Appendix B. Dynamical systems in infinite dimensional spaces

B.1. Banach spaces

An abstract "linear space" X over $\mathbb{R}$ (or $\mathbf{C}$; unless explicitly stated we shall always refer to $\mathbb{R}$) is a collection of elements such that for each $z_1,\, z_2 \in X$ the sum $z_1 + z_2 \in X$ is defined such that

(i) $z_1 + z_2 = z_2 + z_1$

and an element $0 \in X$ exists such that

(ii) $0 + z = z + 0 = z$ for all $z \in X$.

Also, for any number $a \in \mathbb{R}$ and any element $z \in X$ the scalar multiplication is defined $az \in X$ such that

(iii) $1\, z = z$, for all $z \in X$

(iv) $a(bz) = (ab)z = b(az)$, for all $a, b \in \mathbb{R}$ and all $z \in X$

(v) $(a + b)z = az + bz$, for all $a, b \in \mathbb{R}$ and all $z \in X$.

A linear space X is a "normed linear space" if to each $z \in X$ there corresponds a nonnegative real number $\|z\|$ called the "norm" of z which satisfies

(a) $\|z\| = 0 \iff z = 0$

(b) $\|z_1 + z_2\| \le \|z_1\| + \|z_2\|$ for all $z_1,\, z_2 \in X$

(c) $\|az\| = |a|\; \|z\|$ for all $a \in \mathbb{R}$ and all $z \in X$

When confusion may arise, we will write $\|\cdot\|_X$ for the norm on X.
A norm on a linear space X induces a metric via the following distance

$$dist(z_1, z_2) = \|z_1 - z_2\| \;, \quad \text{for all} \quad z_1\,,\; z_2 \in X \;.$$

It is such that

(d1) $dist(z_1, z_2) = 0 \iff z_1 = z_2$

(d2) $dist(z_1, z_2) = dist(z_2, z_1)$

(d3) $dist(z_1, z_3) \leq dist(z_1, z_2) + dist(z_2, z_3)$

for all $z_1, z_2, z_3 \in X$.

X endowed with $dist$ is a "metric space".

A sequence $(z_n)_{n\in\mathbb{N}}$ in a normed linear space X converges in X if a $z \in X$ exists such that $\lim_{n\to\infty} \|z_n - z\| = 0.$

A sequence $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X if for every $\epsilon > 0$ a $\nu \in \mathbb{N}$ exists such that for any $n, m \in \mathbb{N}$ $n, m > \nu$: $\|z_n - z_m\| < \epsilon.$

The space X is "complete" if every Cauchy sequence in X is convergent in X.

A "Banach space" X is a complete normed linear space.

For $z \in X$, a normed linear space, the "open ball" about $z_0 \in X$ with radius $\rho > 0$ is the set $B_\rho(z_0) := \{z \in X \mid \|z - z_0\| < \rho\}$. If $A \subset X$ then z is an "interior point" of A if a $\rho > 0$ exists so that $B_\rho(z) \subset A.$

The set of all interior points of A is called the "interior" of A and is usually denoted by $\overset{\circ}{A}$. Clearly $\overset{\circ}{A} \subset A$. A set $A \subset X$ is "open" if $A = \overset{\circ}{A}.$

A set $A \subset X$ is "bounded" if for any $z \in A$ a $\rho > 0$ exists such that $A \subset B_\rho(z).$

We say that $z \in X$ is a "limit point" of A if a sequence of elements of $A \subset X$ exists such that $z = \lim_{n\to\infty} z_n$. The "closure" $\overline{A}$ of A is the set of all limit points of A. Clearly $A \subset \overline{A}$. A set A is "closed" if $A = \overline{A}$. $A \subset X$ is "dense" in X if $\overline{A} = X$.

Let X be a complete metric space and let $A \subset X$. If $A = \overline{A}$ then A is a complete subspace of X.

A is "precompact" in X if every sequence of elements of A contains a Cauchy subsequence; A is "compact" if every sequence of elements of A contains a subsequence convergent to a point in A.

In a metric space X we say that $A \subset X$ is "relatively compact" if $\overline{A}$ is compact. A subset $A \subset X$ which is relatively compact is also precompact. In a complete metric space X a precompact subset of X is also relatively compact, and thus the two concepts coincide.

In a complete metric space A is compact iff A is precompact and closed.

In a metric space X, every subset $A \subset X$ which is compact is also closed. Every precompact subset A is bounded.

A normed linear space X is "locally compact" if every closed and bounded subset is compact.

Consider $\mathbb{R}^n$ equipped with the Euclidean norm. Then $\mathbb{R}^n$ is complete and locally compact.

A vector space X is said to be "finite dimensional" if there is a positive integer $n \in \mathbb{N}$ such that X contains a set of n linearly independent vectors whereas any set of $n + 1$ or more vectors of X is linearly dependent. This number n is called "dimension" of the space X.

If X is not finite dimensional then it is said to be "infinite dimensional".

If $dim\, X = n$ any set of n linearly independent vectors of X is called a "basis" for X.

If $\{e_1, \cdots, e_n\}$ is a basis for X then any $x \in X$ has a unique representation as a linear combination of the basis vectors.

If X is a finite dimensional normed space

$$M \text{ compact} \quad \Longleftrightarrow \quad M \text{ closed and bounded.}$$

Thus every finite dimensional normed space is "locally compact".

The converse is also true.

Theorem B.1. (F. Riesz) [140] *A locally compact linear normed space has finite dimension.*

Every finite dimensional normed space is complete.

A norm $\|\cdot\|$ on a vector space X is said to be "equivalent" to another norm $\|\cdot\|_o$ on X if there are positive real numbers a and b such that for all $x \in X$ we have

$$a\,\|x\|_o \leq \|x\| \leq b\,\|x\|_o$$

On a finite dimensional vector space all norms are equivalent.

$\mathbb{R}^n$ equipped with the Euclidean norm, is a finite dimensional Banach space.

$C([0,1])$, the set of all continuous real-valued functions defined on the closed interval $[0,1] \subset \mathbb{R}$ is a real Banach space, when equipped with the norm

$$\|u\| = \max_{x \in [0,1]} |u(x)| \ , \quad \text{for} \ \ u \in C([0,1]) \ .$$

$C^k([0,1])$, the set of continuous real-valued functions having $k \in \mathbb{N}$ continuous derivatives on $[0,1] \subset \mathbb{R}$, with norm

$$\|u\|_k = \sum_{s=0}^{k} \max_{x \in [0,1]} \left| u^{(s)}(x) \right|$$

is a real Banach space ($u^{(s)}$ denotes the derivative of order s of the function u).

B.1.1. Ordered Banach spaces

Let E be a real vector space.

An ordering $\leq$ on E is called linear if

(i) $x \leq y \implies x + z \leq y + z$ for all $x, y, z \in E$

(ii) $x \leq y \implies \alpha x \leq \alpha y$ for all $x, y \in E$, $\alpha \in \mathbb{R}_+$.

A real vector space together with a linear ordering is called an "ordered vector space (OVS)" [0].

Let V be an OVS and let $P := \{x \in V \mid x \geq 0\}$. Clearly P has the following properties

(P1) $P + P \subset P$

(P2) $\mathbb{R}_+ P \subset P$

(P3) $P \cap (-P) = \{0\}$

A nonempty subset P of a real vector space V satisfying (P1-P3) is called a "cone".

A cone P is called "generating" if $E = P - P$.

Every cone P in a real vector space E induces a partial linear ordering on E by

$$x \leq_P y \quad \overset{\text{def}}{\iff} \quad y - x \in P \ .$$

The elements in $\dot{P} := P - \{0\}$ are called "positive" and P is called the "positive cone" of the ordering.

