{\Pi_0(a)}{\int_0^{a_\dagger} \gamma(\sigma)\Pi_0(\sigma)d\sigma}$$

Now we might be interested in analyzing how equilibria change as the vital rates vary in some significant way: for instance we introduce a parameter $m > 0$ and change the intrinsic fertility into:

$$(4.10) \qquad \beta^m(a) = \begin{cases} \beta_0(a - m) & \text{if } m < a < a_\dagger \\ 0 & \text{if } 0 < a < m \end{cases}$$

This parameter can be interpreted as the *maturation age*; in fact, if we suppose that for some $\eta > 0$

$$(4.11) \qquad \beta_0(a) > 0 \qquad a.e. \text{ on } [0, \eta]$$

m is viewed as the age at which individuals become fertile. Now we discuss equilibria as m increases, i.e. as $\beta_0(\cdot)$ translates to the right; to this purpose we note that the function

$$(4.12) \qquad R_0(m) = \int_m^{a_\dagger} \beta_0(a - m)\, \Pi_0(a)\, da$$

is decreasing with respect to m and $R_0(a_\dagger) = 0$, so that, defining m_0 and m_1 as follows:

$$m_0 \begin{cases} = 0 & \text{if } R_0(0) \leqslant 1 \\ = \text{the (unique) solution of } R_0(m) = 1 & \text{if } R_0(0) > 1 \end{cases}$$

$$m_1 \begin{cases} = 0 & \text{if } R_0(0)e^{\frac{\varepsilon K}{4}} \leqslant 1 \\ = \text{the (unique) solution of } R_0(m) = e^{\frac{-\varepsilon K}{4}} & \text{if } R_0(0)e^{\frac{\varepsilon K}{4}} > 1 \end{cases}$$

we have one single equilibrium if $0 \leqslant m < m_0$, two equilibria if $m_0 \leqslant m < m_1$, no equilibria if $m > m_1$ (see figure 4.3). Of course, some of the previous claims can be void if either $m_0 = 0$ $m_1 > 0$ or $m_0 = m_1$.

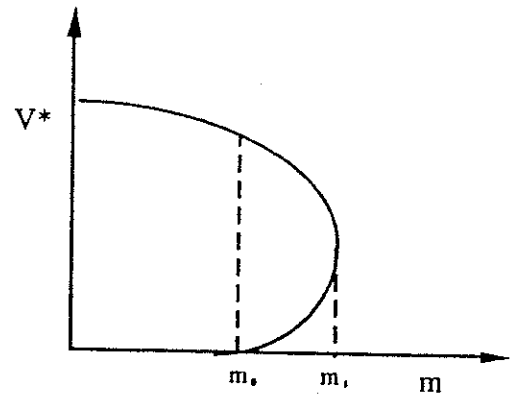

Figure 4.3

Concerning single size models we mention the following further example:

(4.13) $$\beta(a, S) = \beta_0(a), \quad \mu(a, S) = \mu_0(a) + \mu_1(a)S$$

where we suppose again that $\beta_0(\cdot)$, $\mu_0(\cdot)$ satisfy the usual conditions (I.2.6)-(I.2.8) and

(4.14) $$\mu_1(\cdot) \in L^1(0, a_\dagger), \quad \mu_1(a) \geq 0 \text{ a.e. in } [0, a_\dagger]$$

Moreover we assume:

(4.15) $$meas\ (\{a | \beta_0(a) > 0\} \cap [a_\mu, a_\dagger]) > 0$$

where

$$a_\mu = sup\ \{a | \mu_1(a) = 0 \text{ a.e. in } [0, a]\}$$

In this case we have

(4.16) $$R(V) = \int_0^{a_\dagger} \beta_0(a) \Pi_0(a) e^{-M(a)V}\, da$$

where

(4.17) $$M(a) = \int_0^a \mu_1(\sigma)\, d\sigma$$

and we see that, thanks to (4.15)

$$(4.18) \qquad \int_0^{a_\dagger} \beta_0(a) M(a)\, da > 0$$

so that $R(V)$ is decreasing and we have a purely logistic model (see (4.6)) with existence of a unique non trivial equilibrium if and only if the following condition is satisfied

$$(4.19) \qquad R(0) = \int_0^{a_\dagger} \beta_0(a) \Pi_0(a)\, da > 1 .$$

When this nontrivial equilibrium does exists, the stationary solution has the form

$$(4.20) \qquad p^*(a) = V^* \frac{\Pi_0(a) e^{-M(a)V^*}}{\int_0^{a_\dagger} \gamma(a) \Pi_0(a) e^{-M(a)V^*}\, da}$$

The mechanism of growth occurring in this last example, which falls within the purely logistic framework, is produced by a direct increase of mortality, proportional to the population size. This mechanism can be interpreted as the presence of *cannibalism*: in this case the weights $\gamma(a)$ and $\mu_1(a)$, respectively, have the role of selecting cannibals and cannibalized individuals.

5 Two size models

As a first example we consider a population whose vital rates depend on the two variables

$$(5.1) \qquad S(t) = \int_A^{a_\dagger} p(a, t) da, \quad P(t) = \int_0^{a_\dagger} p(a, t) da .$$

The first variable $S(t)$ selects adult individuals, the second one takes the total population. We assume the following forms for β and μ:

$$(5.2) \qquad \beta(a, S, P) = \beta_0(a)\, e^{S\left(1 - \frac{P}{K}\right)}, \quad \mu(a, S, P) = \mu_0(a)$$

defined for $0 \leqslant S \leqslant P$, where we suppose that $\beta_0(\cdot)$, $\mu_0(\cdot)$ satisfy (I.2.6)-(I.2.8).

The form of $\beta(a, S, P)$ is similar to that considered in (4.7): here the use of the two variables accounts for the fact that the Allee effect is supposed to depend on the presence of adults, while the logistic effect is considered as due to all individuals without distinction; in fact we have:

$$\frac{\partial \beta}{\partial S}(a, S, P) = \beta(a, S, P)\left(1 - \frac{P}{K}\right) > 0 \quad for \quad P < K$$

$$\frac{\partial \beta}{\partial P}(a, S, P) = -\beta(a, S, P)\,\frac{S}{K} < 0 \quad for \quad S > 0$$

With the form (5.2) equilibria (S^*, P^*) must satisfy the equations:

$$R_0\, e^{S^*\left(1-\frac{P^*}{K}\right)} = 1 \tag{5.3}$$

$$\frac{S^*}{\int_A^{a_\dagger} e^{-\int_0^a \mu_0(\sigma)d\sigma}da} = \frac{P^*}{\int_0^{a_\dagger} e^{-\int_0^a \mu_0(\sigma)d\sigma}da} \tag{5.4}$$

where R_0 is still given by (4.8). Now (5.3)-(5.4) lead to

$$P^*\left(1-\frac{P^*}{K}\right) = -ln(R_0)\,\frac{\int_0^{a_\dagger} e^{-\int_0^a \mu_0(\sigma)d\sigma}da}{\int_A^{a_\dagger} e^{-\int_0^a \mu_0(\sigma)d\sigma}da}$$

and we can have two, one, or no solutions according to the value of the right hand size. Precisely, we have that if $R_0 \geqslant 1$ there is only one solution and, if $R_0 < 1$ we can discuss existence of equilibria versus the parameter A: in fact we have the situation that is shown in the bifurcation graph shown in figure 5.1.

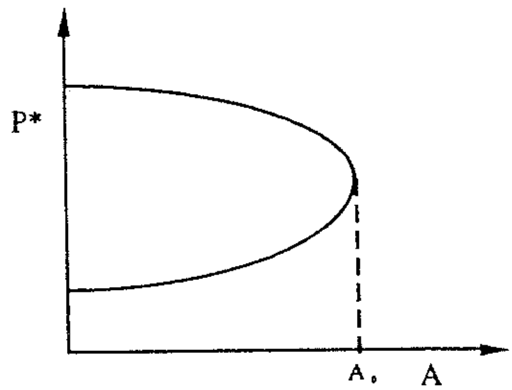

Figure 5.1

We also consider a two size version of the cannibalism model assuming

$$\beta(a,\ K,\ C)=\beta_0(a), \quad \mu(a,\ K,\ C)=\mu_0(a)+\mu_1(a)\ \frac{K}{1+C} \tag{5.5}$$

where $\beta_0(\cdot)$, $\mu_0(\cdot)$ are still the intrinsic rates satisfying (I.2.6)-(I.2.8) and $\mu_1(a)$ fulfills (4.14) and (4.15). Concerning the sizes

$$K(t) = \int_0^{a_\dagger} k(a)p(a,t)da, \quad C(t) = \int_0^{a_\dagger} c(a)Q(a)p(a,t)da \tag{5.6}$$

they respectively represent the size of *cannibals* and that of *potential victims*: in their definition the weights $k(a)$, $c(a)$, $Q(a)$ respectively represent *cannibalistic activity*, *attack rate* and *handling time* as functions of age. We also note that in the definition of $\mu(a,\ K,\ C)$ the term $\dfrac{1}{1+C}$ gives a limit to possible predation.

With these assumptions an equilibrium $(K^*,\ C^*)$ must satisfy

$$R(K^*,\ C^*) = \int_0^{a_\dagger} \beta_0(a)\Pi_0(a)e^{-M(a)\frac{K^*}{1+C^*}}da = 1 \tag{5.7}$$

and

$$\frac{K^*}{\int_0^{a_\dagger} k(a)\Pi_0(a)e^{-M(a)\frac{K^*}{1+C^*}}da} = \frac{C^*}{\int_0^{a_\dagger} c(a)Q(a)\Pi_0(a)e^{-M(a)\frac{K^*}{1+C^*}}da} \tag{5.8}$$

where $M(a)$ is defined as in (4.16).

Now we consider the following function

$$\Phi(\lambda) = \int_0^{a_\dagger} \beta_0(a)\Pi_0(a)e^{-M(a)\lambda}\, da, \quad \lambda \in \mathbf{R}$$

which, thanks to (4.16), is strictly decreasing and $\phi(-\infty) = +\infty$, $\phi(+\infty) = 0$. Thus there exists one and only one solution λ^* of the equation

$$\Phi(\lambda) = 1$$

As a consequence, once that λ^* is found, any equilibrium (K^*, C^*) is determined by the following equations

$$\frac{K^*}{1 + C^*} = \lambda^* \tag{5.9}$$

$$\frac{K^*}{\int_0^{a_\dagger} k(a)\Pi_0(a)e^{-M(a)\lambda^*}da} = \frac{C^*}{\int_0^{a_\dagger} c(a)Q(a)\Pi_0(a)e^{-M(a)\lambda^*}da} \tag{5.10}$$

that is, by the equation

$$\frac{C^*}{1 + C^*} = \lambda^* \frac{\int_0^{a_\dagger} k(a)\Pi_0(a)e^{-M(a)\lambda^*}da}{\int_0^{a_\dagger} c(a)Q(a)\Pi_0(a)e^{-M(a)\lambda^*}da} \tag{5.11}$$

Thus we see that a unique nontrivial equilibrium exists if and only if the right hand side of (5.11) is positive and less than 1.

6 Comments and references

The first attention to non-linear models was paid by Gurtin and McCamy in [46], where they considered a general model with the rates depending on the total population. Since then many different versions of the model have been considered, for general purposes and for specific modeling as well. The version that we have presented in section 1 was essentially the same as the Gurtin McCamy one, extended to consider many size dependence; the proof of existence and uniqueness that we have given in section 2 tries to be general enough to be possibly adopted in other versions of the model.

Specific models have been the subject of several papers, mostly with the purpose of analyzing the behavior of the solutions. We will go through such kind of problems in the following chapters; presently we have been concerned with the preliminary problem of the search for equilibria. The general procedure that we have presented for this latter is also essentially contained in [46]. The examples that we have discussed are of basic importance for the modeling of a single population. In particular the model for cannibalism has attracted the interest of several authors: the version that we have considered is contained in [31] and we refer to this paper for an extended discussion of the modeling aspects (see also [38]).

A few monographies have appeared on non linear structured models. An abstract formulation of the nonlinear case has been developed in [101], with the methods of monotonic operators; an extended treatment of both the modeling aspects and the mathematical methods has been provided in [79]. In this latter text modeling goes beyond the particular case of age structure, considering a more general parameter (size structured models) which evolves with time is some assigned way possibly depending on all the variables of the population: a typical case is the modeling of cell growth.

IV

Stability of equilibria

In this chapter we will treat the stability of the equilibria whose existence was previously discussed. The main tool for this analysis is the characteristic equation that we will derive in Section 2 within the framework of the theory of Volterra integral equations, using the result of this theory as they are presented in Appendix II.

However we must remark that by this approach we are able to state only a sufficient condition for stability while the complete result provides also a condition for instability. The problem is that a proof of the complete result needs a functional analytic setting that cannot be presented here, thus we must limit ourselves to state the condition and use it through the models that we discuss as examples. Some comments on this points are provided in Section 7.

1 Definitions and assumptions

Here we investigate the behavior of the solutions to problem (III.1.2) when initial data are close to equilibria. In particular, we will analyze stability, according to the following definition:

Definition 1.1. *The stationary solution $p^*(\cdot)$ is said to be stable if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that, if $p_0(\cdot)$ satisfies:*

$$|p_0 - p^*|_{L^1} \leqslant \delta$$

then the corresponding solution $p(\cdot, t)$ satisfies:

$$|p(\cdot, t) - p^*(\cdot)|_{L^1} \leqslant \varepsilon \quad \forall\, t \geqslant 0\,.$$

It is said asymptotically stable if it is stable and δ can be chosen such that

$$\lim_{t\to\infty} |p(\cdot,\, t) - p^*(\cdot)|_{L^1} = 0\,.$$

Finally, it is said to be unstable if it is not stable. ∎

Of course, from the point of view of population theory, the behavior of solutions close to the trivial equilibrium is of some special interest because it is related to the problem of sustained growth or extinction.

In the following sections we will be concerned with the analysis of the stability of equilibria, via a linearization procedure of the integral equation (III.2.3). In order to perform this procedure we first need to suppose that, in addition to the main assumptions (III.1.3)-(III.1.7) introduced in the previous chapter, the basic parameters β *and* μ satisfy the following technical conditions:

(1.1) $$\beta(a,x_1,\ldots,x_n)=\beta(a,x_1^0,\ldots x_n^0)+\sum_{i=1}^{n} D_i\beta(a,x_1^0,\ldots,x_n^0)(x_i-x_i^0)+ + R_\beta(a,x^0,x)$$

(1.2) $$\mu(a,x_1,\ldots,x_n)=\mu(a,x_1^0,\ldots x_n^0)+\sum_{i=1}^{n} D_i\mu(a,x_1^0,\ldots,x_n^0)(x_i-x_i^0)+ + R_\mu(a,x^0,x)$$

where R_β and R_μ satisfy:

(1.3) $$|R_\beta(a,x^0,x)| + |R_\mu(a,x^0,x)| \leq M(x - x^0)$$

with $M(\cdot) : \mathbf{R}^n \to \mathbf{R}$ such that:

(1.4) $$\lim_{x\to 0} \frac{M(x)}{|x|} = 0$$

We will assume these conditions throughout the chapter.

2 The basic characteristic equation

The main tool that we use in the stability analysis, is the characteristic equation that arises in the linearization procedure of a system of integral equations originated by our problem (III.1.2). Actually, we consider this system in order to use the theory sketched in Appendix II and finally get results concerning (III.1.2).

First we transform problem (III.1.2) considering the following set of $(n + 1)$ variables:

$$b(t) = p(0, t), \quad S_i(t) \quad i = 1, \ldots, n$$

Namely, we use formulas (III.2.15)-(III.2.16); in fact we start with:

$$p(a, t) = \begin{cases} p_0(a - t)\Pi(a, t, t; S) & \text{if } a \geq t \\ b(t - a)\Pi(a, t, a; S) & \text{if } a < t \end{cases} \tag{2.1}$$

and plug it into:

$$b(t) = \int_0^{a_\dagger} \beta(a, S_1(t), \ldots, S_n(t))p(a, t)da$$

$$S_i(t) = \int_0^{a_\dagger} \gamma_i(a)p(a, t)da$$

getting to the system:

$$\begin{cases} b(t) = \displaystyle\int_0^t K(t, t - \sigma; S)b(\sigma)d\sigma + F(t; S) \\ S_i(t) = \displaystyle\int_0^t H_i(t, t - \sigma; S)b(\sigma)d\sigma + G_i(t, S) \end{cases} \tag{2.2}$$

where $K(t, \sigma; S)$, $F(t; S)$ are defined as in (III.2.4), (III.2.5) and:

$$H_i(t, \sigma; S) = \gamma_i(\sigma)\,\Pi(\sigma, t, \sigma; S)$$

$$G_i(t; S) = \int_t^\infty \gamma_i(a)\Pi(a, t, t; S)\, p_0(a - t)da =$$

$$= \int_0^{+\infty} \gamma_i(a + t)\, \Pi(a + t, t, t; S)\, p_0(a)da\,.$$

Here all the functions are extended as zero outside of $[0, a_\dagger]$.

$$a_{00} = a_{ij} = 0 \quad \forall\, i \neq 0$$

$$a_{0j} = v^* \int_0^{a_\dagger} D_j\beta(\sigma; V_1^*, \ldots, V_n^*)\Pi(\sigma; V^*)d\sigma \quad \forall\, j = 1, \ldots, n$$

(2.7) $$A_{00}(\sigma) = \begin{cases} \beta(\sigma, V_1^*, \ldots, V_n^*)\Pi(\sigma; V^*) & \text{for } \sigma \in [0, a_\dagger] \\ 0 & \text{for } \sigma > a_\dagger \end{cases}$$

$$A_{i0}(\sigma) = \begin{cases} \gamma_i(\sigma)\Pi(\sigma; V^*) & \text{for } \sigma \in [0, a_\dagger] \\ 0 & \text{for } \sigma > a_\dagger \end{cases} \qquad \forall\, i = 1, \ldots, n$$

$$A_{ij}(\sigma) = -v^* \int_0^{a_\dagger} D_j\mu(s, V_1^*, \ldots, V_n^*)\, A_{i0}(\sigma+s)ds \quad \forall\, j \neq 0$$

and each of the non linear functional terms $\mathcal{P}_i$:

$$(U_0(\cdot), \ldots, U_n(\cdot), q_0(\cdot)) \rightarrow \mathcal{P}_i[U_0(\cdot), \ldots, U_n(\cdot), q_0(\cdot)](\cdot)$$

maps $C_0\,([0, +\infty]; \mathbf{R}^{n+1}) \times L^1\,([0, a_\dagger]; \mathbf{R})$ into $C_0\,([0, +\infty]; \mathbf{R})$, satisfying the conditions (3.2)-(3.4) stated in the Appendix II. Thus the system (2.6) is under the form of equation (3.1) of the Appendix II and we are led to study the characteristic equation:

(2.8) $$\det\,((\delta_{ij} - a_{ij} - \hat{A}_{ij}(\lambda))) = 0$$

As a matter of fact, this equation is the tool for the study of the stability of the constant solution $(v^*, V_1^*, \ldots, V_n^*)$ to problem (2.3), but it also allows to study the stability of the corresponding solution to the original problem: this point will be discussed in the next section.

3 Stability and instability

The characteristic equation (2.8), introduced in the previous section, is the main tool to investigate stability of equilibria of problem (III.1.2). First we apply the theory of Appendix II and get the following result:

Theorem 3.1. *Let $p^*(a) = v^*\Pi(a; V^*)$ be a stationary solution of (III.1.2). Then, if the corresponding characteristic equation (2.8) has only roots with negative real part, $p^*(\cdot)$ is asymptotically stable.*

Now it is easy to show that (2.2) has the following limiting system:

$$(2.3)\qquad \begin{cases} v(t) = \int_0^\infty K(t, \sigma; V)v(t-\sigma)d\sigma \\ V_i(t) = \int_0^\infty H_i(t, \sigma; V)v(t-\sigma)d\sigma \end{cases}$$

and that the search for a nontrivial constant solution $(v^*, V_1^*, \ldots, V^*_n)$ of this system, leads to the equations

$$(2.4)\qquad \begin{cases} 1 = \int_0^\infty \beta(\sigma, V_1^*, \ldots, V_n^*)\Pi(\sigma; V^*)d\sigma \\ V_i^* = v^* \int_0^\infty \gamma_i(\sigma)(\sigma; V^*)d\sigma \end{cases}$$

where $\Pi(a; V)$ is the same as in section III.3. As it might be expected, (2.4) corresponds to the systems (III.3.3)-(III.3.4): note that here v^* plays the same role as $v(0)$; in fact, by (2.1), from any nontrivial solution $(v^*, V_1^*, \ldots, V_n^*)$ of (2.4) we get the stationary solution:

$$(2.5)\qquad p^*(a) = v^*\Pi(a; V^*)$$

while the trivial solution $p^* \equiv 0$ corresponds to $v^* = 0$, $V_i^* = 0$.

Then we linearize (2.2) at constant solutions of (2.3) (we include also the trivial solution $v^* = 0$, $V_i^* = 0$); actually, we set:

$$U_0(t) = b(t) - v^*$$
$$U_i(t) = S_i(t) - V_i^* \qquad i = 1, \ldots, n$$
$$q_0(a) = p_0(a) - p^*(a)$$

and we get the system:

$$(2.6)\qquad U_i(t) = \sum_{j=0}^{n} \left[a_{ij}\, U_j(t) + \int_0^t A_{ij}(t-\sigma)U_j(\sigma)d\sigma \right] + \\ + \mathcal{P}_i[U_0(\cdot), \ldots, U_n(\cdot); q_0(\cdot)]\,(t), \quad i = 0, \ldots, n$$

where:

Proof:

Let $(v^*, V_1^*, \ldots, V_n^*)$ be the constant solution of (2.3) corresponding to $p^*(\cdot)$, and consider the linearization (2.6). Then, by Theorem 3.1 of Appendix II, for any $\varepsilon > 0$ there exists $\eta > 0$ such that

(3.1) $$|p_0 - p^*|_{L^1} = |q_0|_{L^1} \leq \eta$$

implies

(3.2) $$|U_i(t)| \leq \varepsilon \quad \forall\, t \geq 0, \quad \lim_{t\to\infty} U_i(t) = 0$$

Now, if (3.1) is satisfied, by (2.1) we have

(3.3) $$\sup_{t\in[0,a_\dagger]} |p(\cdot,t)-p^*(\cdot)|_{L^1} \leq |p_0-p^*|_{L^1}+2a_\dagger[1+a_\dagger v^* H(M)] \sum_{i=0}^{n} \sup_{t\geq 0} |U_i(t)|$$

and, for $t > a_\dagger$,

(3.4) $$|p(\cdot,t)-p^*(\cdot)|_{L^1} \leq [1+a_\dagger v^* H(M)] \sum_{i=0}^{n} \int_{t-a_\dagger}^{t} |U_i(\sigma)|\, d\sigma$$

with $M > \varepsilon + \sum_{i=0}^{n} V_i^*$. These estimates imply

$$|p(\cdot,t) - p^*(\cdot)|_{L^1} \leq \eta + 2(n + 1)a_\dagger[1 + a_\dagger v^* H(M)]\, \varepsilon$$

and

$$\lim_{t\to\infty} p(\cdot,t) = p^*(\cdot) \quad in \quad L^1(0, a_\dagger)$$

so that, since ε is arbitrary and η can be chosen arbitrarily small, the thesis is proved.

■

The result stated in the previous theorem is actually incomplete because the location of the roots of the characteristic equation provides a condition for instability. In fact, in addition to Theorem 3.1 we have

Theorem 3.2. *Let $p^*(a) = v^*\Pi(a; V)$ be a stationary solution of (III.1.2). Then, if the corresponding characteristic equation (2.8) has a root with positive real part, $p^*(\cdot)$ is unstable*

■

However, the proof of this result is not directly feasible by the theory of integral equations that we have sketched in Appendix II, but we would rather need to work within the functional analytic framework of infinite dimensional dynamical systems (see [29]). Thus we must omit the proof of this result though it is of great importance in the analysis of the models and will be used in the following sections.

