6. Age structure

In this section we introduce a dependence of the force of infection upon the chronological age of individuals participating in the epidemic.

Age has been recognized as an important factor in the dynamics of epidemic processes since 1760 by Bernoulli [32] when reporting the data of a smallpox epidemic .

Age dependent models have been analyzed in [70, 121], and great attention has been paid in connection with the analysis of real epidemics by Dietz and Schenzle [84].

From the mathematical point of view a good reference is the monograph [168].

A stochastic version of age-dependent epidemic systems has been proposed more recently in [47] which allows, by means of martingale theory for point processes, the statistical analysis of age-dependent epidemic data.

Here we shall report about the results obtained by Busenberg et al. [41] about the existence and stability of nontrivial endemic states for SIS epidemic systems with constant total population. This is also to show the mathematical methods employed for the analysis of age-dependent epidemic systems.

The total population has an age structure expressed in terms of its age density $p(a;t)$, $a \in \mathbb{R}_+, t \in \mathbb{R}_+$. It is divided into two classes; the susceptible population with age density $s(a;t)$ and the infective population with age density $i(a;t)$, so that

$$p(a;t) = s(a;t) + i(a;t), \qquad a, t \in \mathbb{R}_+. \tag{6.1}$$

Each individual is subject to an age-dependent death rate $\mu(a)$. New individuals are produced at an age-dependent birth rate $\beta(a)$.

The deterministic mathematical model currently accepted for the evolution of the age-dependent populations is based on the so called McKendrick-van Foerster equation [167, 214]

$$\frac{\partial}{\partial t} p(a;t) + \frac{\partial}{\partial a} p(a;t) + \mu(a)\, p(a;t) = 0 \tag{6.2}$$

for $a, t \in \mathbb{R}_+$, subject to the boundary condition

$$p(0;t) = \int_0^\infty \beta(a) p(a;t)\, da, \qquad t \geq 0 \tag{6.3}$$

and to the initial condition

$$p(a;0) = p^o(a), \qquad a \geq 0 \tag{6.4}$$

where $p^o(a), a \in \mathbb{R}_+$ is the initial age distribution of the total population.

Typical assumptions on the parameters are

(H1) β, μ, p^o are nonnegative, piecewise continuous functions on $[0, \infty)$.

(H2) $\beta(a) > 0$ for $a \in (A_o, A)$
$\beta(a) = 0$ for $a \notin (A_o, A)$

(H3) $\displaystyle\int_0^\infty \exp\left(-\int_0^a \mu(\sigma)\,d\sigma\right) da < \infty$.

Note that (H3) implies that $\displaystyle\int_0^\infty \mu(\sigma)\,d\sigma = \infty$.

Under the above assumptions, system (6.2)-(6.4) has a steady state solution (see e.g. [70, 121]) given by

$$p_\infty(a) := b_o \exp\left(-\int_0^a \mu(\sigma)\,d\sigma\right) \quad , \quad a \geq 0 \; , \tag{6.5}$$

iff the net population reproduction rate R equals 1:

$$R := \int_0^\infty \beta(a) \exp\left(-\int_0^a \mu(\sigma)\,d\sigma\right) da = 1 \; . \tag{6.6}$$

($b_o \geq 0$ is an arbitrary parameter related to the total population size).

By referring to the analysis carried out in [41] we shall assume that (6.6) holds and the total population has reached its steady state age-distribution $p_\infty(a)$, $a \in \mathbb{R}_+$.

In such a population we introduce an SIS epidemic.

6.1. An SIS model with age structure

According to the basic definitions in Sect.2 the total population $\{p_\infty(a),\ a \in \mathbb{R}_+\}$ is divided into two subgroups; the susceptible population, with age distribution $\{s(a;t),\ a \in \mathbb{R}_+\}$, and the infective population, with age distribution $\{i(a;t),\ a \in \mathbb{R}_+\}$, at time $t \geq 0$, so that

$$(6.7) \qquad p_\infty(a) = s(a;t) + i(a;t) \ , \qquad a \geq 0 \ , \quad t \geq 0 \ .$$

Given (6.7), the epidemic process can be described by the evolution of the infective population. The basic equations are given in [167] for an early reference; a more detailed discussion can be found in [198]. In accordance with our approach (see Sect.1 and Sect.5.1), as for the space structure the force of infection acting on a susceptible individual of age a, $g(i(\cdot;t))(a)$ depends a priori on the overall distribution of the infective population at time t, $i(\cdot;t)$.