Consequently for every linear space E there is a one-to-one correspondence between the family of linear orderings and the family of cones.

A set A is said "order convex" whenever $x, y \in A$ implies $[x, y] \subset A$, where

$$[x, y] := \{z \in E \mid x \leq z \leq y\} \ .$$

A cone is order convex.

Let $E = (E, \|\cdot\|)$ be a Banach space ordered by a cone P. Then E is called an "ordered Banach space (OBS)" if the positive cone is closed.

Proposition B.2. *Let E be an ordered Banach space with respect to a cone P. If $\overset{\circ}{P} \neq \emptyset$ then P is generating.*

The Euclidean space $\mathbb{R}^n$ is an ordered Banach space with respect to the cone $\mathbb{K} := \mathbb{R}^n_+ := \{z \in \mathbb{R}^n \mid z_i \geq 0\,,\ i = 1, \cdots, n\}$. This cone has a nonempty interior $\overset{\circ}{\mathbb{K}} = \mathbb{R}^{n*}_+ := \{z \in \mathbb{R}^n \mid z_i > 0\,,\ i = 1, \cdots, n\}$, hence it is generating.

Let Ω be a nonempty bounded open subset of $\mathbb{R}^m$, $m \geq 1$. For any $k \in \mathbb{N}$, we denote by $C^k(\overline{\Omega})$ the vector space of all uniformly continuous functions $u : \Omega \longrightarrow \mathbb{R}$ such that all the partial derivatives of order up to k exist and are uniformly continuous on Ω.

Due to the uniform continuity, each of the derivatives $D^\alpha u := D_1^{\alpha_1} \cdots \quad D_m^{\alpha_m} u\,,\quad \alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m\,,\quad |\alpha| = \sum_{i=1}^{m} \alpha_i \leq k$, has a unique continuous extension over $\overline{\Omega}$.

We define

$$\|u\|_k := \sum_{|\alpha| \leq k} \max_{x \in \overline{\Omega}} |D^\alpha u(x)| \quad .$$

Equipped with this norm, each $C^k(\overline{\Omega})$ is a Banach space. With the ordering induced by $C_+(\overline{\Omega}) := \{u \in C(\overline{\Omega}) \mid u(x) \geq 0\,,\ x \in \overline{\Omega}\}$, it is an ordered Banach space.

B.1.2. Functions

Given two sets X and Y, a function F from X to Y is a rule which assigns to any element x of a subset $\mathcal{D}(F) \subset X$ a unique $y \in Y$ that we denote by $F(y)$. This is denoted by

$$F : (\mathcal{D}(F) \subset X) \longrightarrow Y \quad .$$

$\mathcal{D}(F)$ is called the "domain" of F and $\mathcal{R}(F) := \{y \in Y \mid y = F(x)\,,\ \text{for some } x \in \mathcal{D}(F)\}$ is the "range" of F.

If $\mathcal{R}(F) = Y$ the function is "onto". If $F(x) = F(x') \implies x = x'$, then F is "one-to-one"; in this case there exists an "inverse function" $F^{-1} : (\mathcal{R}(F) \subset Y) \longrightarrow X$ such that $F^{-1}F(x) = x$ for all $x \in \mathcal{D}(F) = \mathcal{R}(F^{-1})$ (and $FF^{-1}(y) = y$ for all $y \in \mathcal{D}(F^{-1}) = \mathcal{R}(F)$).

The set $G_F := \{(x, y) \in X \times Y \mid x \in \mathcal{D}(F)\,,\ y = F(x)\}$ is the "graph" of F.

The identity on X is the function $I : X \longrightarrow X$ defined for all $x \in X$: $I(x) = x$.

If X is a linear space on $\mathbb{R}$ a function $F : (\mathcal{D}(F) \subset X) \longrightarrow \mathbb{R}$ is also called a (real) functional on X.

Let X, Y be linear spaces on $\mathbb{R}$. A function $F : (\mathcal{D}(F) \subset X) \longrightarrow Y$ is a "linear operator" on X if $\mathcal{D}(F)$ is a linear subspace of X and

$$F(\alpha x + \beta x') = \alpha F(x) + \beta F(x')$$

for any x, $x' \in \mathcal{D}(F)$, α, $\beta \in \mathbb{R}$ (for a linear function F we shall usually write Fx for $F(x)$).

The "null space", or "kernel" of a linear operator F is the set

$$ker(F) := \{x \in \mathcal{D}(F) \mid Fx = 0\} \ .$$

Clearly $0 \in ker(F)$.

If $F : (\mathcal{D}(F) \subset X) \longrightarrow Y$ is a linear operator between two real linear spaces X and Y , then $\mathcal{R}(F)$ is a vector space. If $dim\mathcal{D}(F) = n < +\infty$, then $dim\mathcal{R}(F) \leq n$. The null space $ker(F)$ is a vector space.

The inverse of a linear operator $F : (\mathcal{D}(F) \subset X) \longrightarrow Y$ exists iff $ker(F) = \{0\}$. If F^{-1} exists it is itself a linear operator. If $dim\mathcal{D}(F) = n < +\infty$, and F^{-1} exists, then $dim\mathcal{R}(F) = dim\mathcal{D}(F)$.

Let $F : (\mathcal{D}(F) \subset X) \longrightarrow Y$, X, Y metric spaces. F is "continuous" at $x_0 \in \mathcal{D}(F)$ if for every $\epsilon > 0$ there exists a $\delta(x_0, \epsilon) > 0$ such that $dist_X(x, x_0) < \delta \implies dist_Y(F(x), F(x_0)) < \epsilon$. Equivalently, if for every sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(F)$, converging to $x_0 \in \mathcal{D}(F)$, the sequence $(F(x_n))_{n\in\mathbb{N}} \subset Y$ converges to $F(x_0)$.

A function F is said to be "continuous" if it is continuous at every $x_0 \in \mathcal{D}(F)$. It is said "uniformly continuous" if, in $\mathcal{D}(F)$, $\delta(\epsilon, x_0)$ can be chosen independently of $x_0 \in \mathcal{D}(F)$.

If S is a compact subset of $\mathcal{D}(F)$ and F is continuous on S then F is uniformly continuous on S.

A function $F : (\mathcal{D}(F) \subset X) \longrightarrow X$, on a metric space X is a "contraction" if $\mathcal{R}(F) \subset \mathcal{D}(F)$ and , for some real $0 \leq \alpha < 1$: $dist(F(x), F(x')) \leq \alpha\, dist(x, x')$ for all x, $x' \in \mathcal{D}(F)$.

Theorem B.3. (Banach-Caccioppoli)[104] *Let $F : X \longrightarrow X$ be a contraction, X a complete metric space. Then there exists a unique "fixed point" $x_0 \in X$ such that*

$$F(x_0) = x_0 \ .$$

A contraction on a metric space X is uniformly continuous on X.

A function $F : (\mathcal{D}(F) \subset X) \longrightarrow Y$, X, Y normed linear spaces, is "Lipschitz continuous" at $x_0 \in \mathcal{D}(F)$ if a real $\alpha(x_0) > 0$ exists such that $\|F(x) - F(x_0)\|_Y \leq \alpha \|x - x_0\|_X$ for all $x \in B_\rho(x_0)$, for some $\rho > 0$.

A function F is said to be Lipschitz continuous on $S \subset \mathcal{D}(F)$ if it is Lipschitz continuous at each $x_0 \in S$.

F is "uniformly Lipschitz continuous on S"$\subset \mathcal{D}(F)$ if $\alpha(x_0)$ can be chosen independent of $x_0 \in S$.

If S is precompact, Lipschitz continuity on S implies uniform Lipschitz continuity on S.

A contraction on a normed linear space X is uniformly Lipschitz continuous on X.