We can soon have a first general result concerning the trivial equilibrium:

Proposition 3.3. *Let the net rate (III.3.6) satisfy* $R(0,\ldots,0) < 1$. *Then the trivial equilibrium* $p^*(a) \equiv 0$ *is asymptotically stable. If instead* $R(0,\ldots;0) > 1$, *then* $p^*(a) \equiv 0$ *is unstable.*

Proof:

We first note that for the trivial solution it is $v^* = 0$ so that equation (2.8) reduces to

$$(3.5) \qquad \hat{A}_{00}(\lambda) = \int_0^\infty e^{-\lambda t}\, \beta(\sigma, 0,\ldots, 0)\, e^{-\int_0^\sigma \mu(a,0,\ldots,0)da}\, d\sigma = 1$$

Then, since $A_{00}(\sigma) \geqslant 0$, by the argument used in Theorem I.5.1 we have that, if $\hat{A}_{00}(0) < 1$, all the roots of this equation have negative real part; if $\hat{A}_{00}(0) > 1$ there is at least one real positive root. Since $\hat{A}_{00}(0) = R(0,\ldots, 0)$, the thesis follows. ■

The result of this proposition concerning the trivial equilibrium is quite simple and we see that in this case the net rate is the key parameter for stability. Thus we are left with the analysis of how the net rate changes as a function of the significant constants of the model. When considering non trivial equilibria the characteristic equation may be more complicated than (3.5) and we need to perform some study on the location of its roots; moreover we are interested in looking at how this location changes when some significant parameters of the model vary. The next section is devoted to a preliminary study of some basic results that can be used in the discussion of the specific models.

4 Some results about the characteristic equation

In view of the applications to the study of stability, we consider the following equation in the complex plane

$$(4.1) \qquad \hat{K}_0(\lambda) + F(\lambda, \tau) = 1$$

where $K_0(\cdot) : [0, +\infty] \to \mathbf{R}$ is such that

(4.2) $$K_0(t) \geqslant 0, \quad K_0(t) = 0 \quad t > T, \quad \int_0^\infty K_0(t)\, dt = 1$$

and $F(\lambda, \tau) : \mathbf{C} \times \mathbf{R} \to \mathbf{C}$ satisfies

(4.3) *$F(\lambda, \tau)$ is continuously differentiable on* $\mathbf{C} \times \mathbf{R}$

(4.4) $$F(\lambda, 0) = 0 \quad \forall \lambda \in \mathbf{C}, \quad \frac{\partial F}{\partial \tau}(0, 0) > 0$$

(4.5) *There exist $M > 0$, $\beta < 0$ such that*

$$|F(\lambda, \tau)| < M|\tau| \quad \text{for } \Re\lambda \geqslant \beta \quad \text{and for } \tau \text{ sufficiently small}.$$

We are interested in the location of the roots of (4.1) with respect to the imaginary axis. We note that in Theorem I.5.1 we have essentially treated the particular case $\tau = 0$; now we have

Proposition 4.1. *There exists δ such that, if $\tau \in [0, \delta]$, equation (4.1) has a real positive root; if $\tau \in [-\delta, 0]$, all the roots of (4.1) have negative real part.*

Proof:

We first recall that the equation

(4.6) $$\hat{K}_0(\lambda) = 1$$

has the real root $\lambda_0 = 0$ which is the unique one in the half plane $\Re\lambda \geqslant \alpha$ for some $\alpha \in (\beta, 0)$. Besides we set:

$$m = \inf_{y \in \mathbf{R}} |1 - \hat{K}_0(\alpha + iy)| > 0$$

and take $L > 0$ such that

$$\frac{1}{2} < |1 - \hat{K}_0(\lambda)| \quad \text{for } |\lambda| > L, \quad \Re\lambda \geqslant \alpha$$

Then, if τ is sufficiently small and such that

$$|\tau| < \frac{\left(m \wedge \frac{1}{2}\right)}{M}$$

we have

$$|F(\lambda, \tau)| < |1 - \hat{K}_0(\lambda)|$$

on the contour of any domain Σ_ϱ such as that shown in figure 3.1, with $\varrho > L$. Consequently, by the Rouché theorem, equation (4.1) has one and only one root in the half plane $\mathcal{R}\lambda \geqslant \alpha$.

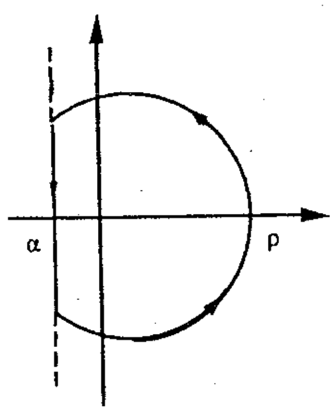

Figure 4.1

In order to locate this root, let $\lambda(\tau)$, be the differentiable path in the complex plane, originating from $\lambda(0) = 0$, such that $\lambda(\tau)$ is a root of (4.1). Then from (4.1)

$$\left.\frac{d\lambda}{d\tau}\right|_{\tau=0} = \frac{\frac{\partial F}{\partial \tau}(0, 0)}{\int_0^\infty tK_0(t)dt} > 0$$

and we see that the path starting from $\lambda(0) = 0$ goes to the right of the imaginary axis as τ *increases from 0*, while it goes to the left if τ decreases. ■

The previous result is only local and we are interested in the crossing of the roots through the immaginary axis, thus we define:

(4.7) $\tau_- = inf\ \{\delta |\ for\ \tau \in [\delta, 0)\ any\ root\ of\ (4.1)\ has\ negative\ real\ part\}$

(4.8) $\tau_+ = sup\ \{\delta |\ for\ \tau \in (0, \delta]\ at\ least\ one\ root\ of\ (4.1)\ has\ positive\ real\ part\}$

As a particular case of (4.1) we consider the equation:

(4.9) $$\hat{K}_0(\lambda) + \tau \hat{K}_1(\lambda) = 1$$

where $K_1(\cdot)$ satisfies

(4.10) $$K_1(t) \geqslant 0, \quad K_1(t) = 0 \quad t > T, \quad \int_0^\infty K_1(t)\, dt = 1$$

With respect to positive values of τ we have

Proposition 4.2. *If $\tau > 0$, equation (4.9) has a real positive root, consequently $\tau_+ = +\infty$.*

Proof:

First we set

$$L(t) = K_0(t) + \tau K_1(t)$$

and write (4.9) as

$$\hat{L}(\lambda) = 1$$

Now, since by the assumptions we have $L(t) \geqslant 0$ and $\hat{L}(0) = 1 + \tau$, proceeding as in the proof of Theorem I.5.1 we see that this equation has a unique real root which is positive. ■

Concerning τ_- the following result states a sufficient condition in order that the roots of (4.9) have negative real part for any $\tau < 0$:

Proposition 4.3. *Let $K_1(t)$ be such that:*

$$(4.11) \qquad \int_0^\infty K_1(\sigma)\cos(\omega\sigma)\,d\sigma \geq 0 \qquad \forall\ \omega \in \mathbf{R}$$

then $\tau_- = -\infty$.

Proof:

Suppose, by contraddiction, that $\tau_- > -\infty$; then, in correspondence with τ_- equation (4.9) must have a purely immaginary root $\lambda_- = i\,\omega$ because, otherwise, proceeding as in the proof of Proposition 4.1, we could prove that the roots of (4.9) have negative real part also for τ in some interval $[\delta, \tau_-]$. Then, from (4.9):

$$1 = \int_0^\infty K_0(\sigma)\cos(\omega\sigma)\,d\sigma + \tau_- \int_0^\infty K_1(\sigma)\cos(\omega\sigma)\,d\sigma$$

and this is impossible because, if $\omega = 0$, it implies

$$1 = 1 + \tau_- < 1$$

if, instead, $\omega > 0$, by (4.11)

$$1 \leq \int_0^\infty K_0(\sigma)\cos(\omega\sigma)\,d\sigma < 1 \qquad \blacksquare$$

Condition (4.11) is verified in some particular cases such as the following one:

Proposition 4.4. *Let $K_1(\cdot) \in C^2[0, +\infty]$ be such that:*

$$(4.12) \qquad K'_1(t) \leq 0, \quad K''_1(t) \geq 0$$

then condition (4.11) is satisfied.

Proof:

Note that $K_1(t) = K'_1(t) = 0$ for $t \geq T$; then integrating by parts we have:

$$\int_0^\infty K_1(\sigma)\cos(\omega\sigma)\,d\sigma = \left[\frac{1}{\omega}K_1(\sigma)\sin(\omega\sigma)\right]_{\sigma=0}^{\sigma=T} -$$

$$-\frac{1}{\omega}\int_0^\infty K'_1(\sigma)\sin(\omega\sigma)d\sigma =$$

$$= \left[\frac{1}{\omega^2}K'_1(\sigma)\cos(\omega\sigma)\right]_{\sigma=0}^{\sigma=T} - \frac{1}{\omega^2}\int_0^\infty K''_1(\sigma)\cos(\omega\sigma)d\sigma =$$

$$= \frac{1}{\omega^2}\left[-K'_1(0) - \int_0^\infty K''_1(\sigma)\cos(\omega\sigma)d\sigma\right] =$$

$$= \frac{1}{\omega^2}\int_0^\infty K''_1(\sigma)\,[1-\cos(\omega\sigma)]d\sigma \geq 0 \qquad \blacksquare$$

However, condition (4.11) is rather restrictive and in general we have $\tau_- > -\infty$: the following example is rather special, but somewhat realistic in the applications:

$$(4.13) \qquad \begin{cases} K_0(t)=K_1(t)=\dfrac{\pi}{2T}\sin\left(\dfrac{\pi t}{T}\right) & \text{for } t\in[0,\ T] \\ K_0(t)=K_1(t)=0 & \text{for } t\in[T,\ +\infty] \end{cases}$$

To discuss this case we first note that, for the unique reason that K_0 and K_1 coincide, we have $\tau_- \leq -2$; in fact (4.9) becomes

$$(4.14) \qquad (1+\tau)\,\hat{K}_0(\lambda) = 1$$

and, if $\Re\lambda \geq 0$ and $\tau \in (-2,\ 0)$, we have

$$|(1+\tau)\,\hat{K}_0(\lambda)| < 1$$

so that λ cannot be a root of (4.9).

Next we look for those values of τ that allow for imaginary roots of equation (4.9). This equation, with the special choice of (4.13), becomes

$$\frac{1+\tau}{2}\quad\frac{1+e^{-\lambda T}}{1+\left(\dfrac{\lambda T}{\pi}\right)^2} = 1$$

Actually, setting $\lambda = i\omega$, we get the equivalent system

$$\begin{cases} \sin(\omega T) = 0 \\ \dfrac{1+\tau}{2} \dfrac{1+\cos(\omega T)}{1-\left(\dfrac{\omega T}{\pi}\right)^2} = 1 \end{cases}$$

which has solutions if and only if $\tau = -4k^2$ $(k = 1, 2, \ldots)$ with the corresponding roots $\lambda_k = \pm \dfrac{2k\pi}{T} i$.

Thus we have $\tau_- = -4$; moreover we can also see that at any $\tau = -4k^2$ two roots actually cross the imaginary axis to the right when τ decreases: in fact it is easy to check that

$$\frac{d}{d\tau} Re\lambda \bigg|_{\tau=-4k^2} < 0$$

so that, for $\tau < -4$, there are at least two roots with positive real part.

5 Back to the Allee-logistic model

We now go back to the Allee-logistic model that we have considered in (III.4.1)-(III.4.2) and apply the results stated in the previous section to analyze the stability of equilibria. We will consider some specific cases and will discuss how stability changes depending on some significant parameters. As a main assumption we will consider S-independent mortality:

(5.1) $\qquad \mu(a, S) = \mu_0(a)$.

In this case, in correspondence with a non trivial equilibrium V^*, the linearization procedure gives (see (2.7))

$$a_{00} = a_{10} = a_{11} = 0, \quad a_{01} = \frac{V^* R'(V^*)}{\int_0^{a_\dagger} \gamma(s)\Pi_0(s)ds}$$

(5.2)
$$A_{00}(\sigma) = \begin{cases} \beta(\sigma, V^*)\Pi_0(\sigma) & \text{for } \sigma \in [0, a_\dagger] \\ 0 & \text{for } \sigma > a_\dagger \end{cases}$$

$$A_{10}(\sigma) = \begin{cases} \gamma(\sigma)\Pi_0(\sigma) & \text{for } \sigma \in [0, a_\dagger] \\ 0 & \text{for } \sigma > a_\dagger \end{cases}$$

$$A_{01}(\sigma) \equiv A_{11}(\sigma) \equiv 0$$

where we have set

$$\Pi_0(t) = e^{-\int_0^t \mu_0(\sigma)d\sigma}.$$

Thus the characteristic equation has the form (4.9) with:

(5.3) $$K_0(t) = A_{00}(t), \quad K_1(t) = \frac{A_{10}(t)}{\int_0^{a_\dagger} \gamma(s)\Pi_0(s)ds}$$

(5.4) $$\tau = V^* R'(V^*)$$

Now, under the assumption (III.4.4) we have that, if there exists a unique nontrivial equilibrium V^*, it follows that

$$R(0) > 1, \quad R'(V^*) < 0$$

and, if there are two equilibria $V_1^* < V_2^*$, it is

$$R(0) < 1, \quad R'(V_1^*) > 0, \quad R'(V_2^*) < 0 .$$

Thus, using also Proposition 3.3, we get the following preliminary result:

Proposition 5.1. *Consider the Allee-logistic model (III.4.1)-(III.4.2) and let (III.4.4) and (5.1) be satisfied; then:*

(5.5) *If there is no nontrivial equilibrium, the trivial one is stable;*

(5.6) *if there is only one nontrivial equilibrium, the trivial one is unstable;*

(5.7) *if there are two nontrivial equilibria, one of them is unstable and the trivial one is stable* ∎

In this Proposition the nature of the equilibrium V_2^* for which $R'(V_2^*) < 0$ is still undecided, but the result can be implemented if, in addition to (5.1), we assume a more specific form for $\beta(a, S)$. In this direction, we consider the *purely logistic model* that we rephrase with the following special assumptions:

$$\beta(a, S) = R_0\beta_0(a)\phi(S), \quad \mu(a, S) = \mu_0(a) \tag{5.8}$$

so that we have the system

$$\left\{\begin{aligned} & p_t(a, t) + p_a(a, t) + \mu_0(a)p(a, t) = 0 \\ & p(0, t) = R_0\phi(S(t)) \int_0^{a_\dagger} \beta_0(\sigma)p(\sigma, t)d\sigma \\ & p(a, 0) = p_0(a) \\ & S(t) = \int_0^{a_\dagger} \gamma(\sigma)p(\sigma, t)d\sigma \end{aligned}\right. \tag{5.9}$$

where we assume that $\phi : [0, +\infty) \to (0, +\infty)$ satisfies:

$$\phi(0) = 1, \quad \phi'(x) < 0, \quad \lim_{x \to +\infty} \phi(x) = 0 \tag{5.10}$$

and

$$\int_0^{a_\dagger} \beta_0(a)\Pi_0(a)\, da = 1 \tag{5.11}$$

With these assumptions, the net rate is given by

$$R(V) = R_0\phi(V) \tag{5.12}$$

and it is decreasing so that, concerning existence of equilibria, we have (see (III.4.6)):

$$\left\{\begin{aligned} & \textit{If } R_0 \leq 1 \textit{ there exists no non-trivial equilibrium} \\ & \textit{If } R_0 > 1 \textit{ there exists one and only one non-trivial equilibrium} \end{aligned}\right. \tag{5.13}$$

Proposition 5.2. *Consider the logistic model (5.8)-(5.11) and moreover assume that*

(5.17) *the function* $\gamma(a)\Pi_0(a)$ *is non increasing and convex.*

Then the nontrivial equilibrium is stable for all $R_0 > 1$. ■

The proof of this result is a direct consequence of Proposition 4.4 which implies $\tau_- = -\infty$. We note that (5.17) is satisfied if:

(5.18) $$\gamma(a) \equiv 1, \quad \mu_0'(a) \leq \mu_0^2(a)$$

Then we consider special kernels based on the example (4.13), for which $\tau_- = -4$, and special cases of ϕ. We have

Proposition 5.3. *Consider the logistic model (5.8)-(5.11) and, moreover, assume that*

(5.19) $$\phi(x) = e^{-x},\ \beta_0(a) = \gamma(a),\ \beta_0(a)\Pi_0(a) = \frac{\pi}{2a_\dagger} \sin\left(\frac{\pi a}{a_\dagger}\right).$$

Then $R_0^* = e^4$. ■

In fact, for $\phi(x) = e^{-x}$ we have $\tau(R_0) = -ln(R_0)$. Moreover

Proposition 5.4. *Consider the logistic model (5.8)-(5.11) and moreover assume that*

(5.20) $$\phi(x) = \frac{1}{1+x},\ \beta_0(a) = \gamma(a),\quad \beta_0(a)\Pi_0(a) = \frac{\pi}{2a_\dagger} \sin\left(\frac{\pi a}{a_\dagger}\right).$$

Then the nontrivial equilibrium is stable for all $R_0 > 1$.

This latter result follows because for $\phi(x) = \frac{1}{1+x}$ we have $\tau(R_0) = \frac{1}{R_0} - 1$.

We note that assumption $\beta_0(a) = \gamma(a)$ is equivalent to assuming $S(t) = B(t)$, i.e. to assuming that the vital parameters depend on the birth rate.

We also consider the cannibalism model (III.4.13)-(III.4.15) defined at the end of Section III.4, setting

First we note that by Proposition 3.3 we have

(5.14) *The trivial equilibrium is stable if* $R_0 < 1$, *unstable if* $R_0 > 1$

Then we consider the case $R_0 > 1$ and perform the linearization at the nontrivial equilibrium $V^* = \phi^{-1}\left(\frac{1}{R_0}\right)$. In this case, in (5.2) a_{01} and A_{00} take the form:

$$a_{01} = \frac{R_0\phi^{-1}\left(\frac{1}{R_0}\right)\ \phi'\left(\phi^{-1}\left(\frac{1}{R_0}\right)\right)}{\int_0^{a_\dagger} \gamma(s)\Pi_0(s)ds}$$

$$A_{00}(\sigma) = \begin{cases} \beta_0(\sigma)\ \Pi_0(\sigma) & \text{for } \sigma \in [0, a_\dagger] \\ 0 & \text{for } \sigma > a_\dagger \end{cases}$$

and, for the characteristic equation in the form (4.9), we have

(5.15) $$K_0(t) = A_{00}(t), \quad K_1(t) = \frac{A_{10}(t)}{\int_0^{a_\dagger} \gamma(s)\Pi_0(s)ds}$$

(5.16) $$\tau = \tau(R_0) = R_0\phi^{-1}\left(\frac{1}{R_0}\right)\ \phi'\left(\phi^{-1}\left(\frac{1}{R_0}\right)\right)$$

We see that both K_0 and K_1 stay unchanged as R_0 varies. Then in the characteristic equation (4.9) τ is the only variable that depends on R_0 and we can trace the stability of the nontrivial equilibrium, as R_0 ranges from 1 to $+\infty$, by looking at the range of τ as a function of R_0.

In particular we note that τ is negative and $\tau(1) = 0$. Thus if R_0 is close to 1 the nontrivial equilibrium is stable, and the problem is to identify the maximal interval $[1, R_0^*]$ prior to the crossing of some root through the imaginary axis, i.e. to find R_0 such that $\tau_- = \tau(R_0^*)$.

Of course R_0^* will depend both on the function ϕ and on the nature of the kernels K_0 and K_1. The following results are based on some special assumptions which allow a full knowledge of the situation.

(5.21) $$\beta(a, S) = \beta_0(a), \quad \mu(a, S) = \mu_0(a) + \mu_1(a)k\phi(S)$$

where we have introduced the parameter k and the function $\phi(\cdot)$ which we suppose increasing.

Now we define the function

$$\Gamma(x) = \int_0^{a_\dagger} \beta_0(a)\Pi_0(a)e^{-M(a)x}da$$

which is decreasing and

$$\Gamma(0) = R_0 = \int_0^{a_\dagger} \beta_0(a)\Pi_0(a)da, \quad \Gamma(+\infty) = 0$$

so that, if $R_0 > 1$, there is one and only one x^* such that $\Gamma(x^*) = 1$. Then we note that

$$R(V) = \Gamma(k\phi(V))$$

so that the unique non trivial equilibrium size V^* is given by

$$V^* = \phi^{-1}\left(\frac{x^*}{k}\right).$$

Concerning the characteristic equation at this equilibrium size we have

$$a_{ij} = 0 \qquad \text{for all } i \text{ and } j$$

$$A_{00}(\sigma) = \begin{cases} \beta_0(\sigma)\Pi_0(\sigma)e^{-M(\sigma)x^*} & \text{for } \sigma \in [0, a_\dagger] \\ 0 & \text{for } \sigma > a_\dagger \end{cases}$$

$$A_{10}(\sigma) = \begin{cases} \gamma(\sigma)\Pi_0(\sigma)e^{-M(\sigma)x^*} & \text{for } \sigma \in [0, a_\dagger] \\ 0 & \text{for } \sigma > a_\dagger \end{cases}$$

$$A_{01}(\sigma) = \tau A^0_{01}(\sigma); \quad A_{11}(\sigma) = \tau A^0_{11}(\sigma)$$

where

$$\tau = -\frac{k\phi^{-1}\left(\frac{x^*}{k}\right)\phi'\left(\phi^{-1}\left(\frac{x^*}{k}\right)\right)}{\int_0^{a_\dagger} \beta_0(\sigma)\Pi_0(\sigma)e^{-M(\sigma)x^*}d\sigma}$$

and

$$A^0_{01}(\sigma) = \int_0^{a_\dagger} \mu_1(s)A_{00}(s+\sigma)ds, \quad A^0_{11}(\sigma) = \int_0^{a_\dagger} \mu_1(s)A_{10}(s+\sigma)ds$$

Thus the characteristic equation takes the form (see (4.1))

$$\hat{K}_0(\lambda) + \tau F(\lambda) = 1$$

with

$$K_0(\sigma)=A_{00}(\sigma), \quad F(\lambda)= \left[\hat{A}^0_{11}(\lambda)+\hat{A}_{00}(\lambda)\hat{A}^0_{11}(\lambda)+\hat{A}_{10}(\lambda)\hat{A}^0_{01}(\lambda)\right]$$

We see that, since x^* depends only on β_0, μ_0, μ_1, both $K_0(t)$ and $F(\lambda)$ do not depend on the parameter k nor on the function $\phi(\cdot)$; then we can discuss stability versus these latters, via the parameter τ given in (5.22), using the results of Section 4 (note that the assumptions (4.2)-(4.5) are satisfied).

6 Bifurcations

In the previous section we have seen examples for which the stationary state looses its stability when, at certain values of some varying parameter, a couple of roots of the characteristic equation crosses the imaginary axis to the right. When this occurs a periodic solution is generated and we have Hopf bifurcation for our model.

A rigorous treatment of this matter would again involve concepts and methods that we cannot use here (see [31] for a reference) so that we just mention the possibility of bifurcations and present a simulated example to show evidence of the phenomenon.

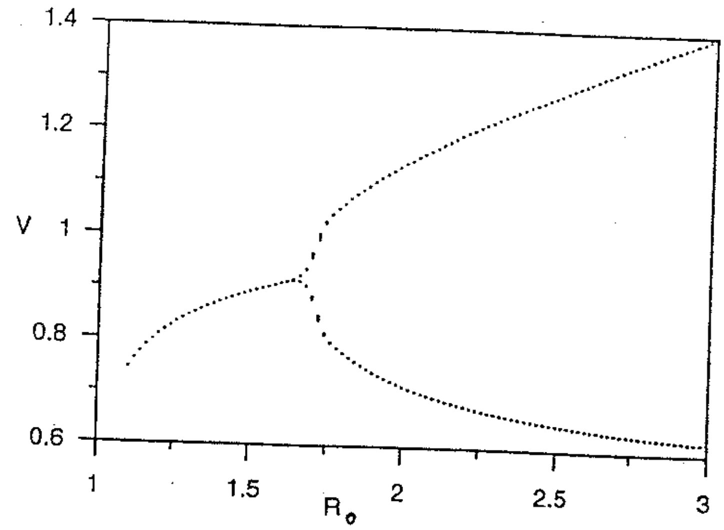

Figure 6.1

In Figures 6.1 and 6.2 we show the simulations for the logistic model (5.9) with the following choices:

(6.1) $$\mu_0(a)= \frac{1}{\pi-a},\ \beta_0(a)\Pi_0(a)= \frac{1}{2}\sin a,\ \phi(x)=e^{-x^{\alpha}},\ \gamma(a)=\beta_0(a)$$

where $a_\dagger = \pi$ and α is a fixed parameter.