A possible choice, suggested by Schenzle [198] is the following

$$(6.8) \qquad g(i(\cdot;t))(a) = \int_0^\infty k(a,a')\, i(a';t)\, da'$$

where $k(a,a')$ describes the action of the infectives of age $a' \in \mathbb{R}_+$ on the susceptibles of age $a \in \mathbb{R}_+$.

As a consequence the infection process is described by

$$(6.9) \qquad g(i(\cdot;t))(a)\, s(a;t) \ .$$

For the case $k(a,a') = \delta(a-a')\, k_o(a)$ (δ is the Dirac function) we get [40.1]

$$(6.10) \qquad g(i(\cdot;t))(a) = k_o(a)\, i(a;t)$$

The case (6.10) is known as "intracohort" infection process; while (6.8) is better known as "intercohort" infection process.

We may wish to remind here that the case $k(a,a') = k$, constant, has been investigated in [80, 98]; $k(a,a') = k(a)$, in [101, 220]; $k(a,a') = k_1(a)\, k_2(a')$, in [84, 99].

If we consider an SIS model, the evolution of the age distribution $\{i(a;t),\ a \in \mathbb{R}_+\}$, $t \in \mathbb{R}_+$, is described by

$$(6.11) \quad \frac{\partial}{\partial t} i(a;t) + \frac{\partial}{\partial a} i(a;t) + \mu(a)\, i(a;t) = g(i(\cdot\,;t))(a)\, s(a;t) - \gamma(a)\, i(a;t)$$

$$(6.12) \qquad i(0;t) = q \int_0^\infty \beta(a) i(a;t)\, da$$

$$(6.13) \qquad i(a;0) = i^o(a)$$

where $\gamma(a)$, $a \in \mathbb{R}_+$ is the recovery rate of infectives (going back into the susceptible class).

The parameter $0 \leq q \leq 1$ in (6.12) is the probability for the disease to be vertically transmitted from infective parents to newborns. When $q = 0$ there is no vertical transmission and hence condition (6.12) becomes

$$i(0;t) = 0 \tag{6.12$'$}$$

that is all newborns are susceptible.

We shall report here the analytical results obtained in [41] about the threshold theorems.

6.1.1. The intracohort case

If we choose the case (6.10) equation (6.11) becomes

$$\frac{\partial}{\partial t} i(a;t) + \frac{\partial}{\partial a} i(a;t) + \mu(a)\, i(a;t) = k_o(a)\, i(a;t)(p_\infty(a) - i(a;t)) \\ -\gamma(a)\, i(a;t) \tag{6.14}$$

where we have taken (6.7) into account.

We shall make the further assumption that

(H4) γ and k_o are nonnegative piecewise continuous functions on $[0, +\infty)$ and k_o is bounded.

System (6.14), (6.12), (6.13) can be explicitly solved along the characteristic lines $t - a = const$, and we obtain the following :

$$i(a;t) = \begin{cases} i_1(a;t)\,, & \text{if} \quad a \geq t \\ i_2(a;t)\,, & \text{if} \quad a < t \end{cases} \tag{6.15}$$

where

$$i_1(a;t) = \frac{i^o(a-t)\, \exp\left(\int_0^t \alpha(a-t+\sigma)\, d\sigma\right)}{1 + i^o(a-t) \int_0^t \exp\left(\int_0^\tau \alpha(a-t+\sigma)\, d\sigma\right) k_o(a-t+\tau)\, d\tau}$$

$$i_2(a;t) = \frac{i(0,t-a)\, \exp\left(\int_0^a \alpha(\sigma)\, d\sigma\right)}{1 + i(0,t-a) \int_0^a \exp\left(\int_0^\tau \alpha(\sigma)\, d\sigma\right) k_o(\tau)\, d\tau}$$

and

$$\alpha(\sigma) = -\mu(\sigma) - \gamma(\sigma) + k_o(\sigma)p_\infty(\sigma)\,, \qquad \sigma \geq 0 \tag{6.16}$$

To completely explicit $i(a;t)$ we need to take into account the boundary condition (6.12).

6.1.1.1. The case $q = 0$

For the case $q = 0$ (no vertical transmission), we apply (6.12') to get

$$i(a;t) = \begin{cases} i_1(a;t)\,, & \text{if} \quad a \geq t \\ 0 & \text{if} \quad a < t \end{cases} \tag{6.17}$$

where $i_1(a;t)$ is defined in the Eqn. (6.15) and following.