If $F : X \longrightarrow Y$ is one-to-one and continuous together with its inverse then we say that F is an "homeomorphism" of X onto Y.

Theorem B.4. (Brower)[104] *Any continuous function of the closed unit ball in $\mathbb{R}^n$ into itself must have a fixed point.*

Corollary B.5. [104] *If $A \subset \mathbb{R}^n$ is homeomorphic to the closed unit ball in $\mathbb{R}^n$ and f is continuous from A into A, then f has a fixed point in A.*

A subset A of a Banach space is "convex" if for any x, $y \in A$ it follows that $tx + (1-t)y \in A$ for any $t \in [0,1]$.

Theorem B.6. (Schauder) [104] *If A is a convex, compact subset of a Banach space X and $f : A \longrightarrow A$ is continuous, then f has a fixed point in A.*

A function $F : (\mathcal{D}(F) \subset X) \longrightarrow Y$, X and Y metric spaces, is "bounded" (resp. "compact") if F maps bounded sets in $\mathcal{D}(F)$ into bounded sets (resp. precompact sets) in Y.

If X and Y are Banach spaces F is compact iff for every bounded set $A \subset \mathcal{D}(F)$, $\overline{F(A)}$ is compact in Y. If in addition F is continuous it is called "completely continuous".

Corollary B.7. [104] *If A is a closed, convex, bounded subset of a Banach space X and $f : A \longrightarrow A$ is completely continuous, then F has a fixed point in A.*

If X, Y are metric spaces, $F : (\mathcal{D}(F) \subset X) \longrightarrow Y$, $\mathcal{D}(F)$ is compact and F is continuous on $\mathcal{D}(F)$, then F is uniformly continuous and $\mathcal{R}(F)$ is compact in Y.

Let $F : (\mathcal{D}(F) \subset X) \longrightarrow Y$ be linear, X, Y normed linear spaces. Then F is bounded iff

$$\sup_{0 \neq x \in \mathcal{D}(F)} \frac{\|Fx\|_Y}{\|x\|_X} = \alpha < +\infty \quad .$$

In this case the quantity α will be called the "norm" of F and shall be denoted by $\|F\|$.

Clearly, for any $x \in \mathcal{D}(F)$:

$$\|Fx\|_Y \leq \|F\| \; \|x\|_X \quad .$$

A real $n \times n$ matrix A is a bounded linear operator on $\mathbb{R}^n$.

If a normed linear space X is finite dimensional, then every linear operator on X is bounded.

Let $F : (\mathcal{D}(F) \subset X) \longrightarrow Y$ be a linear operator, X, Y normed linear spaces. Then F is continuous iff F is bounded. If F is continuous at a point $x_0 \in \mathcal{D}(F)$, then F is continuous, uniformly continuous and uniformly Lipschitz continuous on $\mathcal{D}(F)$.

If F is a bounded linear operator then $ker(F)$ is closed in X.

Let $F : (\mathcal{D}(F) \subset X) \longrightarrow Y$, X, Y complete metric spaces. We say that F is a "closed operator" if its graph is a closed set in $X \times Y$ equipped with the product metric

$$dist_{X \times Y}((x,y)\,,\,(x',y')) := dist_X(x,x') + dist_Y(y,y') \ .$$

Equivalently if given a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(F) \subset X$ such that $(Fx_n)_{n\in\mathbb{N}}$ is Cauchy in Y, then $x_n \longrightarrow x_0 \in \mathcal{D}(F)$ and $Fx_n \longrightarrow Fx_0$ in Y.

B.1.3. Linear operators on Banach spaces

Let now X be a complex Banach space.

For a linear operator $F : (\mathcal{D}(F) \subset X) \longrightarrow X$, the resolvent set $\rho(F)$ consists of all those $\lambda \in \mathbf{C}$ such that

(R1) $\lambda I - F$ is one-to-one

(R2) $\mathcal{R}(\lambda I - F)$ is dense in X

(R3) $(\lambda I - F)^{-1}$ is bounded.

Because of (R2) and (R3), for $\lambda \in \rho(F)$, there exists a unique extension $R(\lambda, F)$ of $(\lambda I - F)^{-1}$ to all of X; we shall call this extension the "resolvent" of F at $\lambda \in \rho(F)$.

We notice that if $\lambda \in \rho(F)$ then F is a closed operator iff $\mathcal{R}(\lambda I - F) = X$, since $(\lambda I - F)^{-1}$ is then closed bounded and densely defined. In this case $R(\lambda, F) = (\lambda I - F)^{-1}$.

The "spectrum" of F is $\sigma(F) := \{\lambda \in \mathbf{C} \mid \lambda \notin \rho(F)\}.$. If is seen that the spectrum may be partitioned as

$$\sigma(F) = \sigma_p(F) \cup \sigma_c(F) \cup \sigma_r(F) \ .$$

The "point spectrum" $\sigma_p(F)$ consists of those $\lambda \in \sigma(F)$ such that $(\lambda I - F)$ is not one-to-one; i.e. $(\lambda I - F)\, g = 0$ for some nonzero $g \in \mathcal{D}(F)$; then we say that λ is an "eigenvalue" of F and g is a corresponding "eigenvector".

The space $ker(\lambda I - F)$ is called the eigenspace of λ and its dimension is known as the "geometric multiplicity" of λ. The "generalized eigenspace" of λ, $\mathcal{M}(\lambda I - F)$ is the smallest closed linear subspace of X that contains $ker((\lambda I - F)^j)$ for $j \in \mathbb{N}^*$; its dimension is called the "algebraic multiplicity" of λ. Evidently the geometric multiplicity of λ is at most equal to its algebraic multiplicity. F is called "semisimple" if these two multiplicities agree for all $\lambda \in \sigma_p(F)$. An eigenvalue is "simple" if its algebraic multiplicity equals 1.

The "continuous spectrum" $\sigma_c(F)$ is the set of those $\lambda \in \sigma(F)$ for which $ker(\lambda I - F) = \{0\}$ and $\mathcal{R}(\lambda I - F)$ is dense in X but $R(\lambda, F)$ is unbounded.

Finally the "residual spectrum" $\sigma_r(F)$ is the set of those $\lambda \in \sigma(F)$ for which $ker(\lambda I - F) = \{0\}$ but $\mathcal{R}(\lambda I - F)$ is not dense in X.

Note that the spectrum of a linear operator on a finite dimensional space is a pure point spectrum , so that the spectrum is made of only eigenvalues.

If $X \neq \{0\}$ is a complex Banach space and F is a linear bounded operator on X, then $\sigma(F) \neq 0$.

The spectrum $\sigma(F)$ of a bounded linear operator on a complex Banach space X is compact and lies in the disk given by

$$|\lambda| \leq \|F\| \ .$$

The "spectral radius" $r(F)$ of a bounded linear operator F on a complex Banach space is the quantity

$$r(F) := \sup_{\lambda \in \sigma(F)} |\lambda| \ .$$

For a bounded linear operator F we have

$$r(F) \leq \|F\|$$

It can be shown further that

$$r(F) = \lim_{n \to \infty} \|F^n\|^{1/n}$$

Let X, Y be normed linear spaces and $F : (\mathcal{D}(F) \subset X) \longrightarrow Y$ a linear operator. Then if F is bounded and $dim\mathcal{R}(F) < +\infty$, the operator F is compact. If $dim(X) < +\infty$, the operator F is compact.

Theorem B.8. [140] *A compact linear operator* F *on a normed linear space* X *has the following properties.*

a) *Every spectral value* $\lambda \neq 0$ *is an eigenvalue. If* $dim X = +\infty$ *then* $0 \in \sigma(F)$ *(but may belong to either* $\sigma_p(F)$ *or* $\sigma_c(F)$ *or even* $\sigma_r(F)$ *).*

b) *The set of all eigenvalues of* F *is countable.* $\lambda = 0$ *is the only possible point of accumulation of that set.*

c) *For* $\lambda \neq 0$ *the dimension of any eigenspace of* F *is finite.*

Let X, Y be OBS's with positive cones P and Q respectively.