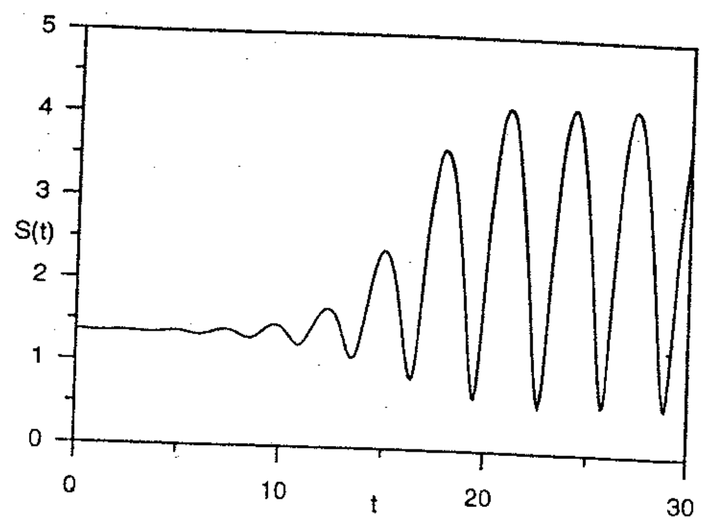

Figure 6.2

Precisely, in Figure 6.1 we can see the bifurcation graph where the size of the stationary state and the amplitude of the bifurcating solution are reported as R_0 varies (the bifurcation occurs at $R_0 = e^{\frac{4}{\alpha}}$). In Figure 6.2 the behavior of a solution is shown at a value of R_0 such that a stable periodic solution exists: we see that the solution is attracted by this periodic solution. Note that the period is approximately equal to $a_\dagger = \pi$.

7 Comments and references

One of the crucial points in modeling a population is of course the local stability of equilibria: in the previous sections we have seen that age structure produces various phenomena in the behavior of a single population, including existence of periodic solutions that bifurcate from an equilibrium becoming unstable.

Thus age-structure is responsible for behaviors that the models for a single homogeneous population are not able to produce. This fact was soon pointed out by Gurtin and MacCamy in their paper [48], by the use of special cases that can be reduced to a system of ordinary differential equations. The analysis performed in Section 5 is then representative of what can happen with age structure.

Stability of equilibria has been considered in many papers ([46]-[49], [52], [53], [82]-[84], [90]); the example (4.13) comes from [83] and the cannibalism model is inspired in [31] (see also [38]) where the characteristic equation is studied for a simplified version of the model described in Section III.5.

In Section 6 we have considered bifurcation of periodic solutions: in this matter we have to give up again and refer to the theory of dynamical systems. The numerical simulations presented in Figures 6.1 and 6.2 show rather well what happens; these have been performed by a discretization scheme proposed in [35], [81].

V

Global behavior

This chapter is focused on the study of the global behavior of the nonlinear model discussed in the previous chapters. Generally speaking, the problem of determining the global behavior of a system is not systematically settled but, usually, any approach takes advantage of some special property presented by the system and rests upon metodologies that provide sufficient conditions for the analysis of the behavior.

Concerning our model, the most usual technology is to take advantage of some specific feature that allows a reduction to those classes of equations for which known methods are available. In this respect, the most natural way to approach the problem is to look at the system of Volterra equations that originate from the model and use methods drawn for the theory of these equations, but more satisfactory results can be drawn for those particular cases that allow reduction to O.D.E.

1 A general approach to a special class of models

Here we consider the single size model of section III.4 and IV.5, with the special assumption of size-independent mortality:

$$(1.1) \qquad \mu(a, S) = \mu_0(a)$$

Namely, we are dealing with the particular case

$$(1.2) \qquad \begin{cases} p_t(a, t) + p_a(a, t) + \mu_0(a)p(a, t) = 0 \\ p(0, t) = \int_0^{a_\dagger} B(\sigma, S(t))p(\sigma, t)d\sigma \\ p(a, 0) = p_0(a) \\ S(t) = \int_0^{a_\dagger} \gamma(\sigma)p(\sigma, t)d\sigma \end{cases}$$

and with the net reproduction rate

$$(1.3) \qquad R(V) = \int_0^{a_\dagger} \beta(a, V) e^{-\int_0^a \mu_0(\sigma)d\sigma} da$$

We have already discussed local stability of equilibria via the equivalent system (IV.2.2) that in the present case becomes:

$$(1.4) \qquad \begin{cases} b(t) = \int_0^t K(a, S(t))b(t-a)da + F(t, S(t)) \\ S(t) = \int_0^t H(a)b(t-a)da + G(t) \end{cases}$$

where we have set:

$$K(a,x) = \beta(a,x)\Pi_0(a), \quad F(t,x) = \int_t^\infty K(a,x) \frac{p_0(a-t)}{\Pi_0(a-t)} da$$

$$H(a) = \gamma(a)\Pi_0(a), \quad G(t) = \int_t^\infty H(a) \frac{p_0(a-t)}{\Pi_0(a-t)} da$$

and

$$\Pi_0(a) = e^{-\int_0^a \mu_0(\sigma)d\sigma}$$

Now we consider the problem of finding conditions in order that the solution be attracted by a given equilibrium. To this purpose, for a fixed equilibrium size $V^* \geq 0$ we consider the following function defined for $(a, x) \in [0, a_\dagger] \times [0, +\infty)$:

$$(1.5) \qquad L_{V^*}(a,x) = \begin{cases} K(a, x) + H(a)\left(\frac{R(x)-1}{x-V^*}\right) v^* & \text{for } x \neq V^* \\ K(a,V^*)+H(a)R'(V^*)v^* & \text{for } x = V^* \end{cases}$$

where we recall that $V^* = v^* \int_0^{a_\dagger} H(a)da$. We also recall that the corresponding equilibrium solution is given by $p^*(a) = v^*\Pi_0(a)$ and note that:

$$b(t) - v^* = \int_0^t L_{V^*}(a,S(t))(b(t-a) - v^*)da + \tag{1.6}$$
$$+ \int_t^\infty L_{V^*}(a,S(t)) \left(\frac{p_0(a-t)-p^*(a-t)}{\Pi_0(a-t)} \right) da$$

$$S(t) - V^* = \int_0^t H(a)(b(t-a) - v^*)da + \tag{1.7}$$
$$+ \int_t^\infty H(a) \left(\frac{p_0(a-t)-p^*(a-t)}{\Pi_0(a-t)} \right) da$$

Then we define the continuous function

$$L_{V^*}(x) = \int_0^\infty |L_{V^*}(a, x)|\, da \qquad x \in [0, +\infty] \tag{1.8}$$

and give the following preliminary result:

Proposition 1.1. *Let $b(t)$ and $S(t)$ be the solution of (1.4) and suppose that:*

$$\lambda = \sup_{t \geqslant T} L_{V^*}(S(t)) < 1 \tag{1.9}$$

for some $T \geqslant 0$. Then

$$\lim_{t \to +\infty} b(t) = v^*, \quad \lim_{t \to +\infty} S(t) = V^* \tag{1.10}$$

Proof:

Let:

$$I_n = [na_\dagger, (n+1)a_\dagger], \quad M_n = \max_{t \in I_n} |b(t) - v^*|$$

Then, by (1.6), for $t \in I_{n+1}$ and $n > \dfrac{T}{a_\dagger}$:

$$|b(t) - v^*| \leqslant \lambda\, (M_{n+1} \vee M_n)$$

that is

$$M_{n+1} \leqslant \lambda \, (M_{n+1} \vee M_n)$$

which, since $\lambda < 1$, yields

$$M_{n+1} \leqslant \lambda \, M_n$$

From this, by induction we get:

$$M_n \leqslant \lambda^{n-N} M_N$$

where $n > \dfrac{T}{a_\dagger}$; thus

$$\lim_{n \to +\infty} M_n = 0$$

and the thesis follows. ■

Concerning Problem (1.2) we then have

Corollary 1.2. *Let $p^*(a) = v^* \, \Pi_0(a)$ be an equilibrium solution to (1.2). Then, if $S(t)$ satisfies (1.9), we have*

$$\lim_{t \to +\infty} |p(t,\cdot) - p^*(\cdot)|_{L^\infty} = 0$$ ■

The previous result can be used to estimate the initial datum p_0 in order that the solution be attracted by a stationary state, we have:

Proposition 1.3. *Let $r > 0$ be such that*

(1.11) $\qquad L_{V^*}(x) < 1 \qquad \text{for } x \in (V^* - r, V^* + r)$

then if $0 \leqslant \delta < r$ and

(1.12) $\qquad |p_0(a) - p^*(a)| \leqslant \dfrac{\delta \, \Pi_0(a)}{\int_0^\infty H(a)da}$

we have

(1.13) $\qquad \lim_{t \to +\infty} |p(\cdot,t) - p^*(\cdot)|_{L^\infty} = 0$

Proof:

First we prove that, if $\bar{\delta} \in (\delta, r)$, then we have:

$$(1.14) \qquad |S(t) - V^*| \leq \bar{\delta} \quad \forall\, t \geq 0$$

In fact, since

$$|S(0)-V^*| = \left| \int_0^\infty H(a) \left(\frac{p_0(a-t)-p^*(a-t)}{\Pi_0(a-t)} \right) da \right| \leq \delta \leq \bar{\delta},$$

if (1.14) is not true there exists $t_0 > 0$ such that

$$|S(t_0)-V^*| = \bar{\delta}, \quad |S(t)-V^*| < \bar{\delta} \quad \text{for } t \in [0, t_0)$$

Now, if $0 < T < t_0$, from (1.6) we have $\forall\, t \in [0, T]$

$$|b(t)-v^*| \leq L_{V^*}(S(t)) \left\{ \left(\max_{a\in[0,T]} |b(a)-v^*| \right) \vee \left(\frac{\delta}{\int_0^\infty H(a)da} \right) \right\}$$

and, consequently:

$$(1.15) \qquad \max_{a\in[0,T]} |b(a)-v^*| \leq \varrho \left\{ \left(\max_{a\in[0,T]} |b(a)-v^*| \right) \vee \left(\frac{\delta}{\int_0^\infty H(a)da} \right) \right\}$$

where

$$(1.16) \qquad \varrho = \max_{x\in[V^*-\bar{\delta},V^*+\bar{\delta}]} L_{V^*}(x) < 1$$

Now (1.15) and (1.16) imply that

$$(1.17) \qquad |b(t)-v^*| \leq \frac{\delta}{\int_0^\infty H(a)da} \quad \forall\, t \in [0, T]$$

and, since T is arbitrary, this estimate is true for all $t \in [0, t_0]$. In conclusion, plugging (1.17) into (1.7) we get:

$$\bar{\delta} = |S(t_0) - V^*| \leq \delta < \bar{\delta}$$

which is impossible.

Once (1.14) is proved the thesis follows from proposition 1.1 because (see (1.16)):

$$\lambda = \sup_{t \geq 0} L_{V'} (S(t)) \leq \varrho < 1. \quad ■$$

The results of this section are rather general, but can be easily used to investigate special cases of the model. In the next section we will consider one of these cases.

2 The purely logistic model

We now consider again the purely logistic model of section IV.5. Namely, we examine the global behavior of problem IV.5.9 with the assumption (IV.5.10)-(IV.5.11).

First we must take care of the following condition on the initial datum p_0 (see I.5.14)

(2.1) $\quad$ *For all* $t \geq 0 \quad \beta_0(a)p_0(a+t) = 0 \quad$ *a.e. for* $a \in [0, a_\dagger]$

in fact, if this is satisfied, in (1.4) we have:

$$F(t, S(t)) = 0 \quad \textit{for all } t \geq 0$$

and, consequently:

$$b(t) = 0 \textit{ for all } t \geq 0.$$

An initial datum satisfying (2.1) is said to be a *trivial initial datum*. In fact we have:

Proposition 2.1. *Let (IV.5.10)-(IV.5.11) be satisfied and suppose that* p_0 *is a trivial initial datum, then the corresponding solution satisfies:*

(2.2) $\quad p(t, a) = 0 \qquad$ *for* $t > a_\dagger$ $\quad$ ■

Further we have:

Theorem 2.2. *Let (IV.5.10)-(IV.5.11) be satisfied and suppose $R_0 < 1$, then the trivial equilibrium is globally attractive, i.e.*

(2.3) $$\lim_{t \to +\infty} |p(t,\cdot)|_{L^\infty} = 0 \qquad \forall\, p_0 \in L^1(0, a_\dagger)$$

Proof:

Since we have

$$L_0(x) = R_0\phi(x) < 1 \qquad \forall x \geqslant 0$$

then condition (1.9) is satisfied and the thesis follows from Corollary 1.2. ■

In order to treat the case $R_0 > 1$, when there exists one and only one nontrivial equilibrium, we have to make some more assumptions, namely:

(2.4) $$\beta_0(a) > 0 \qquad a.e. \text{ in } [a_1, a_2]$$

(2.5) $$\not{R}_0\beta_0(a) = \frac{\gamma(a)}{\int_0^{a_\dagger} \gamma(\sigma)\Pi_0(\sigma)d\sigma} \qquad a.e. \text{ in } [0, a_\dagger]$$

(2.6) *The function $x \to x\phi(x)$ is non-decreasing*

The last two assumptions have the following direct consequence:

Lemma 2.3. *Let (IV.5.10), (IV.5.11), (2.5)-(2.6) be satisfied. Suppose $R_0 > 1$, and let V^* be the non-trivial equilibrium size. Then*

(2.7) $$L_{V^*}(x) < 1 \quad \text{for } x > 0$$

Proof:

From (2.5) it necessarily follows:

$$L_{V^*}(a,x) = R_0\beta_0(a)\Pi_0(a)\phi(x) + v^*\gamma(a)\Pi_0(a)\,\frac{R_0\phi(x) - R_0\phi(V^*)}{x - V^*} =$$

$$= R_0\beta_0(a)\Pi_0(a)\left(\phi(x) + V^*\,\frac{\phi(x) - \phi(V^*)}{x - V^*}\right) =$$

$$=R_0\beta_0(a)\Pi_0(a)\left(\frac{x\phi(x)-V^*\phi(V^*)}{x-V^*}\right)\geqslant 0$$

and consequently, since $\phi(\cdot)$ is decreasing, for $x>0$ we have:

$$L_{V^*}(x) = \int_0^{a_\dagger} L_{V^*}(a,x)\,da =$$

$$= R(x) + V^*\,\frac{R(x)-1}{x-V^*} = \frac{xR(x)-V^*R(V^*)}{x-V^*} < 1 \qquad \blacksquare$$

Moreover we have

Lemma 2.4. *Let (IV.5.10), (IV.5.11), (2.4)-(2.6) be satisfied and suppose $R_0>1$. If p_0 is non-trivial, then*

$$\liminf_{t\to+\infty} S(t) > 0 \tag{2.8}$$

Proof:

We consider the system (1.4) and prove that:

$$\liminf_{t\to+\infty} b(t) > 0 \tag{2.9}$$

from which (2.8) follows via (IV.2.1).

Since p_0 is non-trivial $F(t,\,S(t))$ is not identically vanishing and so is $b(t)$. Thus let $0\leqslant\alpha<\beta$ be such that

$$b(t)>0 \quad \text{for } t\in[\alpha,\,\beta]$$

then for $t\in[\alpha+a_1,\,\beta+a_2]$, by (2.4) we have

$$b(t)\geqslant R_0\phi(S(t))\int_0^t \beta_0(t-a)\Pi_0(t-a)b(a)da\geqslant$$

$$\geqslant R_0\phi(S(t))\min_{a\in[\alpha,\beta]} b(a)\int_\alpha^{t\wedge\beta}\beta_0(t-a)\Pi_0(t-a)da=$$

$$\geqslant R_0\phi(S(t))\min_{a\in[\alpha,\beta]} b(a)\int_{0\vee(t-\beta)}^{t-\alpha}\beta_0(a)\Pi_0(a)da>0$$

in fact $(a_1, a_2) \cap (0 \vee (t - \beta), t - \alpha) \neq \emptyset$ and $\phi(S(t)) > 0$.

Iterating this argument we prove that:

$$b(t) > 0 \quad \text{for } t \in [\alpha + na_1, \beta + na_2]$$

for any positive integer n. Finally, since for n sufficiently large we have that

$$\alpha + (n + 1)a_1 < \beta + na_2$$

then, for some t_0:

$$\bigcup_n [\alpha + na_1, \beta + na_2] \supset [t_0, +\infty)$$

and we see that $b(t)$ is eventually positive.

Let now

$$I_n = [na_\dagger, (n + 1)a_\dagger], \quad m_n = \min_{t \in I_n} b(t)$$

and take $n_0 > \dfrac{t_0}{a_\dagger} + 1$. To prove (2.9) we will show that:

$$m_n \geqslant \bar{m} = m_{n_0} \wedge \frac{v^*}{R_0} \quad \text{for } n > n_0 \tag{2.10}$$

For this purpose, we note that, by (2.5):

$$S(t) = R_0 \int_0^{a_\dagger} \gamma(\sigma)\Pi_0(\sigma)d\sigma \int_0^{a_\dagger} \beta_0(a)\Pi_0(a)b(t-a)\, da \quad \forall\, t > a_\dagger$$

so that

$$b(t) = R_0\phi\left(R_0 \int_0^{a_\dagger} \gamma(\sigma)\Pi_0(\sigma)d\sigma \int_0^{a_\dagger} \beta(a)\Pi_0(a)b(t-a)\, da\right)$$

$$\int_0^{a_\dagger} \beta_0(a)\Pi_0(a)b(t-a)\, da.$$

Further, by (2.6), for $t \in I_{n_0+1}$:

$$b(t) \geq R_0 \phi \left(R_0 \int_0^{a_\dagger} \gamma(\sigma) \Pi_0(\sigma) d\sigma (m_{n_0} \wedge m_{n_0+1}) \right) (m_{n_0} \wedge m_{n_0+1})$$

thus

$$m_{n_0+1} \geq R_0 \phi \left(R_0 \int_0^{a_\dagger} \gamma(\sigma) \Pi_0(\sigma) d\sigma (m_{n_0} \wedge m_{n_0+1}) \right) (m_{n_0} \wedge m_{n_0+1})$$

Now, either $m_{n_0+1} \geq \frac{v^*}{R_0} \geq \bar{m}$ or $m_{n_0+1} < \frac{v^*}{R_0}$. In this latter case $(m_{n_0} \wedge m_{n_0+1}) < \frac{v^*}{R_0}$ and

$$R_0 \phi \left(R_0 \int_0^{a_\dagger} \gamma(\sigma) \Pi_0(\sigma) d\sigma (m_{n_0} \wedge m_{n_0+1}) \right) > R_0 \phi(V^*) = 1$$

so that

$$m_{n_0+1} > (m_{n_0} \wedge m_{n_0+1})$$

which implies

$$m_{n_0+1} > m_{n_0} \geq \bar{m}$$

Thus we have proved (2.10) for $n = n_0 + 1$ and iterating the argument we prove if for all $n > n_0$. ∎

In conclusion by Lemma 2.3 and Lemma 2.4 condition (1.9) is satisfied and we have:

Theorem 2.5. *Let (IV.5.10), (IV.5.11), (2.4)-(2.6) be satisfied. Suppose $R_0 > 1$ and let $p^*(\cdot)$ be the non trivial equilibrium. If p_0 is non-trivial, then*

$$\lim_{t \to +\infty} |p(\cdot,t) - p^*(\cdot)|_{L^\infty} = 0 \tag{2.11}$$

∎

We note that (2.5)-(2.6) are special sufficient conditions that could be weakened in particular cases. For instance, with the special choice

$$\phi(x) = \frac{1}{1+x} \tag{2.12}$$

condition (2.5) can be replaced by

$$R_0\beta_0(a) \geqslant v^*\gamma(a) \qquad a.e.\ in\ [0, a_\dagger] \tag{2.13}$$

Actually, Lemma 2.3 and Lemma 2.4 can be proved under these conditions to get the same result of Theorem 2.5

3 Separable models

A special class of models which allows a complete description of the global behaviour of the solution, is characterized by the following assumptions on β and μ:

$$\begin{cases} \beta(a, x_1,\dots, x_n) = \beta_0(a) \\ \mu(a, x_1,\dots, x_n) = \mu_0(a) + M(x_1,\dots, x_n) \end{cases} \tag{3.1}$$

where $\beta_0(\cdot)$ and $\mu_0(\cdot)$ satisfy the basic assumptions (III.1.3)-(III.1.7) and

$$-M\ (x_1,\dots, x_n) \leqslant M^+ < \inf_{a\in[0,a_\dagger]} \mu_0(a).$$

Thus we have the problem:

$$\begin{cases} p_t(a, t)+p_a(a, t)+\mu_0(a)p(a, t)+M(S_1(t),\dots, S_n(t))p(a, t)=0 \\ p(0, t) = \int_0^{a_\dagger} \beta_0(a)p(a, t)da \\ p(a, 0) = p_0(a) \\ S_i(t) = \int_0^{a_\dagger} \gamma_i(a)p(a, t)da \quad i = 1, \dots, n \end{cases} \tag{3.2}$$

The constitutive form (3.1) leads to separate the analysis of age structure from that of the total population size; it can be interpreted as saying that β_0 and μ_0 rule an intrinsic birth-death process which is age dependent, while $M(S_1(t),\dots, S_n(t))$ models an external mortality which is the same for all ages and depends on the weighted sizes $S_i(t) = \int_0^{a_\dagger} \gamma_i(a)p(a, t)da$.

The key for the treatment of problem (3.2) is to consider the age profile $\omega(a, t) = \frac{p(a,t)}{P(t)}$ and recognize that it obeys the same equation as for the linear problem (I.2.5) with parameters β_0 and μ_0. Namely we have that $\omega(a, t)$ satisfies (see (II.1.6))

$$(3.3) \quad \begin{cases} \omega_t(a, t) + \omega_a(a, t) + \mu_0(a)\omega(a, t) + \omega(a, t) \displaystyle\int_0^{a_\dagger} [\beta_0(\sigma) - \\ - \mu_0(\sigma)]\omega(\sigma, t)d\sigma = 0 \\ \omega(0, t) = \displaystyle\int_0^{a_\dagger} \beta_0(a)\omega(a, t)da; \quad \int_0^{a_\dagger} \omega(a, t)da = 1 \\ \omega(a, 0) = \omega_0(a) \qquad \omega(a_\dagger, t) = 0 \end{cases}$$

This system can be checked proceeding as in (II.1), using the fact that the external mortality $M(S_1(t),..., S_n(t))$ does not depend explicitly on the age a: this implies that the evolution of the age profile is not affected by M.