If we denote, for $a \geq 0$,

$$\begin{aligned} E(a) &:= \exp\left(\int_0^a \alpha(\sigma)\,d\sigma\right) \\ &= \frac{p_\infty(a)}{b_o}\exp\left(\int_0^a [-\gamma(\sigma) + k_o(\sigma)p_\infty(\sigma)]\,d\sigma\right) \end{aligned} \tag{6.18}$$

from (6.17) we get, for the cohort $a_o = a - t \geq 0$

$$i(a_o + t;t) = \frac{i^o(a_o)\dfrac{E(a_o+t)}{E(a_o)}}{1 + i^o(a_o)\displaystyle\int_o^t \left(\frac{E(a_o+\tau)}{E(a_o)}\right) k_o(a_o+\tau)\,d\tau}\,, \qquad t \geq 0 \tag{6.19}$$

Since

$$\lim_{a\to\infty} E(a) = 0 \quad ; \quad \int_0^\infty E(a)\,da < +\infty \tag{6.20}$$

from (6.19) we get, for any cohort $a_o \geq 0$,

$$\lim_{t\to\infty} i(a_o + t;t) = 0 \ . \tag{6.21}$$

That is the cohort of infectives which has age a_o at time $t = 0$ eventually vanishes. We may then conclude that when there is no vertical transmission of the disease, any epidemic dies off due to the aging process.

6.1.1.2. The case $q > 0$

The case $q > 0$ requires a more elaborate treatment based on an integral equation for the quantity

$$u(t) := i(0;t)\,. \tag{6.22}$$

By substituting (6.15) into (6.12) we obtain for $u(t)$ the integral equation

$$u(t) = F(t) + \int_0^t G(a, u(t-a))\, da\,, \qquad t \geq 0 \tag{6.23}$$

where

$$F(t) := \int_t^\infty \frac{q\beta(a)E(a)\, i^o(a-t)}{E(a-t) + i^o(a-t)\displaystyle\int_{a-t}^a E(\tau)k_o(\tau)\, d\tau}\, da\ , \quad t \geq 0 \tag{6.24}$$

$$\begin{aligned} G(a,z) &:= \frac{q\beta(a)E(a)z}{1 + z\displaystyle\int_0^a E(\tau)k_o(\tau)\, d\tau}\ , \qquad a \geq 0, \quad z \geq 0 \\ &= D(a,z)\, z \end{aligned} \tag{6.25}$$

where we have denoted by

$$D(a,z) := \frac{q\beta(a)E(a)}{1 + z\int_0^a E(\tau)k_o(\tau)\, d\tau}\ , \qquad a \geq 0, \quad z \geq 0\ . \tag{6.26}$$

If we make the assumptions (H1)-(H4) we get

$$F(t) = 0 \qquad \text{for} \quad t > A \tag{6.27}$$

$$\begin{aligned} G(a,z) &> 0 \qquad \text{for} \quad a \in (A_o, A) \\ G(a,z) &= 0 \qquad \text{for} \quad a \notin (A_o, A) \end{aligned} \tag{6.28}$$

Furthermore for any $a \in (A_o, A)$:

$$z \to G(a,z) \qquad \text{is an increasing function} \tag{6.29}$$

$$z \to D(a,z) \qquad \text{is a decreasing function.} \tag{6.30}$$

Because of (6.27) and (6.28) the asymptotic behavior of equation (6.23) is the following

$$v(t) = \int_0^A G(a, v(t-a))\, da\,, \qquad t \to +\infty \tag{6.31}$$

so that, if we are interested in constant solutions of (6.31):

$$V = \int_0^A G(a,V)\,da \tag{6.32}$$

or better

$$V = V \int_0^A D(a,V)\,da\,. \tag{6.33}$$

Thus, either $V = 0$, or $V \neq 0$ as given by

$$1 = \int_0^A D(a,V)\,da \tag{6.34}$$

Since the function

$$\Delta(V) = \int_0^A D(a,V)\,da$$

is decreasing with limit zero as $V \to \infty$,equation (6.34) will have one and only one solution, iff the threshold condition

$$T := \Delta(0) > 1 \tag{6.35}$$

is satisfied.