A linear operator $F : X \longrightarrow Y$ is called "positive" if $F(P) \subset Q$, and "strictly positive" if $F(\dot{P}) \subset \dot{Q}$.

If Y is an OBS with respect to Q, and Q has nonempty interior, then F is called "strongly positive" if $F(\dot{P}) \subset \overset{\circ}{Q}$.

The following extends Perron-Frobenius theorem to the infinite dimensional case.

Theorem B.9. (Krein-Rutman) [1] *Let* E *be an OBS whose positive cone* P *has nonempty interior. Let* F *be a strongly positive compact linear operator on* E. *Then the following is true :*

(i) *The spectral radius* $r(F)$ *is positive.*

(ii) $r(F)$ *is a simple eigenvalue of* F *having a positive eigenvector and there is no other eigenvalue with a positive eigenvector.*

(iii) *for every* $y \in \dot{P}$ *the equation*

$$(\lambda I - F)x = y$$

has exactly one positive solution x *if* $\lambda > r(F)$, *and no positive solution for* $\lambda \leq r(F)$.

B.1.4. Dynamical systems and C_o-semigroups

For X a Banach space, a family of continuous operators $\{S(t)\,,\ t \in \mathbb{R}_+\}$ on X is a strongly continuous semigroup of continuous operators if

(i) $S(0) = I$

(ii) $S(t+\tau) = S(t)\,S(\tau)$ for all $t\,,\ \tau \in \mathbb{R}_+$

(iii) $S(\cdot)\,x\ :\ \mathbb{R}_+ \longrightarrow X$ is continuous for any $x \in X$.

Such a family is usually called a C_o-semigroup ((i) and (ii) define a semigroup; C_o refers to (iii)).

Clearly, every C_o-semigroup determines a dynamical system in X, and conversely, by the definition

$$u(t;x) := S(t)\,x \quad , \quad t \in \mathbb{R}_+ \quad , \quad x \in X \ .$$

Hence these two concepts are equivalent [215].

A dynamical system $\{S(t)\ ,\ t \in \mathbb{R}_+\}$ is linear if $S(t)$ is a linear bounded operator on X for every $t \in \mathbb{R}_+$.

If $\{S(t)\ ,\ t \in \mathbb{R}_+\}$ is a linear dynamical system on a Banach space X, there exist numbers $M \geq 1\,,\ \omega \in \mathbb{R}$ such that

$$\|S(t)\| \leq M\,e^{\omega t} \quad , \qquad t \in \mathbb{R}_+$$

With each linear dynamical system $\{S(t)\ ,\ t \in \mathbb{R}_+\}$ on a Banach space X there is associated a certain linear operator $A\ :\ (\mathcal{D}(A) \subset X) \longrightarrow X$ called its "infinitesimal generator" defined as follows

$$\mathcal{D}(A) := \left\{ x \in X \ \middle|\ \text{there exists the limit} \quad \lim_{t \to 0^+} \frac{1}{t}\,[S(t)\,x - x] \right\}$$

$$Ax := \lim_{t \to 0^+} \frac{1}{t}\,[S(t)\,x - x] \ .$$

Theorem B.10. [179, 26] *Let $A\ :\ (\mathcal{D}(A) \subset X) \longrightarrow X$, X a Banach space, be the infinitesimal generator of a linear dynamical system $\{S(t)\ ,\ t \in \mathbb{R}_+\}$. Then*

(i) A is a closed linear operator and $\mathcal{D}(A)$ is dense

(ii) $\mathcal{D}(A)$ *is positive invariant and for every* $x \in \mathcal{D}(A)$

$$\frac{d}{dt}S(t)\,x = A\,S(t)\,x = S(t)\,A\,x\ , \qquad t \in \mathbb{R}_+\ .$$

Theorem B.11. (Hille-Phillips) [129, 26] *A linear operator* $A\ :\ (\mathcal{D}(A) \subset X) \longrightarrow X$ *on a complex Banach space* X *, is the infinitesimal generator of a linear dynamical system* $\{S(t)\ ,\ t \in \mathbb{R}_+\}$ *satisfying* $\|S(t)\| \leq M\,e^{\omega t}$ *for all* $t \in \mathbb{R}_+$ *, iff*

(i) A *is closed and* $\mathcal{D}(A)$ *is dense*

(ii) every real $\mu > \omega$ *is in the resolvent set* $\rho(A)$

(iii) $\|R(\mu, A)^n\| \leq \dfrac{M}{(\mu-\omega)^n}$ *for all* $\mu > \omega,\quad n = 1, 2, \cdots$.

Notice that condition (iii) is "easy" only if $\|R(\mu, A)\| \leq \dfrac{1}{(\mu-\omega)}$ for all $\mu > \omega$.

Here $M > 0$, and $\omega \in \mathbb{R}$.

Let $\{S(t)\ ,\ t \in \mathbb{R}_+\}$ be a linear dynamical system on a complex Banach space with infinitesimal generator $A\ :\ (\mathcal{D}(A) \subset X) \longrightarrow X$.

The following necessary conditions can be stated

Theorem B.12. [215] *If the equilibrium* $x^* = 0$ *is stable, then no eigenvalue of* A *has positive real part. If* $x^* = 0$ *is asymptotically stable, then every eigenvalue has negative real part.*

Remark. The above theorem cannot be reversed into sufficient conditions for stability and asymptotic stability in the infinite dimensional case.

On the other hand for linear systems on a Banach space the following statements are equivalent [215]:

(i) the equilibrium $x^* = 0$ is stable

(ii) every motion is stable

(iii) every positive orbit is bounded.

Further, the following statements are equivalent [215]:

(i) the equilibrium $x^* = 0$ is asymptotically stable

(ii) every motion is GAS.

For nonlinear dynamical systems such coincidences are not generally true.

The next step is to explore the behavior of a linear C_o-semigroup $\{S(t)\ ,\ t \in \mathbb{R}_+\}$ for large t that can be anyway drawn from the knowledge of the spectrum of its infinitesimal generator $A : (\mathcal{D}(A) \subset X) \longrightarrow X$.

Theorem B.13. [168] *Under the above assumptions, for any* $t \in \mathbb{R}_+$

(i) $e^{t\sigma(A)} \subset \sigma(S(t))$

(ii) $e^{t\sigma_p(A)} \subset \sigma_p(S(t)) \subset \left(e^{t\sigma_p(A)} \cup \{0\}\right)$.

Theorem B.14. [168] *If we define*

$$\omega_0 := \inf_{t>0} \frac{1}{t} \ln \|S(t)\|$$

we have

(i) $\lim_{t\to+\infty} \frac{1}{t} \ln \|S(t)\| = \omega_0$

(ii) for any $\omega > \omega_0$ *an* $M(\omega) > 0$ *exists such that* $\|S(t)\| \leq M(\omega)\, e^{\omega t}$, $t \in \mathbb{R}_+$.

(iii) $r(S(t)) = e^{\omega_o t}$, $t \in \mathbb{R}_+$.

The quantity $\omega_0(S)$ is usually called the "growth bound" of the C_o-semigroup $\{S(t)\ ,\ t \in \mathbb{R}_+\}$. For compact semigroups we may relate ω_0 to $s(A) := \sup\{\mathcal{R}e\ \lambda \mid \lambda \in \sigma(A)\}$ as a consequence of Theorem B.8 p.248.