In addition, concerning the total population $P(t) = \int_0^{a_\dagger} p(a, t)da$, it is easy to show that it satisfies the non autonomous problem:

$$(3.4) \quad \begin{cases} \dfrac{d}{dt} P(t) = F(t, P(t)) \\ P(0) = P_0 = \displaystyle\int_0^{a_\dagger} p_0(a)da \end{cases}$$

with:

$$(3.5) \quad F(t, x) = \Big[\alpha(t) - M(\Gamma_1(t)x, ..., \Gamma_n(t)x)\Big] x$$

where:

$$(3.6) \quad \alpha(t) = \int_0^{a_\dagger} [\beta_0(a) - \mu_0(a)]\omega(a, t)da$$

$$(3.7) \quad \Gamma_i(t) = \int_0^{a_\dagger} \gamma_i(a)\omega(a, t)da$$

In fact

$$S_i(t) = \int_0^{a_\dagger} \gamma_i(a)p(a, t)da = \Gamma_i(t)P(t)$$

Now, since the behaviour of $\omega(a, t)$ is completely known by the analysis of section (II.1), we are left with problem (3.4) where we also know that (see II.1)

$$(3.8) \qquad \lim_{t\to+\infty} \alpha(t) = \alpha^* = \int_0^{a_\dagger} [\beta_0(\sigma) - \mu_0(\sigma)]\omega^*(\sigma)d\sigma$$

$$(3.9) \qquad \lim_{t\to+\infty} \Gamma_i(t) = \Gamma_i^* = \int_0^{a_\dagger} \gamma_i(\sigma)\omega^*(a)d\sigma$$

where

$$(3.10) \qquad \omega^*(a) = \frac{e^{-\alpha^* a}\Pi_0(a)}{\int_0^{a_\dagger} e^{-\alpha^*\sigma}\Pi_0(\sigma)d\sigma} = \lim_{t\to\infty} \omega(a, t), \; \Pi_0(a) = e^{-\int_0^a \mu_0(\sigma)d\sigma}$$

and α^* satisfies:

$$(3.11) \qquad 1 = \int_0^{a_\dagger} e^{-\alpha^*\sigma}\beta_0(\sigma)\Pi_0(\sigma)d\sigma$$

As a matter of fact (3.4) has the following limiting equation:

$$(3.12) \qquad \frac{d}{dt} Q(t) = F_\infty (Q(t))$$

with

$$(3.13) \qquad F_\infty(x) = [\alpha^* - M(\Gamma_1^* x, \ldots, \Gamma_n^* x)]\, x = \lim_{t\to+\infty} F(t, x)$$

where the limit is uniform for x in any bounded interval. By this equation, as we will see below, we can determine the asymptotic behaviour of $P(t)$ and finally recover the behaviour of $p(a, t)$, through the formula

$$(3.14) \qquad p(a, t) = P(t)\omega(a, t)$$

We first have to note that equilibria of (3.2) are strictly related to those of (3.12). We have

Proposition 3.1. *Let assumption (3.1) be satisfied, then* Q^* *is a non trivial equilibrium for (3.12) if and only if*

(3.15) $$p^*(a) = Q^*\omega^*(a)$$

is a non trivial equilibrium of (3.2). Moreover letting

(3.16) $$\Lambda^* = -\sum_{i=1}^{n} \Gamma_i^* \frac{\partial M}{\partial x_i} (\Gamma_1^* Q^*, \ldots, \Gamma_n^* Q^*)$$

both Q^* *and* $p^*(a)$ *are asymptotically stable if* $\Lambda^* < 0$*, unstable if* $\Lambda^* > 0$*.*

Proof:

First of all the search for non trivial equilibria for (3.2) gives the equation (see (III.3.3))

(3.17) $$R(V_1^*, \ldots, V_n^*) = \int_0^{a_\dagger} \beta_0(a)\Pi_0(a)e^{-aM(V_1^*,\ldots,V_n^*)}da = 1$$

so that it must be

(3.18) $$M(V_1^*, \ldots, V_n^*) = \alpha^*$$

where α^* is given by (3.11). Moreover, equations (III.3.4) become

$$\frac{V_1^*}{\int_0^{a_\dagger} \gamma_1(a)e^{-a\alpha^*}\Pi_0(a)da} = \ldots = \frac{V_n^*}{\int_0^{a_\dagger} \gamma_n(a)e^{-a\alpha^*}\Pi_0(a)da}$$

and we see that for $i = 1, \ldots, n$

$$V_i^* = v^* \int_0^{a_\dagger} \gamma_i(a)e^{-a\alpha^*}\Pi_0(a)da = v^*\Gamma_i^* \int_0^{a_\dagger} e^{-a\alpha^*}\Pi_0(a)da$$

where v^* is such that $Q^* = v^* \int_0^{a_\dagger} e^{-a\alpha^*}\Pi_0(a)da$ is an equilibrium for (3.12).

Thus the first part of the thesis is proved.

Regarding stability, we note that Λ^*, defined in (3.16), is the key parameter relative to an equilibrium Q^* of problem (3.12); in fact

$$(3.19) \qquad Q^* \Lambda^* = \frac{dF_\infty}{dx}(Q^*)$$

On the other hand the characteristic equation (IV.2.8) for the equilibrium (3.15) takes the following form

$$(3.20) \qquad \frac{1-\hat{A}_{00}(\lambda)}{\lambda}(\lambda + Q^* \Lambda^*)$$

and the thesis is proved. ■

Concerning the global behavior of (3.4) we have

Theorem 3.2. *Assume that the equation (3.12) has exactly $k + 1 \geq 1$ isolated stationary points $0 = Q^*_0 < Q^*_1 < \ldots < Q^*_k < +\infty$, and suppose that*

$$(3.21) \qquad F_\infty(x) < 0 \quad \text{for } x \text{ sufficiently large.}$$

Then we have:

$$(3.22) \qquad \lim_{t \to +\infty} P(t) = Q^*_h$$

for some $h = 0, 1, \ldots, k$. Consequently

$$(3.23) \qquad \lim_{t \to +\infty} \int_0^{a_\dagger} |p(\sigma, t) - p^*_h(\sigma)| d\sigma = 0$$

where

$$(3.24) \qquad p^*_h(a) = Q^*_h \omega^*(a) \quad i = 0, 1, \ldots, k$$

Proof:

Let $P(t)$ be the solution of (3.4); of course $P(t) \geq 0$ and moreover, thanks to (3.21), $P(t)$ is bounded. Let now Ω be the ω-limit set of $P(t)$: we will show that Ω contains a single point. In fact, if this is not true, Ω must contain a whole interval $[A, B]$ such that no point Q_i^* $(i = 0, 1, \ldots, k)$ belongs to it. Consequently $F_\infty(x)$ does not vanish on $[A, B]$ and we can find T_0 such that

$$F(t, x) \neq 0 \quad for\ t > T_0\ and\ x \in [A, B].$$

Suppose for instance (the other case would be similar) that:

$$F(t, x) > 0 \quad for\ t > T_0\ and\ x \in [A, B]$$

and let $T_1 > T_0$ be such that $P(T_1) \in (A, B)$; then it must be:

$$P(t) > P(T_1) > A \quad for\ all \quad t > T_1 \tag{3.25}$$

In fact $P'(t) > 0$ for $t > T_1$ as far as $P(t)$ stays in sufficiently small neighbourhood of $P(T_1)$; thus by (3.22) we have the contradiction that $[A, P(T_1)]$ does not belong to Ω.

Thus, since Ω contains a single point P_∞, and $P(t)$ is bounded, it must be

$$\lim_{t \to +\infty} P(t) = P_\infty$$

and, consequently:

$$F_\infty(P_\infty) = 0$$

That is $P_\infty = Q_h^*$, for some $h = 0, 1, \ldots, k$.

Finally, (3.23) follows from (3.10), (3.14), (3.22). ∎

The previous theorem shows in particular that, within the class of models that we are considering, the total population has an asymptotic behaviour that excludes existence of periodic solutions.

We note that, since $\alpha(t)$ and $\Gamma_i(t)$ depend on the initial age distribution $p_0(a)$, equation (3.4) will also depend on $p_0(a)$ and, in general, if we know only the total initial population P_0, it is not possible to determine which Q_i^* will be attained by $P(t)$. Of course, if in particular $p_0(a) = P_0\omega^*(a)$, then $\alpha(t) \equiv \alpha^*$, $\Gamma_i(t) = \Gamma_i^*$, and equation (3.4) coincides with the limit equation (3.12).

4 The case $a_\dagger = +\infty$

As a final class of reducible models, we consider the case $a_\dagger = +\infty$. In section II.4 we have already considered this case within the context of the linear theory; actually we have shown that special constitutive forms of the vital rates allow reduction of the equations to a system of O.D.E.'s; here we want to use similar assumptions for the non-linear case.

As a first model we consider the case of a single size, assuming:

(4.1) $$\beta(a, x) = \bar{\beta}(x)e^{-\alpha a} > 0, \quad \mu(a, x) = \bar{\mu}(x) > 0, \quad \gamma(a) = 1$$

where $\alpha > 0$, i.e. we consider the model:

(4.2) $$\begin{cases} p_t(a, t)+p_a(a, t)+\bar{\mu}(P(t))p(a, t) = 0 \\ p(0, t) = \bar{\beta}(P(t)) \displaystyle\int_0^{a_\dagger} e^{-\alpha\sigma}p(\sigma, t)d\sigma \\ p(a, 0) = p_0(a) \\ P(t) = \displaystyle\int_0^{a_\dagger} p(\sigma, t)d\sigma \end{cases}$$

These constitutive assumptions describe a very simple phenomenological situation: the fertility function is decreasing with respect to age, while mortality is independent of it; moreover all the rates depend on the total population size.

Now, to perform the reduction we consider the two following variables:

(4.3) $$P(t) = \int_0^{a_\dagger} p(\sigma, t)d\sigma, \quad Q(t) = \int_0^{a_\dagger} e^{-\alpha\sigma}p(\sigma, t)d\sigma$$

Then we have:

$$\frac{d}{dt}P(t) = \int_0^{a_\dagger} p_t(a, t)da = -\int_0^{a_\dagger} p_a(a, t)da - \bar{\mu}(P(t))P(t) =$$

$$= \bar{\beta}(P(t))Q(t) - \bar{\mu}(P(t))P(t)$$

and:

$$\frac{d}{dt} Q(t) = \int_0^{a_\dagger} e^{-\alpha a} p_t(a, t)\, da =$$

$$= - \int_0^{a_\dagger} e^{-\alpha a} p_a(a, t)\, da - \bar{\mu}(P(t)) \int_0^{a_\dagger} e^{-\alpha a} p(a, t)\, da =$$

$$= \bar{\beta}(P(t))Q(t) - \alpha Q(t) - \bar{\mu}(P(t))Q(t)$$

Thus we get the system:

$$(4.4) \qquad \begin{cases} \dfrac{d}{dt} P(t) = \bar{\beta}(P(t))Q(t) - \bar{\mu}(P(t))P(t), & P(0)=P_0 \\ \dfrac{d}{dt} Q(t) = [\bar{\beta}(P(t)) - \alpha - \bar{\mu}(P(t))]\, Q(t), & Q(0)=Q_0 \end{cases}$$

where $P_0 = \int_0^{a_\dagger} p_0(a)\, da$ and $Q_0 = \int_0^{a_\dagger} e^{-\alpha a} p_0(a)\, da$.
This system is equivalent to the original problem because if the couple $(P(t), Q(t))$ solves (4.4), then by setting:

$$b(t) = p(0, t) = \bar{\beta}(P(t))Q(t)$$

we get $p(a, t)$ via the usual formula

$$p(a,t) = \begin{cases} p_0(a - t)e^{-\int_0^t \bar{\mu}(P(\sigma))d\sigma} & \text{if } a \geq t \\ b(t - a)e^{-\int_{t-a}^t \bar{\mu}(P(\sigma))d\sigma} & \text{if } a < t \end{cases}$$

Thus, in order to determine the behavior of the model we can focus on the analysis of (4.4); in particular, if the rates satisfy the assumption considered in the previous sections, the corresponding results can be made more precise and it can be easily shown that existence of periodic solutions occurs when a stationary solution looses its stability. In the following we will develop some general considerations relative to the system (4.4) in order to show its connection with the previous theory.

First we consider the net rate

$$R(V) = \frac{\bar{\beta}(V)}{\alpha+\bar{\mu}(V)}$$

and we see that non-trivial stationary sizes V^* must satisfy:

$$\bar{\beta}(V^*) = \alpha + \bar{\mu}(V^*) \tag{4.5}$$

Moreover, each of the stationary sizes V^* corresponds to an isocline of the system, having the form $P = V^*$. The other isocline is given by

$$Q = \psi(P) = \frac{\bar{\mu}(P)P}{\bar{\beta}(P)} \tag{4.6}$$

so that, for any V^*, we have a stationary point (P^*, Q^*) of (4.4), given by:

$$P^* = V^*, \quad Q^* = \psi(V^*)$$

Further we note that in (4.4) we are interested only in initial data (P_0, Q_0) such that $P_0 > Q_0 > 0$; in this respect we have:

Proposition 4.1. *Let $(P(t), Q(t))$ be the solution of (4.4), relative to the initial datum (P_0, Q_0) such that $P_0 > Q_0 > 0$ then:*

$$P(t) > Q(t) > 0, \quad \forall\, t > 0$$

Proof:

Since $Q_0 > 0$, from the second equation in (4.4) we see that $Q(t) > 0$ for all $t > 0$. Besides, setting $W(t) = P(t) - Q(t)$ we have:

$$\frac{d}{dt} W(t) = -\bar{\mu}(P(t))W(t) + \alpha Q(t), \quad W(0) > 0$$

that implies $W(t) > 0$ for all $t > 0$. ■

Thus we see that, in the phase plane of the system, we are interested only in the open region $\Delta = \{P, Q) | 0 < Q < P\}$, we note that since (4.5) yields $\bar{\beta}(V^*) > \bar{\mu}(V^*)$ then any (P^*, Q^*) belongs to Δ.

We now introduce assumptions on $\bar{\beta}(\cdot)$ and $\bar{\mu}(\cdot)$ so as to consider the purely logistic model of section III.4: namely we want

$$R'(V) < 0 \text{ and } \lim_{V \to +\infty} R(V) = 0$$

The first of these conditions implies:

(4.7) $\bar{\beta}'(V)(\alpha + \bar{\mu}(V)) - \bar{\beta}(V)\bar{\mu}'(V) < 0 \quad \forall\ V \geqslant 0$

and, as for the second condition, we assume that it is fulfilled via the following assumptions

$\bar{\beta}(\cdot)$ *is eventually decreasing and* $\lim_{V \to +\infty} \bar{\beta}(V) = 0$

$\bar{\mu}(\cdot)$ *is eventually increasing and* $\lim_{V \to +\infty} \bar{\mu}(V) > 0$

Note that these assumptions imply that $[\bar{\beta}(\cdot) - \bar{\mu}(\cdot)]$ is eventually negative: it follows that if $P(t)$ is large then $\frac{d}{dt} P(t) < 0$ so that the trajectory remains bounded. Moreover, according to the statement (III.4.6) the system has one and only one non-trivial equilibrium point in the interior of the region Δ if:

$$\bar{\beta}(0) > \alpha + \bar{\mu}(0)$$

otherwise there is only the trivial equilibrium. Let us study the stability of these equilibria. First we consider the trivial one $(0, 0)$ and the Jacobian of the system at this point:

$$J(0, 0) = \begin{pmatrix} -\bar{\mu}(0) & \bar{\beta}(0) \\ 0 & \bar{\beta}(0) - \alpha - \bar{\mu}(0) \end{pmatrix}$$

Thus we see that:

(4.8) *If there exists only the trivial equilibrium, it is a globally attractive node. If there exists also the non-trivial equilibrium the trivial one is a saddle point.*

Concerning the non-trivial equilibrium (P^*, Q^*) a simple calculation shows that the Jacobian at this point has the eigenvalues:

$$\lambda_{\pm} = \frac{1}{2}\left(A \pm \sqrt{A^2 + 4B}\right)$$

where:

$$(4.9) \qquad \begin{cases} A = \tilde{\beta}'(V^*)Q^* - \tilde{\mu}'(V^*)V^* - \tilde{\mu}(V^*) \\ B = \tilde{\beta}(V^*)\,[\tilde{\beta}'(V^*) - \tilde{\mu}'(V^*)]\,Q^* \end{cases}$$

Now, by (4.5) and (4.7) it is always $B < 0$ so that the stability of the point (P^*, Q^*) is determined by the sign of A. Precisely we have:

(4.10) *If $A < 0$ then the equilibrium point (P^*, Q^*) is stable. If $A > 0$ then the equilibrium point is unstable and there exists at least one periodic solution*

We note that the following condition

$$\tilde{\beta}'(V^*) < 0, \quad \tilde{\mu}'(V^*) > 0$$

implies $A < 0$ while the following one

$$\tilde{\beta}'(V^*)V^* > \tilde{\beta}(V^*), \quad \tilde{\mu}'(V^*) < 0$$

implies $A > 0$; this latter condition can be easily produced in simple examples.

Before closing this section we mention more general models that can be treated with the same procedure we have used for the constitutive form (4.2). Namely, the following form,

$$(4.11) \qquad \beta(a, x) = e^{-\alpha a} \sum_{i=1}^{n} a^i \tilde{\beta}_i(x), \quad \mu(a, x) = \tilde{\mu}(x), \quad \gamma(a) = 1$$

can be treated by reducing the problem to a system of $n + 2$ variables.

5 Comments and references

The first part of this chapter is based on the strict relation between the P.D.E. system (III.1.2) and the functional equation (III.2.3). This has been the first approach to the study of the asymptotic behavior, and in particular to the study of global results.

First results can be found in Rorres [82], [83] from which we have drawn the contents of Section 1 and Section 2. Similar methods have been used in [30] and [69]; the latter paper deals with the special case (2.12)-(2.13).

The results that we have presented are not exhaustive of the use that can be done of reduction to integral equations, however, though this approach can take advantage of any technique concerning this type of equations, there are not many results available.

The class of separable models treated in Section 3, has been studied in [11]. This is a very special class of models which, as we have already noticed, in the case of a single population do not allow existence of a periodic solution.

Finally, the models of Section 4 are somewhat classic and the constitutive form (4.11) has been sistematically used in many papers ([48], [49]).

VI

Epidemics through an age structured population

The present chapter and the subsequent one are concerned with the modeling of epidemics with age structure.

Without age structure, the basic models of the theory of epidemics (see for instance [55]) assume that, apart from the differences due to the disease, the population is homogeneous; then for the description of the epidemics it is divided into three main subclasses:

susceptible individuals: *this class includes those individuals who are not sick and can be infected;*

infective individuals: *it includes those individuals who have the disease and can transmit it to others;*

removed individuals: *it is formed by those individuals who have been infective and are now immune, dead or isolated.*

We will denote by $S(t)$, $I(t)$ $R(t)$ the number of individuals that, at the time t, respectively belong to the three classes listed above. Of course, any model must provide the following equality

$$S(t) + I(t) + R(t) = P(t)$$

where $P(t)$ is the total population size.

A fairly general model for the dynamics of a disease assumes that the population is constant, disregarding the vital rates; then the model is represented by the following system:

$$\frac{d}{dt} S(t) = -\lambda(t)S(t) + \delta I(t)$$

$$\frac{d}{dt} I(t) = \lambda(t)S(t) - (\gamma + \delta)I(t) \qquad (0.1)$$

$$\frac{d}{dt} R(t) = \gamma I(t)$$

$$S(0) = S_0,\ I(0) = I_0,\ R(0) = R_0$$

where $S_0 + I_0 + R_0 = P$ and the parameters have the following meaning

the force of infection $\lambda(t)$: *this is the rate at which susceptibles catch the disease, thus entering the infective class;*

the recovery rate δ: *this is the rate at which infectives leave their class and go back to the susceptible one;*

the removal rate γ: *this is the rate at which infectives leave their class to enter the removed one.*

Concerning the force of infection $\lambda(t)$, we need to assume some constitutive law that translates into a mathematical form the infection mechanism of the specific disease. The simplest constitutive form for $\lambda(t)$ is:

$$\lambda(t) = c\phi \frac{I(t)}{P} = kI(t) \qquad (0.2)$$

where the constants c and ϕ have the following meaning:

c = **contact rate** = *the number of contacts that a single individual has, per unit time, with other individuals of the population;*

ϕ = **infectiveness** = *the probability that a contact with an infected individual will transmit the disease.*

We note that in (0.2) the term $\frac{I(t)}{P}$ stands for the probability that the contacted individual is infective. The form (0.2) rests upon the assumption that the population is homogeneously mixing, that the whole population is active (instead, for some disease either part or all of the removed class does not participate in the mixing) and that the contact rate is independent of the size of the active population.

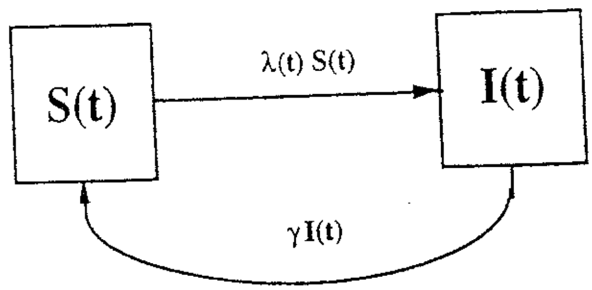

Figure 0.1

Concerning the nature of the disease we may distinguish those which can be caught several times (these are diseases that are not lethal and do not impart immunity such as common cold, influenza and gonorrhea) from those which can instead be caught only one and lead to removal of the infected individual (to this group belong measles, rubella, mumps and other childhood diseases which give immunity, but also lethal diseases such as the recent HIV infection for which the removed class corresponds to individuals who have developed AIDS). In the first case $\gamma = 0$ and the model is called an S-I-S model, because the individual path through the disease is represented by the scheme in Figure 0.1; in the other case $\delta = 0$ and we have a SIR model (see Figure 0.2).

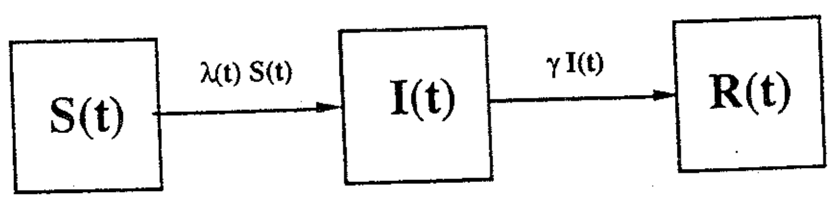

Figure 0.2

The two cases are somewhat alternative and the different conclusions are reported in Figure 0.3 and Figure 0.4 respectively. We see that in both cases we have a *threshold phenomenon*, in fact the value of the parameter $\varrho_0 = \dfrac{c\phi}{\gamma}$ is responsible for an *endemic state* or for the *outbreak* of the epidemics.

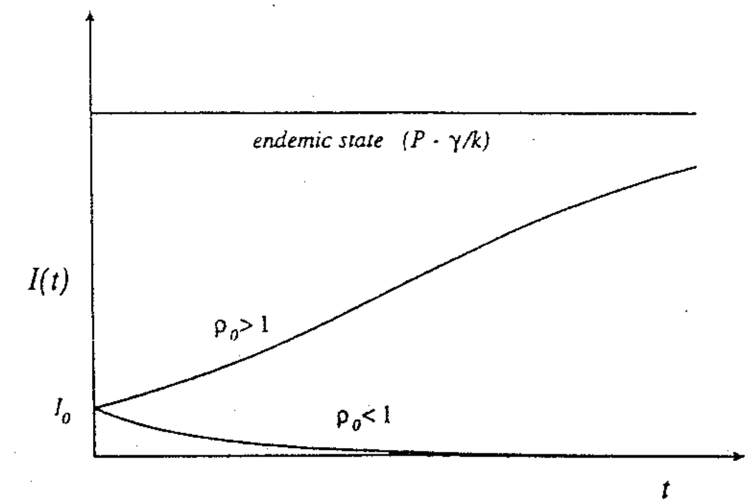

Figure 0.3

In this chapter we want to model the spreading of an epidemic, taking care of the age structure of the population. The importance of considering the age of the individuals in an epidemic model arises from the fact that for many diseases the rate of infection varies significantly with age. In fact, if we consider exanthematic diseases we see that the transmission mainly involves early ages, while for sexual diseases the principal mechanism of infection involves mature individuals. Thus we expect that the vital dynamics of the population and the infection mechanism, interact to produce non trivial behaviors.

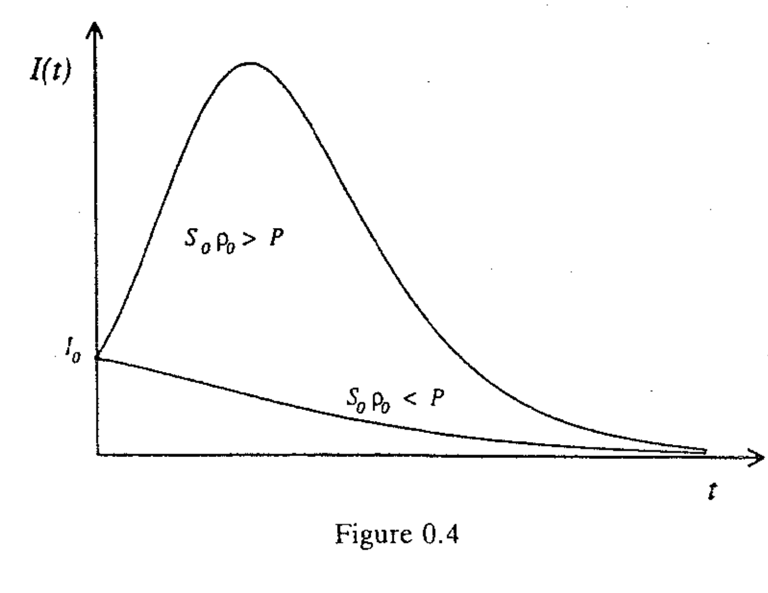

Figure 0.4

In the following sections we will first extend the general model (0.1) and then we will focus on some special cases that can be mathematically treated by the methods of the previous chapters. Actually we will see that the models arising within this setting, though presenting different features, can be treated by the same procedures and methods.