The threshold parameter

$$\begin{aligned} T &= \int_0^A q\beta(a)E(a)\,da \\ &= \frac{q}{b_o}\int_0^A \beta(a)p_\infty(a)\exp\left(\int_0^a [-\gamma(\sigma)+k_o(\sigma)p_\infty(\sigma)]\,d\sigma\right)\,da \end{aligned} \tag{6.36}$$

can be interpreted as a net infection-reproduction rate (number).

By summarizing the above results we get

Theorem 6.1. *Let the "threshold parameter" T be defined as in (6.36).*

(a) If $T \leq 1$, then equation (6.32) has only the trivial constant solution $V \equiv 0$.

b) If $T > 1$, then it also admits a nontrivial solution $V_\infty > 0$ which is obtained as a solution of (6.34).

As far as the stability of the constant solutions 0 and V_∞ is concerned, the following main theorem is shown in [41] by monotone iteraction techniques.

Theorem 6.2. *Let T be defined as before.*

a) If $T \leq 1$ *, then* $\lim_{t\to\infty} u(t) = 0$.

b) If $T > 1$ *, then* $\lim_{t\to\infty} u(t) = V_\infty$.

6.1.2. The intercohort case

The results obtained in Sect. 6.1 for the intracohort case can be extended to the general case in which the force of infection is

$$g(i(\cdot\,;t))\,(a) = k_o(a)\,i(a;t) + \int_0^\infty k(a,a')\,i(a';t)\,da' \tag{6.37}$$

The pure intercohort case ($k_o = 0$) had been analyzed in [41] obtaining only partial local stability results. More recently in [42] a final answer has been given to the above problem by using semigroup theoretic methods in Banach spaces. It is interesting to report about this case here, since, as expected from the analysis in Sect.2.3.1 and Sect. 4.3 , SIS models belong to Class B and show a "quasi-monotone" behavior that implies a monotone evolution operator for which periodic solutions are ruled out (see e.g. [119]), and a nontrivial steady state, whenever it exists, is globally asymptotically stable. Thus showing that quasimonotone systems are "robust" also with respect to age structure.

In this case, with the force of infection (6.37), and $q = 0$, system (6.11)-(6.13) becomes

$$\begin{aligned} &\frac{\partial}{\partial t}u(a,t) + \frac{\partial}{\partial a}u(a,t) + \mu(a)u(a,t) \\ &\qquad = \tilde{g}(u(\cdot,t))(a)\,(1-u(a,t)) - \gamma(a)u(a,t) \end{aligned} \tag{6.38}$$

$$u(0,t) = 0 \tag{6.39}$$

$$u(a,0) = u^o(a) \tag{6.40}$$

for $a \geq 0$ and $t \geq 0$, where we have introduced the adimensional fraction of infectives

$$u(a,t) = \frac{i(a;t)}{p_\infty(a)} \tag{6.41}$$

and

$$\tilde{g}(u(\cdot,t))(a) = k_o(a)p_\infty(a)u(a,t) + \int_0^\infty k(a,a')p_\infty(a')u(a',t)\,da' \tag{6.42}$$

In [42] it is assumed that a maximum age $a^\dagger$ exists such that

(K1) $\beta(\cdot)$ is non-negative and belongs to $L^\infty(0,a^\dagger)$

(K2) $\mu(\cdot)$ is non-negative and measurable

(K3) $R = \int_0^{a^\dagger} \beta(a) e^{-\int_0^a \mu(a')\,da'}\, da = 1$

so that we can assume that the total population has reached its steady state $\{p_\infty(a), a \in \mathbb{R}_+\}$ given by (6.5).

Further it is assumed that

(K4) $\gamma(\cdot)$ and $k_o(\cdot)$ are non-negative and belong to $L^\infty(0, a^\dagger)$

(K5) $k(a,a'), \quad a,a' \in [0,a^\dagger]$, is measurable and there exists a positive constant $\epsilon > 0$ and two non-negative functions $k_1, k_2 \in L^\infty(0, a^\dagger)$ such that

α) $\quad \epsilon k_1(a) k_2(a') \leq k(a,a') p_\infty(a') \leq k_1(a) k_2(a'), \quad$ in $\quad [0, a^\dagger]$.