Theorem B.15. [168, 215] *Suppose that for some* $t_0 > 0$, $S(t_0)$ *is compact. Then*

$$\omega_0 = s(A)\ .$$

Theorem B.16. [215] *Under the above assumptions if the resolvent* $R(\mu, A) = (I - \mu A)^{-1}$ *is compact for some* $\mu \in (0, \lambda_0)$, $\lambda_0 > 0$ *then every bounded orbit is precompact.*

B.2. The initial value problem for systems of semilinear parabolic equations (reaction-diffusion systems)

We shall consider now dynamical systems in Banach spaces defined by systems of semilinear parabolic equations of the form

$$(B.1) \qquad \frac{\partial}{\partial t}u = D\Delta u + f(u) \ , \qquad \text{in } \Omega \times \mathbb{R}_+$$

subject to boundary conditions

$$(B.1b) \qquad \beta_i \frac{\partial}{\partial \nu}u_i + \alpha_i u_i = 0 \ , \qquad \text{in } \partial\Omega \times \mathbb{R}_+ \ ,$$

for $i = 1, \cdots, n$, and initial conditions

$$(B.1o) \qquad u(0) = u_0 \ , \qquad \text{in } \overline{\Omega} \ .$$

(here $\partial/\partial\nu$ denotes the outward normal derivative).

We shall assume that

(H1) $f : G \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a locally Lipschitz continuous function on an open subset G of $\mathbb{R}^n$.

(H2) Ω is a bounded open connected subset of $\mathbb{R}^m$, $m \in \mathbb{N} - \{0\}$ with a sufficiently smooth boundary $\partial\Omega$.

(H3) β_i, α_i are sufficiently smooth nonnegative functions on $\partial\Omega$ such that $\alpha_i + \beta_i > 0$ on $\partial\Omega$. $\beta_i = 0$ means homogeneous Dirichlet boundary conditions, while $\alpha_i = 0$ means homogeneous Neumann boundary conditions.

(H4) $D = diag(d_i)$, $d_i > 0$.

Denote by X the real Banach space $C(\overline{\Omega})$ of continuous vector valued functions $u : \overline{\Omega} \longrightarrow \mathbb{R}^n$, endowed with the norm

$$\|u\| = \sum_{i=1}^{n} \sup_{x \in \overline{\Omega}} |u_i(x)| \ .$$

The space X is a (partially) ordered Banach space with respect to the cone

$$X_+ := \{v \in X \mid 0 \leq v(x)\,,\ x \in \overline{\Omega}\}$$

Note that the order $\leq$ induced on X by the cone X_+, is compatible with the order $\leq$ induced on $\mathbb{R}^n$ by the cone $\mathbb{K}$; i.e.

$$u \leq v \quad \text{in} \quad X \Longleftrightarrow u(x) \leq v(x) \quad \text{in } \mathbb{R}^n \ , \qquad \text{for any } x \in \overline{\Omega} \ .$$

In an analogous way we may then define $u < v$ and $u \ll v$ in X, by the correspondence in $\mathbb{R}^n$.

As for the finite dimensional case, in the Banach space X we may define a "(local) semiflow" as a mapping

$$\varphi \ : \ S \subset \mathbb{R}_+ \times X \ \longrightarrow \ X$$

where

$$S = \{(t,u) \in \mathbb{R}_+ \times X \ \big| \ t \in J(u) := \big[0, \tau^+(u)\big) \ \}$$

($0 < \tau^+(u) \leq +\infty$), having the following properties

(i) $\varphi(0;u) = u$,   for any $u \in X$

(ii) $\varphi(t+s;u) = \varphi(t;\varphi(s;u))$, for any $u \in X$ and for all $s \in J(u)$, and $t \in J(\varphi(s;u))$

(iii) the mapping $t \longrightarrow \varphi(t;u)$ is continuous on $J(u)$, for any $u \in X$

(iv) the mapping $u \longrightarrow \varphi(t;u)$ is continuous in X, for any $t \in J(u)$.

Theorem B.17. [169] *Assume conditions (H1)-(H4) hold. Then problem (B.1), (B.1b), (B.1o) defines a (local) semiflow φ on the Banach space X. For any $u_0 \in X$, $\{\varphi(t;u_0)\,,\ t \in J(u)\}$ provides the unique solution of the problem (B.1), (B.1b), (B.1o) (for homogeneous Dirichlet boundary conditions $X := \{u \in C(\overline{\Omega}) \mid u = 0 \text{ on } \partial\Omega\}$). This semiflow satisfies the following properties.*

(a) (Maximality) For any $u_0 \in X$, such that $\tau^+(u_0) < +\infty$, we have

$$\lim_{t \to \tau^+(u_0)} \|\varphi(t;u_0)\| = +\infty \ .$$

(b) *(Compactness) If* $u_0 \in X$ *is such that the positive orbit* $\Gamma_+(u_0) := \{\varphi(t; u_0) \,|\, t \in \mathbb{R}_+\}$ *is bounded in* X, *then* $\Gamma_+(u_0)$ *is relatively compact.*

A boundary operator B may be defined as

$$(Bu)_i := \beta_i \frac{\partial}{\partial \nu} u_i + \alpha_i u_i \ , \quad i = 1, \cdots, n \quad \text{on } \partial\Omega \ .$$

Global boundedness of solutions of (B.1), (B.1b) may follow e.g. from the existence of a bounded invariant region.

A positively invariant region for the local semiflow defined by (B.1), (B.1b) is a closed subset $\Sigma \subset \mathbb{R}^n$ such that for every initial state $u_0 \in X$, having $u_0(x) \in \Sigma$ for any $x \in \overline{\Omega}$, $\varphi(t; u_0)(x)$ remains in Σ for any $x \in \overline{\Omega}$ and $t \in J(u_0)$.

Actually when we deal with general third type boundary conditions we need to be more precise; an invariant region for the initial-boundary value problem (B1), (B1b) is a set $\Sigma \subset \mathbb{R}^n$ such that if $\{u(x;t) \ , \ x \in \overline{\Omega} \ , t \geq 0\}$ is a solution of (B1) with $u_0(x) \in \Sigma$, for $x \in \Omega$, and $Bu \in \alpha\Sigma$ on $\partial\Omega \times \mathbb{R}_+$ then $u(x;t) \in \Sigma$ for $x \in \overline{\Omega}$, $t \in \mathbb{R}_+$ [39, 204]; here $\alpha\Sigma := \{(\alpha_1 v_1, \cdots, \alpha_n v_n)^T \mid v \in \Sigma\}$.

Typically we consider regions Σ of the form

$$\Sigma := \bigcap_{j=1}^{p} \{z \in \mathbb{R}^n \mid G_j(z) \leq 0\}$$

where G_j are suitable smooth functions on G.

Theorem B.18. [204] *If, at each point* $z \in \partial\Sigma$ *we have*

(a) $\nabla G_j(z)$ *is a left eigenvector of* D , $j = 1, \cdots, p$;

(b) $\nabla G_j(z) \cdot w = 0$ *for any* $w \in \mathbb{R}^n \implies w \cdot H(z)\, w \geq 0$ *for any* $w \in \mathbb{R}^n$, *with* H *the Hessian matrix of* G_j , $j = 1, \cdots, p$;

(c) $\nabla G_j(z) \cdot f(z) < 0$, $j = 1, \cdots, p$;

then Σ *is positively invariant for (B.1), (B.1b).*
Conditions (a)-(c) are also necessary conditions for the invariance of Σ, *with a weak inequality in (c).*

Remark. Provided the vector field f points into Σ on $\partial\Sigma$ we have : (i) if D is a scalar matrix ($D = dI$, $d \in \mathbb{R}_+^*$), and Σ is convex then it is invariant; (ii) otherwise Σ is invariant if and only if it is a (possibly unbounded) parallelogram. The edges of this parallelogram are parallel to the coordinate axes if and only if D is a diagonal matrix. In this case then the invariance of the positive cone X_+ is allowed provided the vector field f points into $\mathbb{K}$ on $\partial\mathbb{K}$.