1 A general model for epidemics

We consider a population that, in the absence of the epidemic that we are going to consider, can be described by the linear model discussed in chapter I, i.e. we consider a population which is isolated, in an invariant habitat, structured by age, with the vital rates $\beta(a)$ and $\mu(a)$ satisfying the assumptions (I.2.6)-(I.2.8).

Because of the epidemics, the population is partitioned into the three classes of *susceptibles, infectives* and *removed* which are described by their respective age-densities $s(a, t)$, $i(a, t)$, $r(a, t)$, at time t. Thus the age-density $p(a, t)$ of the whole population must satisfy

$$p(a, t) = s(a, t) + i(a, t) + r(a, t) \tag{1.1}$$

Denoting by $\gamma(a)$, $\delta(a)$, $\lambda(a, t)$, the age specific *removal rate, cure rate* and *infection rate* respectively, we have the following equations describing the transmission dynamics of the disease:

$$\begin{cases} s_t(a, t)+s_a(a, t)+\mu(a)s(a, t)=-\lambda(a, t)s(a, t)+\delta(a)i(a, t) \\ i_t(a, t)+i_a(a, t)+\mu(a)i(a, t)=\lambda(a, t)s(a, t)-(\gamma(a)+\delta(a))i(a, t) \\ r_t(a, t)+r_a(a, t)+\mu(a)r(a, t)=\gamma(a)i(a, t) \\ s(0, t) = b_1(t),\ i(0,t) = b_2(t),\ r(0, t) = b_3(t) \end{cases} \tag{1.2}$$

Actually, each class undergoes the same demographic evolution determined by the vital rates $\beta(a)$ and $\mu(a)$, while the passage from a class to another is ruled by the rates $\gamma(a)$, $\delta(a)$, $\lambda(a, t)$.

Together with system (1.2), we must consider the initial conditions

$$(1.3) \qquad s(a, 0) = s_0(a),\ i(a, 0) = i_0(a),\ r(a, 0) = r_0(a)$$

and constitutive equations for the birth rates $b_1(t)$, $b_2(t)$, $b_3(t)$. Concerning these latter we assume

$$(1.4) \qquad \begin{cases} b_1(t) = \int_0^{a_\dagger} \beta(a)\,[s(a, t)+(1-q)i(a, t)+(1-w)r(a, t)]da \\ b_2(t) = q \int_0^{a_\dagger} \beta(a)i(a, t)da \\ b_3(t) = w \int_0^{a_\dagger} \beta(a)r(a, t)da \end{cases}$$

where $q \in [0, 1]$, $w \in [0, 1]$ are the *vertical transmission* parameters of infectiveness and immunity, respectively. These parameters indicate the fraction of newborn who are born in the class of their parents; thus if $q = w = 0$ all newborn are susceptible.

In this model we assume that the intrinsic fertility $\beta(a)$ and mortality $\mu(a)$ are not (significantly) affected by the disease, so we expect that the total population (1.1) undergoes the same demographic process of the model of chapter I. In fact, if we add the equations in (1.2) we obtain the following problem for $p(a, t)$

$$(1.5) \qquad \begin{cases} p_t(a, t) + p_a(a, t) + \mu(a)p(a, t) = 0 \\ p(0, t) = \int_0^{a_\dagger} \beta(a)p(a, t)da \\ p(a, 0) = p_0(a) = s_0(a) + i_0(a) + r_0(a) \end{cases}$$

that is, we obtain problem (I.2.5). In this respect we make the following hypothesis on the demography of the population

$$(1.6) \qquad R = \int_0^{a_\dagger} \beta(a)\Pi(a)da = 1$$

i.e. we assume that the population is at zero growth ($\alpha^* = 0$) and consequently it exists the stationary solution

$$(1.7) \qquad p_\infty(a) = P_0\omega^*(a) = b_0\Pi(a)$$

Moreover, we suppose that

$$(1.8) \qquad p(a, t) = p_0(a) = p_\infty(a)$$

i.e. we suppose that the population has reached the steady-state distribution $p_\infty(a)$.

Finally, we must give a constitutive form to the infection rate $\lambda(a, t)$; to this it is usually given the linear form

$$(1.9) \qquad \lambda(a, t) = K_0(a)i(a, t) + \int_0^{a_\dagger} K(a, a')i(a', t)da'$$

where the two terms on the right hand side are called the *intracohort* term and *intercohort* term respectively. The following special cases

$$(1.10) \qquad \lambda(a, t) = K_0(a)i(a, t)$$

$$(1.11) \qquad \lambda(a, t) = K(a) \int_0^{a_\dagger} i(a, t)da$$

correspond to two extreme mechanisms of contagion; in fact (1.10) represents the situation in which individuals can be infected only by those of their own age, while in (1.11) they can be infected by those of any age. When considering these constitutive forms, we will assume

$$(1.12) \qquad K_0(a) \geq 0 \quad a.e. \text{ in } [0, a_\dagger], \quad K_0(\cdot) \in L^\infty(0, a_\dagger)$$

$$(1.13) \qquad K(a) \geq 0 \quad a.e. \text{ in } [0, a_\dagger], \quad K(\cdot) \in L^\infty(0, a_\dagger)$$

A substantial reduction of the problem occurs when we consider the S-I-S epidemic, modeling a disease that does not impart immunity. In fact, when we specialize problem (1.2)-(1.4) assuming $\gamma(a) \equiv 0$, $r_0(a) \equiv 0$, we have

$$(1.14) \qquad \begin{aligned} &s_t(a, t)+s_a(a, t)+\mu(a)s(a, t)=-\lambda(a, t)s(a, t)+\delta(a)i(a, t)\\ &i_t(a, t)+i_a(a, t)+\mu(a)i(a, t)=\lambda(a, t)s(a, t)-\delta(a)i(a, t)\\ &s(0, t)= \int_0^{a_\dagger} \beta(a)[s(a, t)+(1-q)i(a, t)]da, \quad s(a, 0)=s_0(a) \end{aligned}$$

$$i(0, t) = q \int_0^{a_\dagger} \beta(a)i(a, t)da, \quad i(a, 0) = i_0(a)$$

Since, by (1.8),

$$s(a, t) + i(a, t) = p_\infty(a) \tag{1.15}$$

we can set $s(a, t) = p_\infty(a) - i(a, t)$ in the second equation of (1.14) getting the following problem on the single variable $i(a, t)$

$$\begin{cases} i_t(a, t)+i_a(a, t)+\mu(a)i(a, t)=\lambda(a, t)[p_\infty(a)-i(a, t)]-\delta(a)i(a, t) \\ i(0, t) = q \int_0^{a_\dagger} \beta(a)i(a, t)da \\ i(a, 0) = i_0(a) \end{cases} \tag{1.16}$$

and we can limit ourselves to the study of this system.

Another reduction concerns the S-I-R case which corresponds to the assumptions $\delta(a) \equiv 0$ and $w = 1$; in this case we have the following system

$$\begin{aligned} &s_t(a, t) + s_a(a, t) + \mu(a)s(a, t) = -\lambda(a, t)s(a, t) \\ &i_t(a, t)+i_a(a, t)+\mu(a)i(a, t)=\lambda(a, t)s(a, t)-\gamma(a)i(a, t) \\ &s(0, t) = \int_0^{a_\dagger} \beta(a) [s(a, t)+(1-q)i(a, t)]da \\ &i(0, t) = q \int_0^{a_\dagger} \beta(a)i(a, t)da \\ &s(a, 0) = s_0(a) \\ &i(a, 0) = i_0(a) \end{aligned} \tag{1.17}$$

Actually, we can disregard the third equation in (1.2) because the first two are enough to determine the evolution of the two classes of susceptibles and infectives; however, because of the presence of the class of removed individuals, (1.15) is not true and we cannot further reduce the system.

In the coming sections, we will prove some results for (1.16) with the infection rates (1.10) and (1.11). Together with the assumptions (1.12) and (1.13) we will assume

$$\delta(a) \geq 0 \quad \text{a.e. in } [0, a_\dagger], \quad \delta(\cdot) \in L^\infty(0, a_\dagger) \tag{1.18}$$

2 Endemic states for the S-I-S model

Here we consider the S-I-S case (1.14) and discuss existence of endemic states, i.e. non trivial stationary states of the problem.

First we investigate (1.14) assuming the purely intracohort form (1.10) for the infection rate. With this assumption (1.14) becomes

$$\begin{cases} i_t(a, t)+i_a(a, t)+\mu(a)i(a, t)=K_0(a)[p_\infty(a)-i(a, t)]i(a, t)-\delta(a)i(a, t) \\ i(0, t) = q \displaystyle\int_0^{a_\dagger} \beta(a)i(a, t)da \\ i(a, 0) = i_0(a) \end{cases} \tag{2.1}$$

and a stationary state $i^*(a)$ must satisfy

$$\begin{cases} \dfrac{d}{da}\, i^*(a)+\mu(a)i^*(a)=K_0(a)[p_\infty(a)-i^*(a)]i^*(a)-\delta(a)i^*(a) \\ i^*(0) = q \displaystyle\int_0^{a_\dagger} \beta(a)i^*(a)da \end{cases} \tag{2.2}$$

We first note that (2.2) admits the trivial solution $i^*(a) \equiv 0$ and that, if $q = 0$ (i.e. when the disease is not vertically transmitted) this is the only solution. Then, letting $q > 0$ and setting $i^*(0) = v^* > 0$, we see that the first equation in (2.2) yields:

$$i^*(a) = \frac{v^* E(a)}{1+v^* \displaystyle\int_0^a K_0(\sigma)E(\sigma)d\sigma} \tag{2.3}$$

X

where we have set:

$$E(a) = e^{-\int_0^a [\mu(\sigma) + \delta(\sigma) - K_0(\sigma)p_\infty(\sigma)]d\sigma} \tag{2.4}$$

Plugging (2.3) into the second equation in (2.2) we get the following equation for v^*

$$1 = q \int_0^{a_\dagger} \frac{\beta(a)E(a)}{1+v^* \int_0^a K_0(\sigma)E(\sigma)d\sigma} da \tag{2.5}$$

Of course, solving this equation is equivalent to solving (2.2), via formula (2.3).

We note that the right hand side of (2.5) is a decreasing function of v^*, unless the following condition is satisfied:

$$\beta(a) \int_0^a K_0(\sigma)d\sigma = 0 \quad a.e. \text{ for } a \in [0, a_\dagger] \tag{2.6}$$

Then we state the following theorem which gives a threshold condition for the existence of endemic states.

Theorem 2.1. *Let $q > 0$ and assume that (2.6) is not true, then (2.2) has one non trivial solution if and only if*

$$q \int_0^{a_\dagger} \beta(a) e^{\int_0^a \alpha(\sigma)d\sigma} da > 1 \tag{2.7}$$

and, if such a solution exists, it is unique. Moreover, if (2.6) is satisfied, then (2.2) has either no nontrivial solutions or we have

$$q \int_0^{a_\dagger} \beta(a) e^{-\int_0^a [\mu(\sigma)+\delta(\sigma)]d\sigma} da = 1 \tag{2.8}$$

In this last case there is an infinite number of solutions.

Proof:

Suppose that (2.6) is not true, then the function:

$$\Phi(x) = q \int_0^{a_\dagger} \frac{\beta(a)E(a)}{1+x \int_0^a K_0(\sigma)E(\sigma)d\sigma} da, \quad x \in [0, +\infty]$$

is strictly decreasing and:

$$\lim_{x \to +\infty} \Phi(x) = q \int_0^{a_0} \beta(a)E(a)da = q \int_0^{a_0} \beta(a)e^{-\int_0^a [\mu(\sigma)+\delta(\sigma)]d\sigma}da \leqslant$$

$$\leqslant \int_0^{a_0} \beta(a)e^{-\int_0^a \mu(\sigma)d\sigma} < \int_0^{a_\dagger} \beta(a)e^{-\int_0^a \mu(\sigma)d\sigma}da = 1$$

where:

$$a_0 = sup\{a \mid K_0 = 0 \quad a.e. \ in \ [0, a]\}$$

Then there exists $\upsilon^* > 0$ satisfying (2.5) if and only if $\Phi(0) > 1$, and this solution is unique. The threshold condition $\Phi(0) > 1$ is exactly (2.7) and the first part of the theorem is proved.

Finally,if (2.6) is satisfied then $\Phi(x)$ is constant, that is:

$$\Phi(x) = q \int_0^{a_0} \beta(a)e^{-\int_0^a \mu(\sigma)d\sigma}da, \quad x \in [0, +\infty]$$

E(a)

and equation (2.5) has infinitely many solutions if and only if (2.8) is accomplished. ∎

We remark that condition (2.6) means that

$$\beta(a) = 0 \quad a.e. \ for \ a > a_0$$

i.e. the fertility window lies below the infectiveness one. Moreover we note that (2.8) is fulfilled if and only if:

$$q = 1, \quad \delta(a) = 0 \quad a.e. \ on \ S = \{a \mid \beta(a) > 0\}$$

This is a very special situation that will be disregarded.

Now we consider the *purely intercohort* case without *vertical transmission*, that is we assume (1.11) and $q = 0$. Then from (1.14) we get the problem

$$
(2.9) \qquad \begin{cases} i_t(a,t)+i_a(a,t)+\mu(a)i(a,t)=K(a)[p_\infty(a)-i(a,t)]I(t)-\delta(a)i(a,t) \\ I(t) = \int_0^{a_\dagger} i(a,t)da \\ i(0,t) = 0, \quad i(a,0) = i_0(a) \end{cases}
$$

and the following one concerning stationary states

$$
(2.10) \qquad \begin{cases} \frac{d}{da} i^*(a)+\mu(a)i^*(a)=K(a)[p_\infty(a)-i^*(a)]I^*-\delta(a)i^*(a) \\ I^* = \int_0^{a_\dagger} i^*(a)da, \quad i^* \quad (0) = 0 \end{cases}
$$

From the first equation in (2.10) and the condition $i^*(0)= 0$ we get

$$
(2.11) \qquad i^*(a) = I^* \int_0^a H(a,\sigma)e^{-I^*\int_\sigma^a K(s)ds}d\sigma
$$

with

$$
(2.12) \qquad H(a,\sigma) = K(\sigma)p_\infty(\sigma)e^{-\int_\sigma^a[\mu(s)+\delta(s)]ds}
$$

Then, putting (2.11) into the second equation of (2.10), we get the following equation for I^*

$$
(2.13) \qquad 1 = \int_0^{a_\dagger}\int_0^a H(a,\sigma)e^{-I^*\int_\sigma^a K(s)ds}d\sigma da
$$

and we arrive at the following result

Theorem 2.2. *Problem (2.10) has one non trivial solution if and only if*

$$
(2.14) \qquad \int_0^{a_\dagger}\int_0^a H(a,\sigma)d\sigma da > 1
$$

and, if such a solution exists, it is unique.

Proof:

The proof is similar to that of Theorem 2.1; we just note that the function

$$\Phi(x) = \int_0^{a_\dagger} \int_0^a H(a, \sigma) e^{-x \int_\sigma^a K(s)ds} d\sigma da$$

is strictly decreasing since

$$K(\sigma) \int_\sigma^a K(s)ds$$

does not vanish a.e. on the set $\{(a, \sigma) | 0 \leqslant \sigma \leqslant a \leqslant a_\dagger\}$. ∎

In the following section we will consider some results concerning stability for the intra-cohort case.

3 Asymptotic behavior for the intra-cohort case

We are now ready to investigate the asymptotic behaviour of problem (2.1). To do this we integrate the first equation in (2.1) along the characteristic $t - a = const$, obtaining the following formula

$$(3.1) \qquad i(a,t) = \begin{cases} \dfrac{i_0(a-t)E(a)}{E(a-t) + i_0(a-t) \int_0^t K_0(a-\tau)E(a-\tau)d\tau} & \text{if } a \geqslant t \\[2ex] \dfrac{i(0, t-a)E(a)}{1 + i(0, t-a) \int_0^a K_0(\tau)E(\tau)d\tau} & \text{if } a < t \end{cases}$$

where $E(a)$ is defined in (2.4). In fact, setting $U(s) = i(a_0 + s, t_0 + s)$ with $s \geqslant 0$, we have

$$\frac{d}{ds} U(s) = [-\mu(a_0 + s) - \delta(a_0 + s) + K_0(a_0 + s)p_\infty(a_0 + s) -$$

$$- K_0(a_0 + s)U(s)]U(s)$$

so that

$$i(a_0 + s, t_0 + s) = \frac{i(a_0, t_0)E(a_0 + s)}{E(a_0)+i(a_0, t_0) \int_0^s K_0(a_0 + \sigma)E(a_0 + \sigma)d\sigma}$$

from which (3.1) easily follows.

Formula (3.1) is the starting point for the analysis of the model. First we rule out the case $q = 0$ that is the case with no vertical transmission of the disease. In fact, with this condition we have $i(0, t) \equiv 0$ and consequently

$$(3.2) \qquad i(a, t) = 0 \quad for \quad t > a_\dagger$$

thus the disease dies out.

Next we consider $q > 0$. To treat this case we must transform the problem into a Volterra integral equation on the infectives birth rate

$$(3.3) \qquad v(t) = i(0, t)$$

In fact, putting (3.1) into the second equation of (2.1) we get a non-linear integral equation of the form

$$(3.4) \qquad v(t) = F(t) + \int_0^t G(a, v(t - a))da$$

where (we extend all the functions by zero outside of $[0, a_\dagger]$)

$$(3.5) \qquad F(t) = \int_0^\infty \frac{q\ \beta(a+t)E(a+t)i_0(a)}{E(a)+i_0(a) \int_a^{a+t} K_0(\tau)E(\tau)d\tau} da \qquad t \geqslant 0$$

$$(3.6) \qquad G(a, z) = \frac{q\ \beta(a)E(a)z}{1+z \int_0^a K_0(\tau)E(\tau)d\tau} \qquad a \geqslant 0 \quad z \geqslant 0$$

By this reduction to an integral equation, we first prove existence and uniqueness of a solution to the problem. Actually we have

Theorem 3.1. *Let (1.12), (1.14) be satisfied and let $i_0 \in L_1[0, a_\dagger]$, then equation (3.4) has a unique continuous solution $v(t)$.*

Proof:

The solution can be obtained as a fixed point in the space $C[0, T]$ (for any $T > 0$). In fact the mapping $\mathcal{T} : C[0, T] \to C[0, T]$ defined as

$$(3.7) \qquad (\mathcal{T}\, v)\,(t) = F(t) + \int_0^t G(a,\, v(t-a))da$$

leaves the set

$$v(\cdot) \in C[0, T]; \quad 0 \leqslant v(t) \leqslant |F|_{C[0,T]} e^{q|\beta|_\infty t}$$

unchanged and, moreover, for v and $\bar{v}$ belonging to this set,

$$|\mathcal{T}^N v - \mathcal{T}^N \bar{v}|_{C[0,T]} \leqslant \frac{C^N T^N}{N!} \, |v - \bar{v}|_{C[0,T]}$$

where C is a constant. Thus existence and uniqueness follow. ■

Of course, the previous theorem provides existence and uniqueness of a solution to problem (2.1), via formulas (3.1) and (3.3).

We note that, if we consider the limiting equation of (3.4),

$$(3.8) \qquad v(t) = \int_0^{a_\dagger} G(a,\, v(t-s))ds$$

we see that its constant solutions $v^* > 0$ must satisfy

$$v^* = \int_0^{a_\dagger} G(a,\, v^*)da$$

that is the same equation (2.5) which has already been discussed.

We are now able to give a complete description of the asymptotic behaviour of $v(t)$. We suppose that the assumptions of Theorem 3.1 are satisfied and start with the following preliminary result

Proposition 3.2. *Suppose*

$$\beta(a) > 0 \quad a.e. \text{ in } [a_1, a_2] \tag{3.9}$$

and let i_0 be such that

$$\int_0^{a_\dagger} \beta(a+t) i_0(a)\,da > 0 \quad \text{for some } t \geq 0. \tag{3.10}$$

Then the solution of (3.4) is eventually positive.

Proof:

If (2.19) is fulfilled then $F(t)$ and consequently $v(t)$ are not identically zero on $[0, a_\dagger]$. Suppose that $v(t) > 0$ for $t \in [\alpha, \beta] \subset [0, a_\dagger]$; then for $t \in [\alpha + a_1, \beta + a_2]$

$$v(t) = \int_0^t G(t-s, v(s))\,ds \geq$$

$$\geq \min_{t \in [\alpha,\beta]} v(t) \int_{0 \vee (t-\beta)}^{t-\alpha} \frac{q\,\beta(a)E(a)}{1 + v(t-a) \int_0^a K_0(\tau)E(\tau)\,d\tau}\,da > 0$$

because $(a_1, a_2) \cap (0 \vee (t-\beta), t-\alpha) \neq \emptyset$. Thus iterating this argument we get

$$v(t) > 0 \quad \text{for} \quad t \in [\alpha + na_1, \beta + na_2]$$

for any positive integer n and, since

$$\bigcup_n [\alpha + na_1, \beta + na_2] \supset [t_0, +\infty)$$

for some $t_0 > 0$, we have $v(t) > 0$ for $t > t_0$. ∎

Note that condition (3.9) means that the initial datum i_0 has a support which, if translated to the right, hits the fertility window: if this condition is not satisfied then $F(t)$ is identically zero and consequently also $v(t)$ vanishes for $t \geqslant 0$.

Now we analyse the behaviour of $v(t)$ under conditions (3.8), (3.9): this behaviour depends on the threshold condition (2.7). First we have

Theorem 3.3. *Let (3.8), (3.9) be satisfied and assume*

$$q \int_0^{a_\dagger} \beta(a) e^{-\int_0^a \mu(\sigma)d\sigma} da \leqslant 1 \tag{3.11}$$

then

$$\lim_{t \to +\infty} v(t) = 0$$

Proof:

Let $I_n = [na_\dagger, (n+1)a_\dagger]$ for any integer $n \geqslant 0$, then define:

$$M_n = \max_{t \in I_n} v(t), \quad \tilde{M}_n = \max\{M_n, M_{n-1}\} \tag{3.12}$$

Note that, by the proof of Proposition 3.2, we have $M_n > 0$ for any $n \geqslant 0$; then if $t \in I_n$ with $n > 0$ we have

$$v(t) = \int_0^{a_\dagger} G(s, v(t-s))ds \leqslant \int_0^{a_\dagger} G(s, \tilde{M}_n)ds = \tilde{M}_n \Phi(\tilde{M}_n) \tag{3.13}$$

where $\Phi(z)$ is the function defined in the proof of Theorem 2.1. In fact since $s \in [0, a_\dagger]$ we have $t - s \in I_n \cup I_{n-1}$ and, for $a \in [0, a_\dagger]$, $G(a, z)$ is a non decreasing function of z. From (3.12) we get

$$M_n \leqslant \tilde{M}_n \, \Phi(\tilde{M}_n) \quad \forall\, n > 0 \tag{3.14}$$

and, since $\Phi(z)$ is strictly decreasing and $\Phi(0) \leqslant 1$, we have

$$M_n < \tilde{M}_n \Phi(0) \leqslant \tilde{M}_n$$

that is, $M_n < M_{n-1}$.

Thus the sequence $\{M_n\}$ is decreasing and, setting $M_\infty = \lim_{n \to \infty} M_n$ we have, going to the limit in (3.13):

$$M_\infty \leqslant M_\infty \, \Phi(M_\infty)$$

If $M_\infty > 0$ we would have the contradiction

$$1 < \Phi(0)$$

which is absurd because $\Phi(0) \leqslant 1$. So it must be $M_\infty = 0$ and the proof is complete. ■

Besides we have

Theorem 3.4. *Let (3.8), (3.9) be satisfied and assume*

(3.15) $$q \int_0^{a_\dagger} \beta(a) e^{-\int_0^a \mu(\sigma)d\sigma} da > 1.$$

Then

$$\lim_{t \to +\infty} v(t) = v^*$$

Proof:

Let I_n, M_n, $\bar{M}_n$ be defined as before. We first prove the following statement

(3.16) $$\text{if} \quad M_n \leqslant v^* \quad \text{then} \quad M_{n+1} \leqslant v^*$$

In fact, we recall the following inequality already stated in (3.13).

(3.17) $$M_{n+1} \leqslant \bar{M}_{n+1} \, \Phi(\bar{M}_{n+1}) \qquad n \geqslant 0;$$

then if $M_{n+1} > v^*$ we have $\bar{M}_{n+1} = M_{n+1}$ and consequently

$$M_{n+1} \leqslant M_{n+1}\Phi(M_{n+1}) < M_{n+1}\Phi(v^*) = M_{n+1}$$

which is absurd.