In the sequel we shall assume conditions on f that allow the global existence of the semiflow for any $u_0 \in X_+$. Further we shall assume that f is smooth enough so that the solutions u of system (B.1), (B.1b) are classical solutions, i.e.

$$u \in C^{2,1}\left(\Omega \times (0,+\infty)\,,\, \mathbb{R}^n\right) \;\cap\; C^{1,0}\left(\overline{\Omega} \times (0,+\infty)\,,\, \mathbb{R}^n\right)$$

and satisfies the system in a classical sense.

If we denote by $\{U(t)\,u_0\;,\;t \in \mathbb{R}_+\}$ the unique solution of system (B.1), (B.1b), (B.1o) it can be shown [91] that the evolution operator $\{U(t)\,,\;t \in \mathbb{R}_+\}$ is a (nonlinear) C_o-semigroup on X (see Section B.1.4).

Note that whenever we do not have global (in time) solutions, properties (i)-(iii) of Section B.1.4 apply to the maximal interval of existence.

B.2.1. Semilinear quasimonotone parabolic autonomous systems

Here we extend comparison theorems to the parabolic case.

Theorem B.19. [39, 161, 86] *Under the assumptions of Theorem B.17, let further f in system (B.1) be quasimonotone nondecreasing (cooperative) in $\mathbb{K}$. Let $u(x;t) = (u_1(x;t), \cdots, u_n(x;t))^T$ and $v(x;t) = (v_1(x;t), \cdots, v_n(x;t))^T$ be classical solutions of the following two inequalities, respectively,*

$$(B.2) \qquad \frac{\partial}{\partial t}u \leq D\,\Delta u + f(u)$$

$$(B.3) \qquad \frac{\partial}{\partial t}v \geq D\,\Delta v + f(v) \;\;,$$

in $\Omega \times (0,+\infty)$, with boundary conditions

$$(B.4) \qquad \beta_i \frac{\partial}{\partial \nu}u_i + \alpha_i u_i \leq \beta_i \frac{\partial}{\partial \nu}v_i + \alpha_i v_i \;\;, \quad i = 1, \cdots, n \;\;,$$

on $\partial\Omega \times (0,+\infty)$, and initial conditions such that

$$(B.5) \qquad u_0 \le v_0 \qquad in \quad \overline{\Omega} \ .$$

Then

$$(B.6) \qquad u \le v \qquad in \quad \overline{\Omega} \times \mathrm{I\!R}_+ \ .$$

The above theorem implies in particular the monotonicity of the evolution operator

$$(B.7) \qquad u_0 \,,\; v_0 \in X_+ \,,\;\; u_0 \le v_0 \quad \Longrightarrow \quad U(t)\, u_0 \le U(t)\, v_0 \;\;, \quad t \ge 0 \ .$$

If we further assume for $f \in C^1(\mathrm{I\!K}, \mathrm{I\!R}^n)$ the hypotheses (F1)-(F3) of Section A.4.2, then first of all we may actually claim that for any choice of $u_0 \in X_+$, a unique global solution exists for system (B1), (B.1b) subject to the initial condition u_0 at $t = 0$. Hence the family of evolution operators $\{U(t)\,,\; t \in \mathrm{I\!R}_+\}$ define a (global) flow in X_+.

Moreover we may state the following

$$(B.8) \qquad U(t)\, 0 = 0 \ , \qquad t \ge 0$$

$$(B.9) \qquad U(t)\, X_+ \subset X_+ \ , \qquad t \ge 0$$

Lemma B.20. [161] *Under the assumptions of Lemma A.33 the evolution operator $U(t)$ of system (B.1), (B.1b) is strongly positive for any $t > 0$; i.e.*

$$(B.10) \qquad U(t)\, (\dot{X}_+) \subset \overset{\circ}{X}_+ \ , \qquad t > 0$$

Lemma B.21. [161] *Under the assumptions of Lemma A.33*

(B.11) for any $u_0 \,,\; v_0 \in X_+$

$$u_0 \le v_0 \ , \;\; u_0 \ne v_0 \quad \Longrightarrow \quad U(t)\, u_0 \ll U(t)\, v_0$$

Lemma B.22. [161] *Under the assumptions of Lemma A.34, the evolution operator $U(t)$ of system (B.1), (B.1b) is strongly concave, for any $t > 0$.*

(B.12) for any $u_0 \in \overset{\circ}{X}_+$ *, and for any* $\sigma \in (0,1)$ *an* $\alpha = \alpha(u_0, \sigma) > 0$

exists such that

$$U(t)\,(\sigma u_0) \geq (1+\alpha)\,\sigma\, U(t)\, u_0 \ , \quad t > 0 \ .$$

Thus Theorem A.36 applies also for the PDE case, excluding the existence of more than one nontrivial equilibrium for system (B.1), (B.1b) under the assumptions of Lemma B.22.

B.2.1.1. The linear case

Let B be a real $n \times n$ quasimonotone (cooperative) matrix, i.e. such that $b_{ij} \geq 0$ for $i \neq j$, $i, j = 1, \cdots, n$, and consider the linear quasimonotone parabolic system

$$(B.13) \qquad \frac{\partial}{\partial t} v = D\Delta v + Bv \ , \qquad \text{in } \Omega \times (0, +\infty)$$

subject to the boundary conditions (B.1b).

The evolution operator associated with system (B.13), (B.1b) will be denoted by $\{T(t)\,,\ t \in \mathbb{R}_+\}$; it generates a (global) semiflow in X_+. Moreover, since equation (B.13) is linear, $T(t)$ will be, for any $t \in \mathbb{R}_+$, a linear operator.

Theorem B.23. [1] *Consider the following eigenvalue problem for the Laplace operator Δ (actually any strongly uniformly elliptic operator)*

$$(B.14) \qquad \begin{cases} \Delta\phi + \lambda\phi = 0 \ , & \textit{in} \ \ \Omega \\ \beta \dfrac{\partial}{\partial \nu}\phi + \alpha\phi = 0 \ , & \textit{on} \ \ \partial\Omega \end{cases}$$

where α, β are sufficiently smooth functions in $\partial\Omega$, itself sufficiently smooth. The eigenvalue problem (B.14) admits a smallest eigenvalue λ_α which is real and nonnegative. A unique (normalized) eigenfunction ϕ_α is associated with λ_α and it can be chosen strictly positive ($\phi_\alpha \gg 0$ in Ω). If $\alpha \geq 0$ on $\partial\Omega$, is not identically zero, then $\lambda_\alpha > 0$; if $\alpha \equiv 0$ on $\partial\Omega$, then $\lambda_\alpha = 0$. If $\beta \neq 0$, $\phi_\alpha \gg 0$ in $\overline{\Omega}$.

Theorem B.24. (Separation of variables) [161] *Consider system (B.13) subject to the same boundary conditions on both components of v:*

$$(B.15) \qquad \beta \frac{\partial}{\partial \nu} v + \alpha v = 0 \qquad \textit{on} \ \partial\Omega \times (0, +\infty)$$

where α , $\beta \geq 0$ *are sufficiently smooth real functions defined in* $\partial\Omega$ *. Suppose that the (B.13), (B.15) is subject to an initial condition of the form*

$$(B.16) \qquad v(x;0) = v_0(x) = \phi_\alpha(x)\,\xi \ , \qquad in\ \overline{\Omega}$$

where ϕ_α *is the unique eigenfunction associated with the first eigenvalue of problem (B.14), and* $\xi \in \mathbb{K}$ *, then the solution of system (B.13), (B.15), (B.16) is given by*

$$(B.17) \qquad v(x;t) = [T_\alpha(t)\, v_0]\,(x) = \phi_\alpha(x)\, w_\xi(t)$$

in $\overline{\Omega} \times [0,+\infty)$ *, where* $w_\xi(t)$ *,* $t \geq 0$ *is the unique solution of the following ODE system*

$$(B.18) \qquad \frac{d}{dt}w = (-\lambda_\alpha D + B)w \ , \qquad t > 0$$

subject to the initial condition

$$(B.19) \qquad w(0) = \xi \ .$$

Note that, in the above theorem, we have denoted by $\{T_\alpha(t)\,,\ t \in \mathbb{R}_+\}$ the evolution semigroup of linear operators associated with system (B.13), (B.15).

Lemma B.25. *Let* $v(t) = T(t)\, v_0$ *,* $t \in \mathbb{R}_+$ *be the solution of the linear system (B.13), (B.1b) subject to the initial condition* $v_0 \in X_+$ *. If* B *is quasimonotone (cooperative) irreducible then*

$$(B.20) \qquad v_0 \in X_+ \ , \ \ v_0 \neq 0 \quad \Longrightarrow \quad v(t) = T(t)\, v_0 \gg 0 \ , \qquad t > 0$$

i.e.

$$(B.20') \qquad T(t)\, \dot{X}_+ \subset \overset{\circ}{X}_+ \ .$$

Theorem B.26. *Consider system (B.13), (B.15).*

(i) If, for any $\lambda \in \sigma(-\lambda_\alpha D+B)$, we have $\mathcal{R}e\ \lambda < 0$, then the trivial solution is GAS in X_+ for system (B.13), (B.15)

(ii) if $\mu = \max\{\mathcal{R}e\ \lambda \mid \lambda \in \sigma(-\lambda_\alpha\, D + B)\} > 0$ and B is irreducible, then the trivial solution is unstable. Moreover

$$(B.21) \qquad \text{for any} \quad v_0 \in \dot{X}_+ \ : \ \liminf_{t\to+\infty} \|T_\alpha(t)\, v_0\|\, e^{-\mu t} > 0 \ .$$

B.2.1.2. The nonlinear case

From now on we shall denote by

$$\alpha_m = \min_{x\in\partial\Omega}\ \min_{1\le i\le n}\ \{\alpha_i(x)\}$$

$$\beta_m = \max_{x\in\partial\Omega}\ \max_{1\le i\le n}\ \{\beta_i(x)\}$$

and by

$$\alpha_M = \max_{x\in\partial\Omega}\ \max_{1\le i\le n}\ \{\alpha_i(x)\}$$

$$\beta_M = \min_{x\in\partial\Omega}\ \min_{1\le i\le n}\ \{\beta_i(x)\}$$

and by $\lambda_m\,(\phi_m)$, and $\lambda_M\,(\phi_M)$ the corresponding eigenvalues (eigenvectors) of problem (B.14).

Theorem B.27. [161] *Let B be a quasimonotone (cooperative) matrix such that*

$$(B.22) \qquad \text{for any} \quad \xi \in \mathbb{K} \ : \ f(\xi) \le B\xi \ .$$

If, for any $\lambda \in \sigma(-\lambda_m D + B)$, we have $\mathcal{R}e\ \lambda < 0$, then the trivial solution is GAS in X_+ for system (B.1), (B.1b).

Theorem B.28. [161] *Let B be a quasimonotone (cooperative) irreducible matrix such that a $\delta > 0$ exists for which*

$$(B.23) \qquad \textit{for any} \quad \xi \in \mathbb{K}\,, \ \ \|\xi\| < \delta \ : \ f(\xi) \geq B\xi \ .$$

If, $\mu = \max\{\mathcal{R}e\ \lambda \mid \lambda \in \sigma(-\lambda_M D + B)\} > 0$ then the trivial solution is unstable for system (B.1), (B.1b).

B.2.1.3. Lower and upper solutions. Existence of nontrivial equilibria

If we deal with quasimonotone reaction-diffusion systems of the form (B.1), subject to homogeneous Neumann boundary conditions ($\alpha_i = 0\,,\ \beta_i \neq 0\,,\ \ i = 1, \cdots, n$, in (B.1b)), space homogeneous equilibria are possible, which are critical points of the vector function f .

Under this assumption it is not difficult to extend Theorem A.37 (Nested Invariant Rectangles) and Theorem A.38 (Nested Contracting Rectangles) to the PDE case [39].

When we have general boundary conditions (B.1b), space homogeneous equilibria are not any more allowed, and we are obliged to extend the definition of lower and upper solution to the PDE case, explicitly.

An equilibrium solution of system (B.1), (B.1b) is a classical solution $\phi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of system

$$(B.24) \qquad D\,\Delta\phi + f(\phi) = 0 \ , \qquad \text{in } \Omega$$

subject to the boundary condition

$$(B.1b) \qquad \beta_i \frac{\partial}{\partial\nu} u_i + \alpha_i u = 0 \ , \qquad i = 1, \cdots, n \ , \quad \text{on } \partial\Omega \ .$$

A "lower solution" of system (B.24), (B.1b) is a classical solution $\underline{\phi} \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of the following inequalities

$$(B.25) \qquad D\,\Delta\underline{\phi} + f(\underline{\phi}) \geq 0 \ , \qquad \text{in } \Omega$$

$$(B.25b) \qquad \beta_i \frac{\partial}{\partial \nu} \underline{\phi}_i + \alpha_i \underline{\phi}_i \leq 0 \ , \qquad i = 1, \cdots, n \ , \quad \text{on } \partial\Omega \ .$$

The notion of "upper solution" is obtained by reversing the inequalities in (B.25) and (B.25b).

Lemma B.29. [161] *Let* $\underline{\phi} \geq 0$ *be a lower solution of system (B.25), (B.25b). Then* $\{U(t)\,\underline{\phi},\ t \in \mathbb{R}_+\}$ *is monotone nondecreasing in* t*. Further, if we set*

$$(B.26) \qquad \begin{aligned} \mathcal{N}_+(\underline{\phi}) := \big\{\psi \in C^2(\Omega) \cap C^1(\overline{\Omega}) \mid \psi \ \textit{is an equilibrium} \\ \textit{for } \text{(B.1), (B.1b)}, \ \textit{and} \ \underline{\phi} \leq \psi\big\} \end{aligned}$$

and $\mathcal{N}_+(\underline{\phi}) \neq \emptyset$*, then a* $\phi_- = \min \mathcal{N}_+(\underline{\phi})$ *exists such that*

$$lim_{t \to +\infty}\, dist\,\big(U(t)\,\underline{\phi}\ ,\ \phi_-\big) = 0$$

Lemma B.30. [161] *Let* $\overline{\phi} \leq 0$ *be an upper solution of system (B.25), (B.25b). Then* $\big\{U(t)\,\overline{\phi},\ t \in \mathbb{R}_+\big\}$ *is monotone nonincreasing in* t*. Moreover, if we set*

$$(B.27) \qquad \begin{aligned} \mathcal{N}_-(\overline{\phi}) := \big\{\psi \in C^2(\Omega) \cap C^1(\overline{\Omega}) \mid \psi \ \textit{is an equilibrium} \\ \textit{for } \text{(B.1), (B.1b)}, \ \textit{and} \ \psi \leq \overline{\phi}\big\} \ , \end{aligned}$$

then a $\phi_+ = \max \mathcal{N}_-(\overline{\phi})$ *exists such that*

$$\lim_{t \to +\infty} dist\,\big(U(t)\,\overline{\phi}\ ,\ \phi_+\big) = 0 \ .$$