Next we prove that

(3.18) $$\text{if} \quad M_n > v^* \quad \text{for} \quad n > N \quad \text{then} \quad M_{n+1} < M_n \quad \text{for } n > N$$

and

$$\lim_{n \to +\infty} M_n = v^*$$

In fact, if $M_n > v^*$ and consequently $\bar{M}_{n+1} > v^*$, we have from (3.16)

$$M_{n+1} \leqslant \bar{M}_{n+1} \Phi(\bar{M}_{n+1}) < \bar{M}_{n+1} \Phi(v^*) = \bar{M}_{n+1}$$

That is $M_{n+1} < M_n$. Then, letting $M_\infty = \lim_{n \to \infty} M_n \geqslant v^*$, we go to the limit in (3.16)

$$M_\infty \leqslant M_\infty \Phi(M_\infty)$$

so that, if $M_\infty > v^*$ we have $M_\infty < M_\infty$, which is absurd and then necessarily $M_\infty = v^*$.

In addition we define

$$m_n = \min_{t \in I_n} v(t), \quad \bar{m}_n = \min\{m_n, m_{n-1}\}$$

and, noticing that the sequence $\{m_n\}$ must eventually be positive by Proposition 3.2, we can also prove (the proof is similar to that of (3.15) and (3.17))

$$\text{(3.19)} \qquad \text{if} \quad m_n \geqslant v^* \quad \text{then} \quad m_{n+1} \geqslant v^*$$

$$\text{(3.20)} \qquad \text{if} \quad m_n < v^* \quad \text{for } n > N \quad \text{then} \quad m_{n+1} > m_n \quad \text{for} \quad n > N \quad \text{and} \quad \lim_{n \to +\infty} = v^*$$

Finally, putting together (3.16), (3.17), (3.18), (3.19), we get the proof of the theorem. ■

4 Comments and references

The spread of an epidemic through an age structured population has received attention in connection with the modeling of childhood diseases like measles, mumps, rubella, chickenpox. The general model that we have presented is essentially due to Anderson-May [1]-[2], Dietz [33] and Schenzle [88], but the mathematical results on these models are rather recent ([3], [4], [8]-[10], [13], [14], [17], [39], [40], [54], [58], [63], [93], [96]).

The S-I-S model that we have presented has been studied in [9] where also local stability results for the inter-cohort case were given. Actually, in some subsequent papers ([13], [14]) more general results have been given, showing that the main result concerning the existence of a unique endemic state, attracting all solutions, is true for the general form (1.9) of the infection rate. The methods to prove these results actually belong to the theory of positive operators and cannot be presented here.

Though the analysis of the S-I-S case is sufficiently settled and shows a simple behavior of the model, this is not the case for the S-I-R model. Many partial results exist, but the situation is far from clear. Some sufficient conditions for existence, uniqueness and stability of endemic states have been given in [4], [63], while in [3] and [93] some cases in which the endemic equilibrium is unstable are shown. Some partial results on the simple intra-cohort case has been given in [17], but even the question of uniqueness of an endemic state is not settled.

VII

Class-age structure for epidemics

In the previous chapter we have considered the spread of an epidemic stressing the interaction between the demographic processes and the epidemiological mechanisms. To that purpose we have considered the age of the individuals of the population as the main structure operating this interaction. However, this demographic age, or *chronologicala age* as it is often called, is not the only age that is necessary to consider when modeling epidemics. In fact we can also consider the age of the disease (*class age*), i.e. the time elapsed since an individual has become infected. The very first and celebrated epidemic model by Kermack and McKendrick ([65]-[67]), that we considered in Section 1, is actually structured by class-age: this is important when modelling a disease for which an infected individual has a variable chance of recovery or death and his infectiveness also depend on the time spent as an infected. The recent epidemics by HIV/AIDS infection well represents this latter phenomenology and in Sections 5 and 6 we will be concerned with a model that is inspired by this disease.

1 The Kermack McKendrick model

Let us consider a closed population in which no migration is present and no births or deaths occur by natural reasons. This assumption, of course, is realistic as long as we want to describe the single outbreak of an epidemic through a period of time such that demographic changes can be disregarded. Thus, the population size is constant with time and the total number of individuals, that we denote by P, is as usual partitioned into the three subclasses of susceptibles, infectives and removed individuals.

For a member of the infected class we will denote by $\vartheta \in [0, \vartheta_{\dagger}]$ the time elapsed since infection ($\vartheta_{\dagger}$ denotes a maximum age of infection); then the situation is described by the following variables

$S(t)$ = *number of susceptible individuals at time t*
$i(\vartheta, t)$ = *ϑ-density of infected individuals at time t*
$R(t)$ = *number of removed individuals at time t*

Of course we must have

$$(1.1) \qquad S(t) + \int_0^{\vartheta_f} i(\vartheta, t)d\vartheta + R(t) = P$$

In addition we consider the following parameters:

$\gamma(\vartheta)$ = *age-specific removal rate*
$\lambda(t)$ = *infection rate (force of infection)*

thus

$$\gamma(\vartheta)i(\vartheta, t)d\vartheta dt$$

denotes the number of infected individuals, with class age in the interval $[\vartheta, \vartheta + d\vartheta]$, that pass into the removed class during the time interval $[t, t + dt]$; moreover

$$\lambda(t)S(t)dt$$

is the number of susceptible individuals becoming infected during the time interval $[t, t + dt]$.

Actually the force of infection $\lambda(t)$ must be given a constitutive form describing the mechanism of infection. The simplest form for $\lambda(t)$ is the following:

$$(1.2) \qquad \lambda(t) = \int_0^{\vartheta_f} K(\vartheta)i(\vartheta, t)d\vartheta$$

which we will use here, while in a subsequent section we will discuss more general and significant constitutive laws for $\lambda(t)$.

With all these preliminaries, the equations of the model turn out to be:

$$i) \qquad \frac{d}{dt} S(t) = -\lambda(t)S(t)$$

$$(1.3)\quad \begin{aligned} &ii)\quad i_t(\vartheta, t) + i_\vartheta(\vartheta, t) + \gamma(\vartheta)i(\vartheta, t) = 0 \\ &iii)\quad i(0, t) = \lambda(t)S(t) \\ &iv)\quad \frac{d}{dt} R(t) = \int_0^{\vartheta_t} \gamma(\vartheta)i(\vartheta, t)d\vartheta \end{aligned}$$

with the initial conditions:

$$(1.4)\qquad S(0) = S_0,\quad i(\vartheta, 0) = i_0(\vartheta),\quad R(0) = R_0$$

In (1.3) equations *ii), iii)* are derived by a similar argument as in Section I.2. Actually, in the present case, we can start from the balance (see I.2.1)

$$\int_0^{\vartheta+h} i(\sigma, t + h)d\sigma = \int_0^{\vartheta} i(\sigma, t)d\sigma + \int_t^{t+h} \lambda(\sigma)S(\sigma)d\sigma -$$

$$\int_0^h \int_0^{\vartheta+s} \gamma(\sigma)i(\sigma, t + s)d\sigma ds$$

We note that, with the constitutive form (1.2) the equations *i), ii), iii)* of system (1.3) are decoupled from *iv)*, so that for the study of the system, it is enough to consider the first three equations.

In the following section we will treat problem (1.2), (1.3) under the following assumptions on the parameters:

$$(1.5)\qquad \gamma(\vartheta) \geq 0,\quad K(\vartheta) \geq 0\quad a.e.\ in\ [0, \vartheta_t]$$

$$(1.6)\qquad \gamma(\cdot) \in L^1_{loc}(0, \vartheta_t);\quad \int_0^{\vartheta_t} \gamma(\sigma)d\sigma = +\infty$$

$$(1.7)\qquad K(\cdot) \in L^\infty(0, \vartheta_t);\quad K(\vartheta) > 0\quad a.e.\ in\ [\vartheta_1, \vartheta_2]$$

2 Reduction of the system

We will reduce (1.3) into a system of integro-differential equations, starting from the integrated formula

$$(2.1)\qquad \begin{cases} i_0(\vartheta - t)\ \dfrac{B(\vartheta)}{B(\vartheta-t)} & \text{if } \vartheta \geq t \\ i(0,\ t - \vartheta)B(\vartheta) & \text{if } \vartheta < t \end{cases}$$

where we have set

$$(2.2)\qquad B(\vartheta) = e^{-\int_0^\vartheta \gamma(\sigma)d\sigma}$$

Formula (2.1) follows by integration of (1.3, *ii*) using the initial datum $i_0(\vartheta)$. Then, we consider the following variable

$$(2.3)\qquad v(t) = \lambda(t)S(t) = i(0,\ t)$$

from which we can get $i(\vartheta,\ t)$, using formula (2.1). We have (see (1.2))

$$v(t) = \int_0^{\vartheta_t} K(\vartheta)i(\vartheta,\ t)d\vartheta\ S(t) =$$

$$= \left[\int_0^t K(\vartheta)B(\vartheta)v(t-\vartheta)d\vartheta + \int_t^\infty K(\vartheta)\ \frac{B(\vartheta)}{B(\vartheta-t)}\ i_0(\vartheta-t)d\vartheta\right] S(t)$$

where $K(\vartheta)$, $B(\vartheta)$, $i_0(\vartheta)$ are extended by zero outside of $[0,\ \vartheta_t]$. Thus we have the following system on the variables $v(t)$ and $S(t)$

$$(2.4)\qquad \begin{cases} \dfrac{d}{dt} S(t) = -v(t) \\ v(t) = \left[\displaystyle\int_0^t A(t - s)v(s)ds + F(t)\right] S(t) \end{cases}$$

where

$$(2.5)\qquad \begin{cases} A(t) = K(t)\ B(t) \\ F(t) = \displaystyle\int_0^\infty K(t + s)\ \frac{B(t+s)}{B(s)}\ i_0(s)ds \end{cases}$$

with the initial condition:

(2.6) $\quad S(0) = S_0 > 0$

Concerning existence and uniqueness of a global solution to (2.4)-(2.6) we have

Theorem 2.1. *Let (1.5)-(1.7) be verified and let* $i_0 \in L^1[0, \vartheta_t]$, *then problem (2.4) with (2.5), (2.6) has a unique solution* $(v(t), S(t))$ *with:*

$$v(t) \geqslant 0, \qquad v(\cdot) \text{ continuous on } [0, +\infty)$$

$$S(t) \geqslant 0, \qquad S(\cdot),\ S'(\cdot) \text{ continuous on } [0, +\infty)$$

Proof:

To prove the thesis of the theorem it is convenient to transform (2.4) into a single equation, in fact, since

$$\frac{d}{dt} S(t) = -\left[\int_0^t A(t-s)v(s)ds + F(t) \right] S(t)$$

we have

(2.7) $$S(t) = S_0 e^{-\left[\int_0^t A_1(t-\sigma)v(\sigma)d\sigma + F_1(t)\right]}$$

where $A_1(t) = \int_0^t A(\sigma)d\sigma$, $F_1(t) = \int_0^t F(\sigma)d\sigma$; then (2.4) is equivalent to

(2.8) $$v(t) = S_0 \left[\int_0^t A(t-\sigma)v(\sigma)d\sigma + F(t) \right] e^{-\left[\int_0^t A_1(t-\sigma)v(\sigma)d\sigma + F_1(t)\right]}$$

Now $F(t)$, $F_1(t)$ are non negative continuous and bounded on $[0, +\infty)$, while $A(t)$, $A_1(t)$ are non negative a.e. and belong to $L^\infty(0, +\infty)$; thus the mapping $\mathcal{T} : C[0, T]$ defined as

(2.9) $$(\mathcal{T}\, v)(t) = S_0 \left[\int_0^t A(t-\sigma)v(\sigma)d\sigma + F(t) \right] e^{-\left[\int_0^t A_1(t-\sigma)v(\sigma)d\sigma + F_1(t)\right]}$$

leaves the set

$$v(\cdot) \in C[0,\ T]; \quad 0 \leqslant v(t) \leqslant S_0|F|_\infty\ e^{S_0|A|_\infty t}$$

unchanged and moreover, for v and $\bar{v}$ belonging to this set,

$$|\mathcal{T}^N v - \mathcal{T}^N \bar{v}|_{C[0,T]} \leqslant \frac{C^N T^N}{N!} \quad |v - \bar{v}|_{C[0,T]}$$

where C is a constant. Thus, existence and uniqueness of a continuous solution to (2.8) follow as a fixed point of $\mathcal{T}$.

Finally, by (2.7) we get $S(t)$. ∎

In the following section we will use (2.4) for analyzing the asymptotic behavior of (1.3).

3 Behavior of the solution

Concerning the behaviour of the epidemic we first have:

Theorem 3.1. *Let $(v(t), S(t))$ be the solution to (2.4) by Theorem 2.1; then we have*

$$\lim_{t \to +\infty} v(t) = 0, \quad \lim_{t \to +\infty} S(t) = S_\infty \tag{3.1}$$

where S_∞ satisfies

$$S_\infty = S_0\, exp \left[S_\infty \int_0^\infty A(\sigma) d\sigma + \int_0^\infty F(\sigma) d\sigma - S_0 \right] \tag{3.2}$$

Proof:

We first note that, from (2.4)

$$S(t) = S_0 - \int_0^t v(s) ds > 0$$

so that

$$\int_0^\infty v(s) ds \leqslant S_0 \tag{3.3}$$

and

$$\lim_{t\to+\infty} S(t) = S_\infty = S_0 - \int_0^\infty v(s)ds \geqslant 0$$

Also

$$F(t) = 0 \quad for \quad t > \vartheta_\dagger$$
$$A(t) = 0 \quad for \quad t > \vartheta_\dagger$$

so that, since by (3.3) $v \in L^1(0, +\infty)$, we have

$$\lim_{t\to+\infty} \int_0^t A(t-s)v(s)ds = 0$$

In conclusion, going to the limit in (2.4), (3.1) is proved.

Concerning the final size of the susceptible class, since $v(t) = -\frac{d}{dt}S(t)$, from (2.7) we get

$$S(t) = S_0 \exp\left[\int_0^t A_I(t-\sigma)\,\frac{dS}{dt}(\sigma)d\sigma + F_I(t)\right] =$$

$$= S_0 \exp\left[\int_0^t A(\sigma)S(t-\sigma)d\sigma + F_I(t) - S_0\right]$$

so that passing to the limit we have (3.2). ■

The previous theorem states two main facts about the single epidemic: the infection eventually dies out but the susceptible class is not depleted by the epidemic. In fact

$$(3.4) \qquad \lim_{t\to+\infty} I(t) = \lim_{t\to+\infty} \int_0^{\vartheta_\dagger} i(\vartheta, t)d\vartheta = \lim_{t\to+\infty} \int_0^\infty v(t-\vartheta)B(\vartheta)d\vartheta = 0$$

and also, by (3.2), it must be $S_\infty > 0$.

Another important aspect of the dynamics of an epidemic is the existence of a threshold in order that the infection be sustained. To introduce this threshold we must first prove:

Proposition 3.2. *Under the conditions of Theorem 2.1, $v(t)$ is either identically zero or eventually positive. If, in addition*

(3.5) $$K(\vartheta) > 0 \quad a.e. \text{ in } [0, \vartheta_+]$$

then $v(t)$ is positive for all $t \geqslant 0$.

Proof:

The proof is close to that of Proposition VI.3.2. In fact, if $v(t)$ is not identically zero, let

$$v(t) > 0 \quad for \quad t \in [\alpha, \beta]$$

then for $t \in [\alpha + \vartheta_1, \beta + \vartheta_2]$ (see (1.7))

$$v(t) \geqslant S(t) \int_0^t A(t-\vartheta)v(\vartheta)d\vartheta \geqslant S(t) \int_\alpha^{t\wedge\beta} A(t-\vartheta)v(\vartheta)d\vartheta \geqslant$$

$$\geqslant S(\beta+\vartheta_2) \min_{\vartheta\in[\alpha,\beta]} v(\vartheta) \int_\alpha^{t\wedge\beta} A(t-\vartheta)d\vartheta =$$

$$= S(\beta+\vartheta_2) \min_{\vartheta\in[\alpha,\beta]} v(\vartheta) \int_{0\vee(t-\beta)}^{t-\alpha} K(\vartheta)B(\vartheta)d\vartheta > 0$$

in fact $(\vartheta_1, \vartheta_2) \cap (0 \vee (t-\beta), t-\alpha) \neq \emptyset$ and $S(\beta + \vartheta_2) > 0$.

Iterating this argument we prove that:

$$v(t) > 0 \quad for \quad t \in [\alpha + n\vartheta_1, \beta + n\vartheta_2]$$

for any positive integer n and consequently $v(t)$ is eventually positive.

Let now (3.5) be satisfied, then $F(0) > 0$ and consequently $v(0) > 0$. If $v(t)$ vanishes somewhere there must exist t_0 such that

$$v(t_0) = 0, \quad v(t) > 0 \quad for \quad t \in [0, t_0)$$

Then we must have:

$$0 = v(t_0) = S(t_0)\left[\int_0^{t_0} A(t_0 - \sigma)v(\sigma)d\sigma + F(t_0)\right] \geqslant$$

$$\geqslant S(t_0)\int_0^{t_0} A(t_0 - \sigma)v(\sigma)ds > 0$$

which is impossible; so it must be $v(t) > 0$ for all $t \geqslant 0$. ■

If we now set $I_k = [k\vartheta_\dagger, (k + 1)\vartheta_\dagger]$ $(k = 1, 2...)$ and define

(3.6) $$m_k = \min_{t\in I_k} v(t), \quad M_k = \max_{t\in I_k} v(t), \quad S_k = S(k\vartheta_\dagger)$$

we have the following immediate consequence of the previous proposition

Proposition 3.3.

$M_k > 0$ for all $k \geqslant 0$ and $m_k > 0$ eventually

If (3.5) is satisfied, then $m_k > 0$ for all $k \geqslant 0$.

Proof:

First we note that, since $F(t)$ is not identically zero on $[0, \vartheta_\dagger]$, neither is $v(t)$ and we have $M_0 > 0$. Besides, assume $M_k > 0$ and let $[\alpha, \beta] \subset I_k$ be such that $v(t) > 0$ on $[\alpha, \beta]$; then, by the proof of Proposition 3.2 $v(t) > 0$ on $[\alpha + n\vartheta_1, \beta + n\vartheta_2]$. Since it is possible to find n such that $(k + 1)\vartheta_\dagger < \alpha + n\vartheta_1 < (k + 2)\vartheta_\dagger$, it is also $v(t) > 0$ somewhere in I_{k+1} and consequently also $M_{k+1} > 0$.

The last part of the thesis is a straighforward consequence of Proposition 3.2. ■

Now we define the following threshold value:

(3.7) $$T = \frac{1}{\int_0^{\vartheta_\dagger} A(\vartheta)d\vartheta}$$

then:

Theorem 3.4. *Let (3.5) be satisfied. Then for $k > 0$:*

(3.8) $\quad$ *if* $S_k < T$ *then* $M_k < M_{k-1}$

(3.9) $\quad$ *if* $S_{k+1} > T$ *then* $m_k > m_{k-1}$

moreover we have

(3.10) $\quad S_\infty < T$

Proof:

Let $t \in I_k$ with $k > 0$, then

$$v(t) = S(t) \int_0^{\vartheta_t} A(s)v(t-s)ds$$

Since $(t-s) \in I_k \cup I_{k-1}$ for $s \in [0, \vartheta_t]$ we have

$$v(t) \leq S(t) \int_0^{\vartheta_t} A(s)ds \; max[M_k, M_{k-1}]$$

$$M_k \leq \frac{S_k}{T} \; max[M_k, M_{k-1}]$$

so that, since $M_k > 0$ we have

$$M_k < max[M_k, M_{k-1}]$$

and (3.8) is proved. The proof of (3.9) is analogous. Concerning (3.10), assume by contradiction that $S_k > T$ for all k, then, by (3.9) the sequence m_k is increasing, which is impossible by (3.1). ∎

The previous theorem allows to give a detailed description of how the epidemic evolves through the sequence of time steps whose length is ϑ_t. For instance we see that, if at the end of the first step the number of susceptibles is under the threshold T, then the epidemics will not be sustained. Also by (3.10) we see that the number of susceptibles will eventually go under this threshold. In general, if $S_0 > T$, let $\bar{t} \in I_{\bar{k}}$ be such that $S(\bar{t}) = T$, then if $\bar{k} > 0$ we will have

$$m_{k+1} > m_k \quad for \; k = 0, 1, \ldots, \bar{k} - 1$$

$$M_{k+1} < M_K \quad for \; k \geq \bar{k}$$

In conclusion we have the following gross description of the epidemic process

The infection blows up and goes on increasing for a finite period of time, as long as the number of susceptibles keeps over the threshold, then it starts decreasing to finally die out.

4 On the constitutive form of the infection rate

In the previous section we have assumed (1.2) as the constitutive form of the infection rate $\lambda(t)$; actually (1.2) is the form adopted by Kermack and McKendrick in their model, but a more realistic assumption for $\lambda(t)$ should include a description of the mechanism of contagion. Thus we consider the general form

$$\lambda(t) = C[S(t)+I(t)+\alpha R(t)] \frac{\int_0^{\vartheta_t} \varphi(\vartheta) i(\vartheta, t) d\vartheta}{S(t) + I(t) + \alpha R(t)} \tag{4.1}$$

Here the function $C[x] : \mathbf{R}_+ \to \mathbf{R}_+$ denotes the number of contacts that an individual has in the unit time when the total size of the *active population* is x. By *active* we mean those individuals who take part in the usual life. In fact, in (4.1) the term $S(t) + I(t) + \alpha R(t)$ accounts for the total active population at time t: the constant $\alpha \in [0, 1]$ is the fraction of removed people who is still active, namely it denotes those removed individuals for whom the disease ended into immunity and, consequently, have a standard behaviour.

Concerning the form of the function $C[x]$, it must reflect the social behaviour and the way people mixe together; the simple assumption of mere proportionality:

$$C[x] = C_0 x \quad (C_0 > 0) \tag{4.2}$$

leads to the form (1.2) with $K(\vartheta) = C_0\varphi(\vartheta)$. In the general case $C[x]$ is assumed to be a non decreasing function of x.

Moreover, in (4.1) the function $\varphi(\vartheta)$ denotes the age specific infectiveness, i.e. the probability of being infected through a contact with an infected individual of age ϑ, thus the term:

$$\frac{\int_0^{\vartheta_t} \varphi(\vartheta) i(\vartheta, t) d\vartheta}{S(t) + I(t) + \alpha R(t)}$$

(5.5) $\varphi \in L^{\infty}(0, a_{\dagger}), \quad \varphi(\vartheta) > 0 \quad a.e. \text{ in } [\vartheta_1, \vartheta_2]$

and the function $C[\cdot]$ is continuously differentiable and such that

(5.6) $C[0] \geqslant 0, \quad C'[x] \geqslant 0, \quad C'[0] > 0$

(5.7) *the function* $x \longrightarrow \dfrac{C[x]}{x}$ *is non-increasing*

Note that, since $\alpha = 0$, we are dealing again with the case in which the first three equations in (5.1) do not depend on $R(t)$, so that the equation for this last can be neglected.

For the analysis of (5.1), (5.2) we proceed as before. Integration along characteristic gives

(5.8)
$$i(\vartheta,t) = \begin{cases} i_0(\vartheta - t)\, e^{-\mu t} \dfrac{B(\vartheta)}{B(\vartheta-t)} & \text{if } \vartheta \geqslant t \\ v(t - \vartheta) e^{-\mu\vartheta} B(\vartheta) & \text{if } \vartheta < t \end{cases}$$

where $B(\vartheta$ and $v(t)$ are defined as in Section 2; then the problem is transformed into the following system

(5.9)
$$\begin{cases} \dfrac{d}{dt} S(t) = \Lambda - \mu S(t) - v(t) \\ I(t) = \displaystyle\int_0^t B_1(t-s) v(s) ds + G(t) \\ v(t) = S(t) \dfrac{C[S(t)+I(t)]}{S(t)+I(t)} \left[\displaystyle\int_0^t A_1(t-s) v(s) ds + F_1(t) \right] \end{cases}$$

with

$$B_1(t) = e^{-\mu t} B(t), \quad A_1(t) = e^{-\mu t} \varphi(t) B(t)$$

$$F_1(t) = e^{-\mu t} \int_0^{\infty} \varphi(t+s) \frac{B(t+s)}{B(s)} i_0(s) ds$$

$$G(t) = e^{-\mu t} \int_0^\infty \frac{B(t+s)}{B(s)} i_0(s)ds$$

where, as usual, φ, B, i_0 are extended by zero outside $[0, \vartheta_\dagger]$.

Here again, the standard theory provides existence of a global solution to (5.9), such that

$$S(t) \geqslant 0, \quad I(t) \geqslant 0, \quad v(t) \geqslant 0$$

$$S(t), \quad S'(t), \quad I(t), \quad v(t) \quad \text{are continuous}$$

We omit the details but for some estimates which imply globality of the solution and will also be used later. First we note the following equality

$$(5.10) \qquad S(t) = S_0 e^{-\mu t} + \frac{\Lambda}{\mu}(1-e^{-\mu t}) - \int_0^t e^{-\mu(t-s)} v(s)ds$$

From this

$$(5.11) \qquad \limsup_{t\to+\infty} \int_0^t e^{-\mu(t-s)} v(s)ds \leqslant \frac{\Lambda}{\mu}$$

Moreover, since by (5.9) and (5.10) we have

$$S(t) + I(t) = S(t) + \int_0^t B_1(t-s)v(s)ds + G(t) \leqslant$$

$$\leqslant S(t) + \int_0^t e^{-\mu(t-s)} v(s)ds + G(t) \leqslant$$

$$\leqslant S_0 e^{-\mu t} + \frac{\Lambda}{\mu}(1-e^{-\mu t}) + G(t)$$

we get

$$(5.12) \qquad \limsup_{t\to+\infty} (S(t) + I(t)) \leqslant \frac{\Lambda}{\mu}$$

Finally

$$\limsup_{t\to+\infty}\int_0^t A_1(t-s)v(s)ds \leqslant |\varphi|_\infty \limsup_{t\to+\infty}\int_0^t e^{-\mu(t-s)}v(s)ds \leqslant |\varphi|_\infty \frac{\Lambda}{\mu}$$

implies

(5.13) $$\limsup_{t\to+\infty} v(t) < +\infty$$

From these we get

Theorem 5.1. *Let*

(5.14) $$C\left[\frac{\Lambda}{\mu}\right]\int_0^{\vartheta_t} e^{-\mu\vartheta}\varphi(\vartheta)B(\vartheta)d\vartheta < 1,$$

then

(5.15) $$\lim_{t\to+\infty} v(t) = 0$$

Proof:

By (5.9) we have

$$v(t) \leqslant C[S(t)+I(t)]\left[\int_0^t A_1(t-s)v(s)ds + F_1(t)\right]$$

and, since

$$\limsup_{t\to+\infty}\int_0^t A_1(t-s)v(s)ds \leqslant \int_0^\infty A_1(s)ds \limsup_{t\to+\infty} v(t)$$

we get

$$\limsup_{t\to+\infty} v(t) \leqslant C\left[\frac{\Lambda}{\mu}\right]\int_0^\infty A_1(s)ds \limsup_{t\to+\infty} v(t)$$

so that (5.15) necessarily follows from (5.13) and (5.14) ∎

Condition (5.14) is a threshold condition and the parameter

$$R_0 = C\left[\frac{\Lambda}{\mu}\right]\int_0^{\vartheta_f} e^{-\mu\vartheta}\varphi(\vartheta)B(\vartheta)d\vartheta \tag{5.16}$$

is called the *basic reproductive number* of the epidemic. It can be interpreted as the average number of new cases that an infective individual can produce when the active population size is $\frac{\Lambda}{\mu}$. Actually, $S^* = \frac{\Lambda}{\mu}$, $i^*(\vartheta) \equiv 0$ is a stationary state for the system (5.1) and it is called the *disease free state*. By (5.10), Theorem 5.1 yields

If $R_0 < 1$, the epidemics goes extinct and the susceptible class attains the stationary state $\frac{\Lambda}{\mu}$.

Note that, with the special assumption (4.2) and $\mu = 0$, R_0 is the same as the threshold for the model of section 3; when $R_0 > 1$, existence of non trivial stationary states in the present model occurs (endemic states of the epidemics): we will investigate this point in the next section.

6 Endemic states

We now look for stationary states of problem (5.1), (5.3); namely we look for a solution $(S^*, i^*(\vartheta))$ of the problem

$$\begin{cases} \Lambda - \mu S^* - \lambda^* S^* = 0 \\ i^*_\vartheta(\vartheta) + \gamma(\vartheta) i^*(\vartheta) + \mu\, i^*(\vartheta) = 0 \\ i^*(0) = \lambda^* S^* \\ \lambda^* = C[S^* + I^*] \dfrac{\int_0^{\vartheta_f} \varphi(\vartheta) i^*(\vartheta) d\vartheta}{S^* + I^*} \\ I^* = \int_0^{\vartheta_f} i^*(\vartheta) d\vartheta \end{cases} \tag{6.1}$$

We remark that the disease free equilibrium $\left(\frac{\Lambda}{\mu}, 0\right)$ is actually a solution of (6.1), as we have already noted in the previous section. In order to investigate existence of other solutions such that $i^*(\vartheta) \not\equiv 0$ (i.e. endemic states of the disease) we have to transform (6.1). Actually, since from the second equation in (6.1) we have

$$(6.2) \qquad i^*(\vartheta) = v^* e^{-\mu\vartheta} B(\vartheta)$$

where we have set $v^* = i^*(0) > 0$, it is easy to see that via (6.2) problem (6.1) is equivalent to

$$(6.3) \qquad \begin{cases} \Lambda - \mu S^* - v^* = 0 \\ I^* = v^* \int_0^{\vartheta_f} e^{-\mu\vartheta} B(\vartheta) d\vartheta \\ 1 = \dfrac{S^*}{S^* + I^*} C[S^* + I^*] \int_0^{\vartheta_f} \varphi(\vartheta) e^{-\mu\vartheta} B(\vartheta) d\vartheta \end{cases}$$

Comparing this with (5.9), we also see that (v^*, S^*, I^*) is a constant solution of this latter. To solve (6.3) we introduce the variable

$$(6.4) \qquad \xi = \frac{I^*}{S^* + I^*}$$

so that

$$(6.5) \qquad I^* = \xi(S^* + I^*), \quad S^* = (1 - \xi)(S^* + I^*)$$

From the first two equation in (6.3) we get

$$\Lambda - \mu(1-\xi)(S^*+I^*) - \frac{\xi}{\int_0^{\vartheta_f} e^{-\mu\vartheta} B(\vartheta) d\vartheta} (S^* + I^*) = 0$$

that is

$$(6.6) \qquad (S^*+I^*) = \frac{\Lambda}{\mu + \left(\dfrac{1}{\int_0^{\vartheta_f} e^{-\mu\vartheta} B(\vartheta) d\vartheta} - \mu \right) \xi}$$

Thus, plugging (6.6) and (6.5) into the third equation we finally get

$$(6.7) \quad 1=(1-\xi)C\left[\frac{\Lambda}{\mu+\left(\dfrac{1}{\int_0^{\vartheta_\dagger} e^{-\mu\vartheta}B(\vartheta)d\vartheta}-\mu\right)\xi}\right]\int_0^{\vartheta_\dagger}\varphi(\vartheta)e^{-\mu\vartheta}B(\vartheta)d\vartheta$$

This last equation is equivalent to (6.3) via formulas (6.5), (6.6) so that we are left with the search of a $\xi \in (0, 1)$ satisfying (6.7) (note that $\xi = 0$ would give the disease free equilibrium). Now, since $C[x]$ is non decreasing and

$$\int_0^{\vartheta_\dagger} e^{-\mu\vartheta} B(\vartheta)d\vartheta \leq \frac{1}{\mu}$$

the whole right hand side of (6.7) is a decreasing function of ξ with values R_0 and 0 at $\xi = 0$ and $\xi = 1$ respectively; this implies that

> *Equation (6.7) has a solution $\xi^* \in (0, 1)$ if and only if $R_0 > 1$. This solution is unique.*

Thus the threshold parameter R_0 is responsible for the existence of endemic states. We collect the previous results in the following:

Theorem 6.1. *If $R_0 \leq 1$ the system (5.1), (5.3) admits only the disease free stationary state $\left(\frac{\Lambda}{\mu}, 0\right)$. If $R_0 > 1$ there is also another stationary state $(S^*, v^* e^{-\mu\vartheta} B(\vartheta))$ with $S^* < \frac{\Lambda}{\mu}, v^* > 0$.* ∎

We have already seen that $R_0 < 1$ implies the extinction of the epidemics (Theorem 6.1); if $R_0 > 1$ the behaviour of the solution to (5.1) is not completely known. A detailed analysis of this behavior has been performed in [95]-[96], showing that with $R_0 > 1$ the epidemic is persistent and the endemic state can be stable or unstable according with the properties of the various parameters involved in the system. In particular it is shown that periodic solutions may arise when the endemic state loses its stability.

7 Comments and references

With this last chapter we have gone back again to the very first age structured models. In fact the Kermack-McKendrick model presented in Section 1 is part of the earliest production in the field together with the papers by Lotka. Though this first step in the description of epidemics included age structured in the model, the names of the authors have been usually associated with the simplified version that disregards age structure.

Later on, class age for epidemics has been considered in [56] where a general model is formulated including both chronological and class age. Results on this kind of models have been stated by several authors, but only recently, in connection with the HIV/AIDS epidemics, it has received more serious attention. The model of Section 5 was introduced and widely analyzed in [15], [94], [95], while in [59], [60] it was numerically implemented and used to attempt a description of the AIDS epidemic in Italy.

Appendix I

Laplace transform

In this Appendix we present some definitions and results concerning Laplace transform theory. They are of course well known, but we think it useful to collect them here so as to have some precise statements to refer to, when needed in the text. We will not go through the proof of these results, but will refer to textbooks on the theory; in particular we suggest the monography by G. Doetsch [34] to which this appendix is inspired.

1 Definitions and properties

Let $f(\cdot) \in L^1_{loc}$ $(\mathbf{R}_+; \mathbf{R})$ *and* $\lambda \in \mathbf{C}$. Then $f(\cdot)$ is said to be *Laplace transformable* at λ, if the integral

$$(1.1) \qquad \hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt$$

exists as an improper integral i.e. if the following limit exists

$$\lim_{T \to +\infty} \int_0^T e^{-\lambda t} f(t)dt$$

Moreover $f(\cdot)$ is said to be *absolutely Laplace transformable* at λ, if the integral in (1.1) is absolutely convergent.

It is easy to see that if f is Laplace transformable (resp. absolutely Laplace transformable) at λ_0, then it is Laplace transformable (resp. absolutely Laplace transformable) at any λ such that $\Re\lambda > \Re\lambda_0$. Thus we can define the *abscissa of convergence*

$$\sigma = inf\ \{\lambda_0 \in \mathbf{R} | f \textit{ is Laplace transformable at } \lambda_0\}$$

so that (1.1) defines a complex function in the half plane $S_\sigma = \{\lambda | Re\lambda > \sigma\}$; this function turns out to be analytical in S_σ. Thus

> *The analytic function $\hat{f}(\lambda)$, defined by (1.1) on S_σ, is called the Laplace transform of f.*

The Laplace transform, thanks to its properties, is a useful tool to treat differential and integral problems; here we list some basic statements

Theorem 1.1. *Let $f(t)$ be Laplace transformable at $\lambda_0 > 0$ and consider*

$$F(t) = \int_0^t f(s)ds \quad for \quad t \geqslant 0$$

Then $F(t)$ is absolutely Laplace transformable for $\Re\lambda > \lambda_0$ and

$$\hat{F}(\lambda) = \frac{\hat{f}(\lambda)}{\lambda} \quad for \quad \Re\lambda > \lambda_0$$

■

Theorem 1.2. *Let $f(t)$ be absolutely continuous and suppose that $f'(t)$ is Laplace transformable at $\lambda_0 > 0$. Then $f(t)$ is absolutely Laplace transformable for $\Re\lambda > \lambda_0$ and*

$$\hat{f}'(\lambda) = \lambda\hat{f}(\lambda) - f(0^+) \quad for \ \Re\lambda > \lambda_0$$

■

Theorem 1.3. *Let $f(t)$ be Laplace transformable at $\lambda_0 > 0$ and $g(t)$ be absolutely Laplace transformable at $\lambda_0 > 0$. Consider the convolution*

$$F(t) = \int_0^t f(t-s)g(s)ds \quad for \ t \geqslant 0.$$

Then $F(t)$ is Laplace transformable for $\Re\lambda > \lambda_0$ and

$$\hat{F}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda) \quad for \ \Re\lambda > \lambda_0$$

■

2 The inversion formula

A central problem in the theory of the Laplace transform is that of recovering the original function $f(t)$ when its transform $\hat{f}(\lambda)$ is known; a main result in this respect is the following

$$(3.1) \qquad \hat{f}(\lambda) = \sum_{i=-m}^{+\infty} c_i(\lambda - \lambda_0)^i$$

Suppose in addition that there exists $\sigma_1 < \Re\lambda_0$ *such that*

$$(3.2) \qquad \lim_{\substack{|\lambda|\to+\infty \\ \sigma_1\leq\Re\lambda\leq\sigma}} \hat{f}(\lambda) = 0$$

Then there exists $\delta < \Re\lambda_0$ *such that*

$$(3.3) \qquad f(t) = e^{\lambda_0 t} \sum_{i=1}^{m} c_{-i} \frac{t^{i-1}}{(i-1)!} + \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} e^{\lambda t} \hat{f}(\lambda) d\lambda \qquad \blacksquare$$

Of course (3.3) determines the asmptotic behavior of $f(t)$ provided we are able to determine the behavior of the last integral; however the following reasonable assumption

$$(3.4) \qquad \int_{-\infty}^{+\infty} |\hat{f}(\delta + iy)| dy < +\infty$$

is enough to have

$$\lim_{t\to+\infty} e^{-\Re\lambda_0 t} \int_{\delta-i\infty}^{\delta+i\infty} e^{\lambda t} \hat{f}(\lambda) d\lambda = 0$$

A repeated application of the previous theorem can provide an asymptotic expansion of the original function.

Appendix II
Integral equations theory

This Appendix is devoted to present some results from the theory of Volterra integral equations and is intended to introduce, in a somewhat complete form, those aspects of the theory that underlie the methods used in this book. Thus the outcome of this presentation is a rather special collection of results, including a few that are not easy to find in the literature and that will be presented in some detail. As a reference we suggest the book by Miller [80] and the more recent treatise by Gripenberg, Londen and Staffans [41].

1 The linear theory

Here we consider the linear Volterra convolution system

$$u(t) = \int_0^t K(t-s)u(s)ds + f(t) \tag{1.1}$$

where the unknown $u(t)$ and the input datum $f(t)$ are n-vectors, while $K(t)$ is an $n \times n$ matrix. We assume

$$K(\cdot) \in L^1([0, \infty); \mathcal{L}(\mathbf{R}_n)), \quad f(\cdot) \in L^1([0, \infty); \mathbf{R}^n) \tag{1.2}$$

Some special results on (1.1), in the one dimensional case, have been discussed in Chapter I; here we want to present a general treatment of the problem based on the use of the resolvent equations

$$R(t) = -K(t) + \int_0^t K(t-s)R(s)\, ds \tag{1.3}$$

$$R(t) = -K(t) + \int_0^t R(t-s)K(s)\, ds \tag{1.4}$$

In fact, concerning these equations we have

Theorem 1.1. *Let $K(\cdot)$ satisfy (1.2), then there exists a unique $R(\cdot) \in L^1_{loc}$ $([0, \infty])$ satysfying (1.3)-(1.4) and such that for any $f(\cdot)$ satisfying (1.2)*

$$(1.5) \qquad u(t) = f(t) - \int_0^t R(t-s) f(s)\, ds$$

is the unique solution of (1.1). ∎

The proof of this theorem is based on the usual iterative procedure (see I.4) which yields the solution of (1.3)-(1.4) in the following form

$$(1.6) \qquad R(t) = - \sum_{i=1}^{\infty} (K * K * K * \overset{i \text{ times}}{\cdots} * K)(t)$$

Where $*$ denote convolution and the series converges in L^1_{loc} $([0, \infty); \mathcal{L}(\mathbf{R}_n))$. The main interest of the theorem, besides existence and uniqueness of a solution to (1.1), is the representation formula (1.5) which gives this solution versus the input function $f(\cdot)$ and allow to get properties of the solution in connection with those of $f(\cdot)$. In this respect a special situation occurs when the resolvent kernel $R(\cdot)$ is integrable over the whole semiline $[0, \infty)$; in fact we have

Proposition 1.2. *Let*

$$(1.7) \qquad R(\cdot) \in L^1([0, \infty); \mathcal{L}(\mathbf{R}^n))$$

then, if $f(\cdot) \in C_B([0, \infty); \mathbf{R}^n)$ we have

$$(1.8) \qquad |u(t)| \leqslant (1 + \|R\|_{L^1}) \, |f|_\infty \quad \forall\, t > 0$$

If moreover $\lim_{t \to +\infty} f(t) = 0$ then

$$(1.9) \qquad \lim_{t \to +\infty} u(t) = 0$$

∎

The previous Proposition is actually a stability result for the trivial solution $u(t) \equiv 0$ of (1.1) (corresponding to the trivial input $f(t) \equiv 0$, according with the following definition

Definition 1.3. *The trivial solution of (1.1) is said to be stable if* $\forall\ \varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\text{(1.10)} \qquad \text{if} \quad |f|_\infty < \delta \quad \text{then} \quad |u|_\infty < \varepsilon$$

It is said to be asymptotically stable if it is stable and

$$\text{(1.11)} \qquad \text{if} \quad \lim_{t\to+\infty} f(t) = 0 \quad \text{then} \quad \lim_{t\to+\infty} u(t) = 0$$

■

This definition, which is relative to the case of continuous inputs, is the concept of stability that we need for our purposes and that we will extend to the nonlinear case. Here we want to stress a crucial point concerning condition (1.7). In fact Proposition 1.2 states that this condition implies asymptotic stability of (1.1); actually we have more

Theorem 1.4. *The trivial solution of (1.1) is stable if and only if* $R(\cdot)$ *satisfies (1.7).*

Proof:

We give the proof for the scalar case, since the extension to the general case is trivial. First we note that, by (1.8), condition (1.7) implies stability. Then we suppose that stability occurs and prove (1.7).

To this purpose, by Definition 1.3, we let δ be such that for $f \in C_B([0, \infty), \mathbf{R})$ and $|f|_\infty \leq \delta$ the solution $u(t)$ to (1.1) satisfies $|u(t)| \leq 1$. Then for any $f \in C_B([0, \infty), \mathbf{R})$ we set $g(t) = \dfrac{\delta}{|f|_\infty} f(t)$ so that, since

$$\text{(1.12)} \qquad (R * f)\,(t) = f(t) - \frac{|f|_\infty}{\delta}\,[g(t) - (R * g)\,(t)]$$

and the square bracket term is the solution to (1.1) with input $g(\cdot)$, we conclude

$$\text{(1.13)} \qquad |(R * f)\,(t)| \leq \left(1 + \frac{1}{\delta}\right) |f|_\infty$$

Now we suppose by contraddiction that (1.7) is not satisfied and let t_n be such that

$$\lim_{t \to \infty} t_n = +\infty, \quad \int_0^{t_n} |R(s)| \, ds > n$$

Next we set

$$\phi_n(s) = \text{sign}\,(R(t_n - s)), \quad \forall s \in [0, t_n]$$

so that

$$\int_0^{t_n} R(t_n - s)\phi_n(s) \, ds > n, \quad |\phi_n(s)| \leq 1 \quad \text{a.e. in} \quad [0, t_n]$$

Then, for any fixed n we consider the sequence $\{\phi_n^k(s)\}_k$ of continuous functions on $[0, t_n]$ such that

$$|\phi_n^k(s)| \leq 1, \; \phi_n^k(t_n) = 0, \; \lim_{k \to \infty} \phi_n^k(s) = \phi_n(s) \quad \text{a.e. in } [0, t_n]$$

and choose $\bar{k}$ such that

$$\text{(1.14)} \qquad \int_0^{t_n} R(t_n - s)\phi_n^{\bar{k}}(s) \, ds > n$$

Finally, setting

$$f_n(t) = \begin{cases} \phi_n^{\bar{k}}(t) & \text{for } t \in [0, t_n] \\ 0 & \text{for } t > t_n \end{cases}$$

we have $f_n \in C_B([0, \infty), \mathbf{R})$, $|f_n|_\infty \leq 1$ and, by (1.13),

$$|(R * f_n)(t)| \leq \left(1 + \frac{1}{\delta}\right)$$

which contradicts (1.14). ■

As a consequence of this theorem and of Proposition 1.2, we also have

Corollary 1.5. *The trivial solution of (1.1) is stable if and only if it is asymptotically stable.* ■

2 The Paley-Wiener theorem

In the previous section we have seen that in order to study the asymptotic stability of (1.1) we are led to consider conditions on the kernel $K(\cdot)$ so that (1.7) be satisfied: here we present a classical result that is usually known as the Paley-Wiener Theorem and that is basic tool for investigating stability of integral equations (see [80]). In this respect let

$$\hat{K}(\lambda) = \int_0^\infty e^{\lambda t} K(t)\, dt$$

denote the Laplace transform of $K(\cdot)$ which, by (1.2), exists absolutely for $\Re\lambda \geq 0$; then we have

Theorem 2.1. *The following condition*

$$\text{(2.1)} \qquad \det\,(I - \hat{K}(\lambda)) \neq 0 \quad \text{for} \quad \Re\lambda \geq 0$$

is necessary and sufficient in order that (1.7) be satisfied.

Proof:

We prove the theorem in the scalar case since the general case is analogous (note that in the scalar case the two equations (1.3)-(1.4) coincide). Let us first prove that (2.1) is necessary. In fact, (1.7) implies that $R(\cdot)$ is absolutely Laplace transformable for $\Re\lambda \geq 0$ and, moreover, from (1.3) we have

$$\hat{R}(\lambda) - \hat{K}(\lambda)\hat{R}(\lambda) = -\hat{K}(\lambda), \quad \Re\lambda \geq 0$$

that is

$$(1 - \hat{K}(\lambda))\,(1 - \hat{R}(\lambda)) = 1, \quad \Re\lambda \geq 0$$

so that (2.1) must be satisfied.