Lemma B.31. [161] *Let* $(\chi_k)_{k \in \mathbb{N}}$ *be a monotone increasing sequence of equilibria of system (B.1), (B.1b) such that for any* $k \in \mathbb{N}$, $\chi_k \gg 0$ *in* $\overline{\Omega}$ *(in* Ω *for homogeneous Dirichlet boundary conditions). Then a* $\chi \gg 0$ *in* $\overline{\Omega}$ *(in* Ω *for homogeneous Dirichlet boundary conditions) exists such that* $\lim\limits_{k \to +\infty} \chi_k = \chi$*, in* X*.* χ *is itself an equilibrium of system (B.1), (B.1b).*

Theorem B.32. [137, 139, 161] *Let* f *in system (B.1) satisfy the following assumptions*

(i) a quasimonotone (cooperative), irreducible matrix B *exists for which a* $\delta > 0$ *exists such that*

$$\textit{for any} \quad \xi \in \mathbb{K}\,, \ \ \|\xi\| < \delta \ : \ f(\xi) \geq B\xi$$

and

$$\mu = \max\{\mathcal{R}e\ \lambda \mid \lambda \in \sigma(-\lambda_M D + B)\} > 0$$

(ii) a quasimonotone (cooperative) matrix C *exists for which a* $\delta > 0$ *exists such that*

$$\textit{for any} \quad \xi \in \mathbb{K}\,, \ \ \|\xi\| \geq \delta \ : \ f(\xi) \leq C\xi$$

and

$$\textit{for any} \quad \lambda \in \sigma(C) \ : \ \mathcal{R}e\ \lambda < 0$$

(iii) (F4) and (F5) of Section A.4.2 .

Then system (B.1), (B.1b) admits a unique nontrivial equilibrium solution $\phi \gg 0$ *in* $\overline{\Omega}$ *(in* Ω *for homogeneous Dirichlet boundary conditions) which is GAS in* $\dot{X}_+$ *.*

B.2.2. Lyapunov methods for PDE 's, LaSalle Invariance Principle in Banach space

Suppose we are given a semidynamical system on a closed subset D of a Banach space X.

As we have seen in the previous sections it may be defined by a C_o-semigroup of nonlinear operators $\{U(t)\,;\ t \in \mathbb{R}_+\}$ acting on D (see Section B.1.4).

For any $u \in D$, we may define the positive orbit starting from u at $t = 0$, as usual

$$\Gamma_+(u) := \{U(t)\,u \in D \mid t \in \mathbb{R}_+\} \quad .$$

We say that $\phi \in D$ is an equilibrium point for $\{U(t)\,,\ t \in \mathbb{R}_+\}$ if $\Gamma_+(\phi) = \phi$.

Stability concepts can be rephrased in a Banach space, with respect to its norm.

A "Lyapunov functional" for the dynamical system $\{U(t)\,;\ t \in \mathbb{R}_+\}$ on $D \subset X$ is a continuous real valued function $\mathcal{V} : D \subset X \longrightarrow \mathbb{R}$ such that

$$\dot{\mathcal{V}}(u) := \limsup_{t \to 0+} \frac{1}{t} \{\mathcal{V}(U(t)\,u) - \mathcal{V}(u)\} \le 0$$

for all $u \in D$.

Theorem B.33. [108] *Let $\{U(t)\,,\ t \in \mathbb{R}_+\}$ be a dynamical system on D, and let 0 be an equilibrium point in D. Suppose $\mathcal{V}$ is a Lyapunov function on D which satisfies*

(i) $\mathcal{V}(0) = 0$

(ii) $\mathcal{V}(u) \ge c(\|u\|)\,,\ u \in D$ *where c is a continuous strictly increasing function such that $c(0) = 0$ and $c(r) > 0$ for $r > 0$.*

Then 0 is stable. Suppose in addition that

(iii) $\dot{\mathcal{V}}(u) \le -c_1(\|u\|)\,,\ u \in D$ *where c_1 is also continuous increasing and positive with $c_1(0) = 0$.*

Then 0 is uniformly asymptotically stable.

A set $\Sigma \subset D$ is "positively invariant" for the dynamical system $\{U(t)\,,\ t \in \mathbb{R}_+\}$ if $U(t)\,\Sigma \subset \Sigma$, for any $t \in \mathbb{R}_+$.

If $u_0 \in D$ and $\Gamma_+(u_0)$ is its positive orbit, then the ω-limit set of u_0, or of $\Gamma_+(u_0)$, is

$$\omega(u_0) = \omega(\Gamma_+(u_0)) := \big\{ u \in D \mid \text{a sequence } \; t_n \in \mathbb{R}_+ \; \text{ exists such that } \\ t_n \longrightarrow \infty \text{ , and } \; U(t_n) u_0 \longrightarrow u \text{ , for } \; n \longrightarrow \infty \big\} \; .$$

Theorem B.34. [108, 215] *Suppose* $u_0 \in D$ *is such that its orbit* $\Gamma_+(u_0)$ *is precompact (lies in a compact set of* D *); then* $\omega(u_0)$ *is nonempty, compact, invariant and connected. Moreover*

$$\lim_{t \to +\infty} dist(U(t)\, u_0 \;,\; \omega(u_0)) = 0 \; .$$

Theorem B.35. (LaSalle Invariance Principle) [108, 215] *Let* $\mathcal{V}$ *be a Lyapunov functional on* D *(so that* $\dot{\mathcal{V}}(u) \leq 0$ *on* D *). Define*

$$E := \Big\{ u \in D \;\Big|\; \dot{\mathcal{V}}(u) = 0 \Big\}$$

and let M *be the largest (positively) invariant subset of* E *. If for* $u_0 \in D$ *the orbit* $\Gamma_+(u_0)$ *is precompact (lies in a compact set of* D *), then*

$$\lim_{t \to +\infty} dist(U(t)\, u_0 \;,\; M) = 0 \; .$$

Remark. For dynamical systems generated by evolution equations

$$\frac{d}{dt} u = Au + f(u)$$

where A is a strongly elliptic operator, bounded orbits are generally precompact [108, 219], and boundedness of orbits frequently follows from the existence of a Lyapunov functional such that $\{u \in D \mid \mathcal{V}(u) < k\}$ is a bounded set for a suitable choice of $k > 0$.

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Notation

$\mathbb{N}$	$:= \{0, 1, 2, \cdots\}$ is the set of all natural numbers
$\mathbb{N}^*$	$:= \mathbb{N} - \{0\}$
$\mathbb{R}$	is the set of all real numbers
$\mathbf{C}$	is the set of all complex numbers
$\mathbb{R}_+$	$:= [0, +\infty)$ is the set of all nonnegative real numbers
$\mathbb{R}_+^*$	$:= \mathbb{R}_+ - \{0\} = (0, +\infty)$ is the set of all positive real numbers
$\mathbb{R}^n$	$:= \mathbb{R} \times \cdots \times \mathbb{R}$ (n times) is the n-dimensional Euclidean space
$\mathbb{K}$	$:= \mathbb{R}_+^n = \mathbb{R}_+ \times \cdots \times \mathbb{R}_+$ is the positive cone of the n-dimensional Euclidean space
$\mathbb{R}_+^{n*}$	$:= \mathbb{R}_+^* \times \cdots \times \mathbb{R}_+^*$ (n times) $= \overset{\circ}{\mathbb{K}}$
$\\| \cdot \\|$	denotes in general the Euclidean norm in $\mathbb{R}^n$, but it can also denote the norm in an arbitrary normed vector space (depending upon the context)
$B_\rho(z_0)$	$:= \{z \in E \mid \\|z - z_0\\| < \rho\}$ denotes the open ball with center $z_0 \in E$ and radius $\rho > 0$ in a normed vector space E
A^T	denotes the transpose of the matrix A

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