To prove that (2.1) is sufficient we first prove that $R(\cdot)$ is absolutely Laplace transformable for $\Re\lambda$ sufficiently large. In fact, let $\lambda > 0$ be sufficiently large in order that

$$a = \int_0^\infty e^{-\lambda t} |K(t)|\, dt < 1$$

Then, from (1.3)

$$\int_0^T e^{-\lambda t}|R(t)|\,dt \leq a + \int_0^T e^{-\lambda t}\int_0^t |K(t-s)||R(s)|\,ds\,dt$$

$$\leq a + \int_0^T e^{-\lambda s}|R(s)| \int_s^T e^{-\lambda(t-s)}|K(t-s)|\,dt\,ds$$

$$\leq a + a\int_0^T e^{-\lambda s}|R(s)|\,ds$$

and

$$\int_0^T e^{-\lambda s}|R(s)|\,ds \leq \frac{a}{1-a}$$

which implies absolute Laplace transformability of $R(\cdot)$ for $\Re\lambda$ sufficiently large. We also note that by (1.3) we also have

(2.2) $$\hat{R}(\lambda) = \frac{\hat{K}(\lambda)}{\hat{K}(\lambda)-1} \quad \textit{for} \quad \Re\lambda \quad \textit{sufficienlty large}$$

We now need a basic result from Fourier transformation theory; namely, denoting by $f^*(x)$ the Fourier transform of $f \in L^1(\mathbf{R})$

$$f^*(x) = \int_{-\infty}^{+\infty} e^{-ixt} f(t)\,dt, \quad x \in \mathbf{R}$$

we recall

(2.3) *Let $F(z)$ be analytic in the connected open set $A \ni 0$ and such that $F(0) = 0$. Let $f \in L^1(\mathbf{R})$ be such that $f^*(x) \in A$ for $x \in \mathbf{R}$. Then there exists $g \in L^1(\mathbf{R})$ such that*

$$g^*(x) = F(f^*(x)) \quad \forall\, x \in \mathbf{R}.$$

Then we call $\tilde{K}(\cdot)$ the extension of $K(\cdot)$ by setting $\tilde{K}(t) = 0$ for $t < 0$; of course, by (2.1) we have

$$\tilde{K}^*(x) = \hat{K}(ix) \neq 1, \quad \forall\, x \in \mathbf{R}$$

so that, since $F(z) = \dfrac{z}{z-1}$ is analytic in $\mathbf{C} - \{1\}$, there exists $g \in L^1(\mathbf{R})$ such that

$$(2.4) \qquad g^*(x) = \frac{\hat{K}(ix)}{\hat{K}(ix)-1} \qquad x \in \mathbf{R}$$

Now we consider the two functions

$$\phi_1(z) = \int_{-\infty}^{0} e^{-zt} g(t)\, dt \quad \text{in the half plane} \quad \Re z \leqslant 0$$

$$\phi_2(z) = \frac{\hat{K}(z)}{\hat{K}(z)-1} - \int_0^{\infty} e^{-zt} g(t)\, dt \quad \text{in the half plane} \quad \Re z \geqslant 0$$

These are analytic in their respective domains, moreover by (1.16) they satisfy

$$\phi_1(ix) = \phi_2(ix) \quad x \in \mathbf{R}$$

$$\phi(z) = \phi_1(z) \quad for \quad \Re z < 0$$

$$\phi(z) = \phi_2(z) \quad for \quad \Re z < 0$$

Now, ϕ is bounded because so are ϕ_1 and ϕ_2, consequently it is constant and, since $\lim_{x \to -\infty} \phi(x) = 0$, it is $\phi(z) = 0 \quad \forall\, z \in \mathbf{C}$. This yields

$$\hat{g}(\lambda) = \frac{\hat{K}(\lambda)}{\hat{K}(\lambda)-1} \quad for \quad \Re\lambda \geqslant 0$$

which, compared with (1.15) implies $R(t) = g(t)$ and (1.7) is proved. ■

3 A class of non-linear functional perturbations

Now we consider the following perturbation of equation (1.1)

$$(3.1) \qquad u(t) = \int_0^t K(t-s)u(s)\, ds + \mathcal{P}[u(\cdot), c(\cdot)](t)$$

where the kernel $K(\cdot)$ is supposed to verify the assumption (1.2) and the non-linear functional term

$$\mathcal{P} : C_0\ ([0,\ +\infty);\ \mathbf{R}^n) \times L^1([a,\ b];\ \mathbf{R}^m) \to C_0\ ([0,\ +\infty);\ \mathbf{R}^n)$$

satisfies the following conditions

(3.2) $\mathcal{P}[0,\ 0] = 0$

(3.3) *There exists a function* $R \to L(R)$, $\lim_{R \to 0} L(R) = 0$ *such that*

$$|\mathcal{P}[u(\cdot),\ c(\cdot)] - \mathcal{P}[\bar{u}(\cdot),\ c(\cdot)]|_\infty \leqslant L(R)|u - \bar{u}|$$

for $|u|_\infty,\ |\bar{u}|_\infty,\ |c|_{L^1} \leqslant R$

(3.4) *There exists a constant* $K > 0$ *such that*

$$|\mathcal{P}[0,\ c(\cdot)]|_\infty \leqslant K\ |c|_{L^1} \quad \forall\ c \in L^1\ ([a,\ b];\ \mathbf{R}^m)$$

In equation (3.1) the term $c \in L^1\ ([a,\ b];\ \mathbf{R}^m)$ acts as an input variable and is supposed to be assigned.

We let $R(\cdot)$ be the resolvent kernel relative to $K(\cdot)$ and transform (3.1) into

$$u(t) = \mathcal{P}[u(\cdot),\ c(\cdot)](t) - \int_0^t R(t-s)\ \mathcal{P}[u(\cdot),\ c(\cdot)]\ (s)\ ds \tag{3.5}$$

The following result is strictly related to the condition (2.1) of the Paley-Wiener Theorem

Theorem 3.1. *Let* K *and* $\mathcal{P}$ *satisfy the assumptions (1.2), (3.2)-(3.4) and suppose that (1.7) holds. Then for any* $\varepsilon > 0$ *there exist* δ *such that for any* $c \in L^1\ ([a,\ b];\ \mathbf{R}^m)$ *satisfying* $|c|_{L^1} \leqslant \delta$ *equation (3.1) has one and only one solution* $u \in C_0\ ([0,\ +\infty];\ \mathbf{R}^n)$ *such that* $|u|_\infty \leqslant \varepsilon$.

Proof:

Take $\varepsilon > 0$; then, since $R(\cdot) \in L^1\ ([0,\ \infty);\ \mathcal{L}\ (\mathbf{R}^n))$, we can chose $\eta < \varepsilon$ such that

$$L(\eta) < \frac{1}{2(1 + K)(1 + \|R\|_{L^1})}\ ;$$

then we set

$$\delta = L(\eta)\ \eta$$

Now we define the set

$$\mathcal{K} \equiv \{u \in C_0\ ([0,\ +\infty);\ \mathbf{R}^n);\ |u|_\infty \leqslant \eta\}$$

and for any fixed $c \in L^1\ ([a,\ b];\ \mathbf{R}^m)$, such that $|c|_{L^1} \leqslant \delta$, we consider the mapping $\mathcal{T}$ defined as

$$(\mathcal{T}u)(t) = \mathcal{P}[u(\cdot),\ c(\cdot)](t) - \int_0^t R(t-s)\mathcal{P}[u(\cdot),\ c(\cdot)]\ (s)\ ds \quad \forall\, u \in \mathcal{K} \tag{3.6}$$

Actually, for any $u \in \mathcal{K}$ we have

$$|(\mathcal{T}u)|_\infty \leqslant (1+\|R\|_{L^1})(L(\eta)|u|_\infty + K|c|_{L^1}) \leqslant (1+\|R\|_{L^1})(1+K)L(\eta)\eta < \eta$$

and for $u,\ \bar{u} \in \mathcal{K}$

$$|(\mathcal{T}u) - (\mathcal{T}\bar{u})|_\infty \leqslant (1+\|R\|_{L^1})\ L(\eta)\ |u-\bar{u}|_\infty < \frac{1}{2}\ |u-\bar{u}|_\infty$$

Thus $\mathcal{T}$ is a contraction such that $\mathcal{T}(\mathcal{K}) \subset \mathcal{K}$ and, consequently, it has one and only one fixed point in $\mathcal{K}$. This proves the theorem. ∎

References

[1] ANDERSON, R. and MAY, R.: *Vaccination against rubella and measles: quantitative investigations of different policies*. J. Hyg. Camb. 90, 259-325 (1983).

[2] ANDERSON, R. and MAY, R.: *Age-related changes in the rate of disease transmission: implication for the designing of vaccination programs*. J. Hyg. Camb. 94, 365-436 (1985).

[3] ANDREASEN, V. A.: *Instability in a S-I-R model with age-dependent susceptibility*. To appear (1992).

[4] ANDREASEN, V. A.: *The effect of age dependent host mortality on the dynamics of an endemic disease*. Math. Biosc. to appear (1992).

[5] BACCHETTI, P. and MOSS, A. R.: *Incubation period of AIDS in San Francisco*. Nature 338, 251-253 (1989).

[6] BERNOULLI, D.: *Essai d'une nouvelle analyse de la mortalité causée par la petite vérole et les advantages de l'inoculation pour la prévenir*. Histoire de l'Académie Royale des Sciences, Année 1760, (1766).

[7] BLYTHE, S. P. and ANDERSON, R. M.: *Variable infectiousness in HIV Transmission Models*. IMA Journal of Mathematics Applied in Medicine e Biology 5, 181-200 (1988).

[8] BUSENBERG, S. and COOKE, K.: *Vertically transmitted diseases, Models and Dynamics*. Springer, Biomathemathics V. 23 (1993).

[9] BUSENBERG, S., COOKE, K. and IANNELLI, M.: *Endemic threshold and stability in a class of age-structured epidemics*. SIAM J. Appl. Math. 48, 1379-1395 (1988).

[10] BUSENBERG, S., COOKE, K. and IANNELLI, M.: *Stability and threshold in some age-structured epidemics*. Lecture Notes in Biomathematics 81, C. Castillo-Chavez, S. Levin and C. Shoemaker, Ed., Springer-Verlag, Berlin-Heidelberg-New York, 124-141 (1989).

[11] BUSENBERG, S. and IANNELLI, M.: *Separable models in age-dependent population dynamics*. J. Math. Biol. 22, 145-173 (1985).

[12] BUSENBERG, S. and IANNELLI, M.: *A class of nonlinear diffusion problems in age-dependent population dynamics*. J. Nonlinear Analysis T.M.A., vol. 7 n. 5, 501-529 (1983).

[13] BUSENBERG, S., IANNELLI, M. and THIEME, H.: *Global behaviour of an age-structured S-I-S epidemic model.* SIAM J. Appl. Math. 22, 1065-1080 (1991).

[14] BUSENBERG, S., IANNELLI, M. and THIEME, H.: *Dynamics of an age-structured epidemic model.* In «Dynamical systems» S-T Liao, T-R Ding, Y-Q Ye eds., Nankai series in Pure, Applied Mathematics and Theoretical Physics, Vol. 4, World Scientific (1993).

[15] CASTILLO-CHAVEZ, C., COOKE, K., HUANG, W. and LEVIN, S. A.: *On the role of long incubation periods in the dynamics of acquired immunodeficiency syndrome (AIDS). 1. Single population models.* J. Math. Biol. 27, 373-398 (1989).

[16] CHIPOT, M.: *On the equations of age-dependent population dynamics.* Arch. Rat. Mech. Anal. 82, 13-26 (1983).

[17] CHIPOT, A., IANNELLI, M. and PUGLIESE, A.: *Age Structured S-I-R Epidemic Model with Intracohort Transmission.* U.T.M. 407 November 1992.

[18] CLEMENT, PH., DIEKMANN, O., GYLLENBERG, M., HEIJMANS, H. J. A. M. and THIEME, H. R.: *Perturbation Theory for dual semigroups. I. The sun-reflexive case.* Math. Ann. 277, 709-725 (1987)

[19] CLEMENT, PH., DIEKMANN, O., GYLLENBERG, M., HEIJMANS, H. J. A. M. and THIEME, H. R.: *Perturbation Theory for dual semigroups. II. Time-dependent perturbations in the sun-reflexive case.* Proc. Roy. Soc. Edinburgh 109A, 145-172 (1988).

[20] CLEMENT, PH., DIEKMANN, O., GYLLENBERG, M., HEIJMANS, H. J. A. M. and THIEME, H. R.: *Perturbation Theory for dual semigroups. III. Non-linear Lipschitz continuous perturbations in the sun-reflexive case.* In «Volterra Integrodifferential Equations in Banach Spaces and Applications» (G. Da Prato and M. Iannelli eds.) 67-89, Longman, New York (1989).

[21] COALE, A. J.: *The growth and structure of human populations. A mathematical investigation.* Princeton University Press, Princeton, New Jersey, (1972).

[22] CUSHING, J.: *Model Stability and instability in age structured population dynamics.* J. Theor. Biol. 86, 709-730 (1980).

[23] CUSHING, J. M.: *Stability and maturation periods in age structured populations.* In: Differential Equations and Applications in Ecology, Epidemics, and Population Problems. (S. N. Busenberg and K. L. Cooke eds.). Academic Press, New York (1981).

[24] Cushing, J. M.: *Bifurcation of time periodic solutions of the McKendrick equations with applications to population dynamics*. Comp. e Maths. with Appls. 9, 459-478 (1983).

[25] Cushing, J. M.: *Existence and stability of equilibria in age-structured population dynamics*. J. Math. Biol. 20, 259-276 (1984).

[26] Cushing, J. M.: *Global branches of equilibrium solutions of the McKendrick equations for age-structured population growth*. Comp. e Math. with Appl. 11, 459-478 (1985a).

[27] Cushing, J. M. and Saleem, M.: *A predator-prey model with age structure*. J. Math. Biol. 14, 231-251 (1982).

[28] de Ross, A. M.: *Numerical methods for structured population models: the escalator boxcar train*. Numer. Methods for Partial Differential Equations 4, 173-195 (1988).

[29] Desch, W. and Schappacher, W.: *Linearized stability for non-linear semigroups*. In «Differential Equation in Banach Spaces» Proceedings Bologna 1985 (A. Favini and E. Obrecht eds.), Lecture Notes in Mathematics 1223, Springer Verlag (1986).

[30] Di Blasio, G., Iannelli, M. and Sinestrari, E.: *Approach to equilibrium in age structured population with an increasing recruitment process*. J. Math. Biol. 13, 371-382 (1982).

[31] Diekmann, O., Nisbet, R. M., Gurney, W. S. C. and Bosch, F. Van Den: *Simple mathematical models for cannibalism: a critique and a new approach*. Math. Biosc. 78, 21-46 (1986).

[32] Diekmann, O., Heesterbeck, H. and Metz, H.: *On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations*. J. Math. Biol. 28, 365-382 (1990).

[33] Dietz, K. and Schenzle, D.: *Proportionate mixing models for age-dependent infection transmission*. J. Math. Biol. 22, 117-120 (1985).

[34] Doetsch, G.: *Introduction to the Theory and Application of the Laplace Transformation*. Springer-Verlag (1974).

[35] Douglas, Jr., J. and Milner, F. A.: *Numerical methods for a model of population dynamics*. Calcolo, 24, 247-254 (1987).

[36] Euler, L.: *Recherches générales sur la mortalité et la multiplication du genre humaine*. Mémoires de l'Académie Royale des Sciences et Belles Lettres, vol. 16 (1760).

[37] Feller, W.: *On the integral equation of renewal theory*. Annals of Mathematical Statistics 12, 243-267 (1941).

[38] FRAUENTHAL, J. C.: *Some simple models of cannibalism*. Math. Biosc. 63, 87-98 (1983).

[39] GREENHALGH, D.: *Analytic results on the stability of age-structured recurrent epidemic models*. IMA J. Math. Appl. Med. Biol. 4, 109-144 (1987).

[40] GREENHALGH, D.: *Threshold and stability results for an epidemic model with an age-structured meeting rate*. IMA J. Math. Appl. Med. Biol. 5, 81-100 (1988).

[41] GRIPENBERG, G., LONDEN, S.-O. and STAFFANS, O.: *Volterra Integral and Functional Equations*. Cambridge University Press, Cambridge (1990).

[42] GURNEY, W. S. C. and NISBET, R. M.: *Age and density-dependent population dynamics in static and variable environments*. Theor. Pop. Biol. 17, 321-344 (1980).

[43] GURNEY, W. S. C. and NISBET, R. M.: *The systematic formulation of delay-differential models of age or size structured populations*. In: Populations Biology (H. I. Freedman e Strobeck, eds.) Springer Lect. Notes in Biomath. 52, 163-172 (1983).

[44] GURTIN, M. E.: *A system of equations for age-dependent population diffusion*. J. Theoret. Biol. 40, 389-392 (1973).

[45] GURTIN, M. E. and LEVINE, D. S.: *On predator-prey interaction with predation dependent on age of prey*. Math. Biosci. 47, 207-219 (1979).

[46] GURTIN, M. E. and MACCAMY, R. C.: *Nonlinear age-dependent population dynamics*. Archive for Rational Mechanics and Analysis 54, 281-300 (1974).

[47] GURTIN, M. E. and MACCAMY, R. C.: *On the diffusion of biological population*. Math. Biosci. 38, 35-49 (1977).

[48] GURTIN, M. E. and MACCAMY, R. C.: *Some simple models for nonlinear age-dependent population dynamics*. Math. Biosci. 43, 199-211 (1979).

[49] GURTIN, M. E. and MACCAMY, R. C.: *Population dynamics with age dependence*. In: Nonlinear Analysis and Mechanics. Herriot-Watt Symposium III. Pitman, Boston (1979).

[50] GURTIN, M. E. and MACCAMY, R. C.: *Diffusion models for age structured populations*. Math. Biosc. 54, 49-59 (1981).

[51] GURTIN, M. E. and MACCAMY, R. C.: *Product solutions and asymptotic behaviour for age dependent, dispersing populations*. Math. Biosc. 62, 157-167 (1982).

[52] GYLLENBERG, M.: *Nonlinear age-dependent population dynamics in continuously propagated bacterial cultures*. Math. Biosc. 62, 45-74 (1982).

[53] GYLLENBERG, M.: *Stability of a nonlinear age-dependent population model containing a control variable*. SIAM J. Appl. Math. 43, 1418-1438 (1983).

[54] HEESTERBECK, H.: R_0. Dissertation, Amsterdam (1992).

[55] HETHCOTE, H. W.: *Three basic Epidemiological Models*. Applied Mathematical Ecology S. A. Levin, T. Hallam and L. J. Gross (eds.), Springer Verlag (1989).

[56] HOPPENSTEADT, F.: *An age-dependent epidemic model*. J. Franklin Inst. 197, 325-333 (1974).

[57] HOPPENSTEADT, F.: *Mathematical Theories of Populations: Demographics, Genetics and Epidemics*. Philadelphia: S.I.A.M. (1975).

[58] IANNELLI, M., MILNER, F. A., PUGLIESE, A.: *Analytical and numerical results for the age structured SIS epidemic model with mixed inter-intra-cohort transmission*. SIAM J. Math. Anal. 23, 662-688 (1992).

[59] IANNELLI, M., LORO, R., MILNER, F. A., PUGLIESE, A. and RABBIOLO, G.: *An AIDS model with distributed incubation and variable infectiousness: application to IV-drug users in Latium, Italy*. Eur. J. Epidemiol. 8, 585-593 (1992).

[60] IANNELLI, M., LORO, R., MILNER, F. A., PUGLIESE, A. and RABBIOLO, G.: *Numerical Analysis of a Model for the Spread of HIV/AIDS*. U.T.M. 398 September 1992.

[61] IMPAGLIAZZO, J.: *Deterministic Aspects of Mathematical Demography*. Biomathematics Vol. 13, Berlin: Springer (1985).

[62] INABA, H.: *Weak ergodicity of population evolution processes*. Math. Biosc. Vol. 96, 195-219 (1989).

[63] INABA, H.: *Threshold and stability results for an age-structured epidemic model*. J. Math. Biol. 28, 411-434 (1990).

[64] INABA, H.: *Strong ergodicity for perturbed semigroups and application to age-dependent population dynamics*. J. Math. Anal. and Appl., vol. 165, 102-132 (1992).

[65] KERMACK, W. O. and MCKENDRICK, A. G.: *A Contribution to the Mathematical Theory of Epidemics*. Proc. Roy. Soc. (A) 115, 700-721 (1927).

[66] KERMACK, W. O. and MCKENDRICK, A. G.: *Contributions to the mathe-*

matical theory of epidemics II. The problem of endemicity. Proc. Roy. Soc. A 138, 55-83 (1932).

[67] KERMACK, W. O.-MCKENDRICK, A. G.: *Contributions to the mathematical theory of epidemics III. Further studies on the problem of endemicity.* Proc. Roy Soc. A 141, 94-122 (1933).

[68] KEYFITZ, N.: *Introduction to the Mathematics of Populations with Revisions.* Reading, Mass.: Addison-Wesley (1977).

[69] LAMBERTI, L. and VERNOLE, P.: *Existence and asymptotic behaviour of solutions of an age structured population model.* Boll. UMI, Anal. Funz. Appl., Serie Vol. XVIII-C (1981).

[70] LANGHAAR, H.: *General population theory in the age-time continuum.* Journal of the Franklin Institute 293 (3), 199-214 (1972).

[71] LANGLAIS, M.: *A nonlinear problem in age dependent population diffusion.* SIAM J. Math. Anal. 16, 510-529 (1985).

[72] LANGLAIS, M.: *Large time behavior in a nonlinear age-dependent population dynamics problem with spatial diffusion.* J. Math. Biol. 26, 319-346 (1988).

[73] LOPEZ, A.: *Problems in Stable Population Theory.* Office of Population Research, Princeton (1961).

[74] LOTKA, A. J.: *On a integral equation in population analysis.* Ann. Math. Stat. 10, 1-25 (1939).

[75] LOTKA, A. J.: *The stability of the normal age distribution.* Proceedings of the National Academy of Sciences 8, 339-345 (1922).

[76] LOTKA, A. J.: *Théorie Analytique des Associations Biologiques. Deuxième Partie: Analyse Démographique avec Application Particulière à l'Espèce Humaine.* (Actualités Scientifiques et Industrielles, No. 780) Paris: Hermann-Cie (1939).

[77] MALTHUS, T. R.: *An Essay on the Principle of Population.* (First edition), London (1798).

[78] MCKENDRICK, A. C.: *Applications of mathematics to medical problems.* Proceedings of the Edinburgh Mathematical Society 44, 98-130 (1926).

[79] METZ, J. A. J. and DIEKMANN, O.: *The Dynamics of physiologically structured populations.* Springer Lecture Notes in Biomathematics, 68 (1986).

[80] MILLER, R. K.: *Nonlinear Volterra Integral Equations.* W. A. Benyamin, Menlo Park Ca. (1971).

[81] MILNER, F. A. and RABBIOLO, G.: *Rapidly converging numerical methods for models of population dynamics.* J. Math. Biol. (1992).

[82] RORRES, C.: *Stability of an age specific population with density dependent fertility*. Theor. Population Biol. 10, 26-46 (1976).

[83] RORRES, C.: *A nonlinear model of population growth in which fertility is dependent on birth rate*. SIAM J. Math. 37, 423-432 (1979).

[84] RORRES, C.: *Local stability of a population with density-dependent fertility*. Theor. Pop. Biol. 16, 283-300 (1979).

[85] SALEEM, M.: *Predator-prey relationships: egg-eating predators*. Math. Biosc. 65, 187-197 (1983).

[86] SALEEM, M.: *Egg-eating age-structured predators in interaction with age-structured prey*. Math. Biosc. 70, 91-104 (1984).

[87] SHARPE, F. R. and LOTKA, A. J.: *A problem in age-distribution*. Philosophical Magazine, Ser. 6, 21, 435-438 (1911).

[88] SCHENZLE, D.: *An age structured model for pre and post-vaccination measles transmission*. IMA J. Math. Appl. Biol. Med. 1, 169-191 (1984).

[89] STREIFER, W.: *Realistic models in population ecology*. Advances in ecological research, A. Macfadyen, Ed., vol. 8, Academic Press, New York, (1974).

[90] SWICK, K. E.: *Stability and bifurcation in age-dependent population dynamics*. Theor. Pop. Biol. 20, 80-100 (1981).

[91] THIEME, H. R.: *Renewal theorems for linear discrete Volterra equations*. J. Reine e Angew. Math. 353, 55-84 (1984).

[92] THIEME, H. R.: *Renewal theorems for linear periodic Volterra integral equations*. J. Int. Eq. 7, 253-277 (1984).

[93] THIEME, H.: *Stability change of the endemic equilibrium in age-structured models for the spread of S-I-R type infectious diseases*. In Differential Equations Models in Biology Epidemiology and Ecology, S. Busenberg and M. Martelli, Eds., Springer Lecture Notes in Biomathematics 92, 139-158 (1991).

[94] THIEME, H. R. and CASTILLO-CHAVEZ, C.: *On the role of variable infectivity in the dynamics of the human immunodeficiency virus epidemic*. Mathematical and Statistical Approaches to AIDS Epidemiology, C. Castillo-Chavez (Ed.), Lecture Notes in Biomathematics, vol. 83, Springer (1989).

[95] THIEME, H. R. and CASTILLO-CHAVEZ, C.: *How may infection-age dependent infectivity affect the dynamics of HIV/AIDS?*. SIAM J. Appl. Math., to appear.

[96] TUDOR, D.: *An age dependent epidemic model with application to measles*. Math. Biosci. 73, 131-147 (1985).

[97] VERHULST, P. F.: *A note on the law of population growth*. Correspondence Mathématique et Physique Publiée par A. Quetelet, vol. 10, Brussels, (1838).

[98] VON FOERSTER, H.: *Some remarks on changing populations*. In The Kinetics of Cellular Proliferation, F. Stohlman, Ed. Grunne e Stratton, New York, 382-407 (1959).

[99] WEBB, G.: *Nonlinear semigroups and age-dependent population models*. Annali di Matematica Pura et Applicata, Sr. 4, 129, 43-55 (1981).

[100] WEBB, G.: *A semigroup proof of the Sharpe-Lotka theorem*. In: Infinite Dimensional System, F. Kappel and W. Schappacher, (eds.), Lecture Notes in Mathematics 1076, 254-268, Berlin: Springer (1984).

[101] WEBB, G.: *Theory of Nonlinear Age-Dependent Population Dynamics*. New York: Marcel Dekker (1985).

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