β) $\quad$ there exist $\quad 0 \leq a_1, a_2, b_1, b_2 \leq a^\dagger \quad$ such that

$$a_1 < b_1, \quad a_2 < b_2, \quad a_1 < b_2$$

and

$$k_1(a) > 0, \qquad \text{if} \quad a_1 < a < b_1$$

$$k_2(a) > 0, \qquad \text{if} \quad a_2 < a < b_2 \ .$$

The mathematical set up which has been used to analyze this problem refers to the Banach space $L^1(0, a^\dagger)$ so that the solutions $\{u(\cdot,t), t \in \mathbb{R}_+\}$ of problem (6.38)-(6.40) are looked for as elements of the closed convex set

$$C = \left\{ f \in L^1(0, a^\dagger) \mid 0 \leq f(a) \leq 1 \quad \text{a.e.} \right\} \tag{6.43}$$

The following theorem has been proven [42].

Theorem 6.3. *Let the initial condition $u^o \in C$. Then problem (6.38)- (6.40) has a unique mild solution [179] in C. This defines a flow $\{S(t)u^o, t \in \mathbb{R}_+\}$ [119] which has the following properties*

$$S(t)C \subset C \tag{6.44}$$

$$\text{if} \quad u^o \leq v^o \quad \text{then} \quad S(t)u^o \leq S(t)v^o \tag{6.45}$$

$$\text{if} \quad 0 \leq \xi \leq 1 \quad \text{then} \quad \xi S(t)u^o \leq S(t)(\xi u^o) \tag{6.46}$$

We wish to point out that a mild solution of system (6.38)-(6.40) is essentially a solution of a suitable corresponding integral formulation of the same problem [26], so that it is not required that $\{S(t)u^o, t \in \mathbb{R}_+\}$ admits time derivatives (for a more detailed discussion we refer to [42] and [179]; see also [168] and [220]).

Condition (6.45) gives the monotonicity of the evolution operator $\{S(t), t \in \mathbb{R}_+\}$, while (6.46) expresses its sublinearity.

An endemic solution u_∞ of system (6.38)-(6.40) is a fixed point of the evolution operator $\{S(t), t \in \mathbb{R}_+\}$:

$$S(t)u_\infty = u_\infty\,, \qquad t \in \mathbb{R}_+ \ . \tag{6.47}$$

Unfortunately the threshold theorem as given in [42] refers to an abstract formulation of system (6.38)-(6.40).

Anyway a parameter ρ , the spectral radius of a suitable operator defined on the Banach space $L^1(0, a^\dagger)$, is introduced so that

Theorem 6.4.

a) *If* $\rho \le 1$ *then no nontrivial endemic states exist for system (6.38)-(6.40).*

b) *if* $\rho > 1$ *then a unique nontrivial endemic state* u_∞ *exists for system (6.38)-(6.40), in addition to the trivial one.*

The stability properties of the endemic equilibria are established by the following

Theorem 6.5. *Assume no nontrivial endemic equilibrium exists, then for any initial condition* $u^o \in C$ *we have*

$$\lim_{t\to\infty} S(t)u^o = 0 \ .$$

To consider the nontrivial case, we introduce the concept of nontrivial initial condition, i.e. a $u^o \in C$ such that

$$\int_t^{a^\dagger} k_2(a)\, u^o(a-t)\, da > 0, \qquad \text{for some} \quad t \ge 0 \ .$$

Theorem 6.6. *Let* u_∞ *be the unique nontrivial equilibrium. Then for any nontrivial initial condition* u^o *we have*

$$\lim_{t\to\infty} S(t)u^o = u_\infty \ .$$

If u^o *is not nontrivial then*

$$S(t)\, u^o = 0\,, \qquad \text{for any} \quad t \ge a^\dagger$$

.

The monotonicity of the evolution operators $\{S(t), t \in \mathbb{R}_+\}$ implies the GAS of the trivial endemic state in Theorem 6.5 and of the nontrivial endemic state in Theorem 6.6.

Further extension to the case of vertically transmitted diseases is considered in [43].

6.2. An SIR model with age structure [123]

In this section we report the results obtained recently by Inaba [123] about the existence and stability of an SIR model with age structure as formulated by Greenhalgh [99].

The total population has an age structure expressed in terms of its age density $p(a;t)$, $a \in [0, a^\dagger]$, $t \geq 0$ (it is assumed a maximum demographic age $a^\dagger > 0$).

It is divided into three classes; the susceptible class, with age density $s(a;t)$; the infective class with age density $i(a;t)$, and the removed (immune) class, with age density $r(a;t)$, so that

$$p(a;t) = s(a;t) + i(a;t) + r(a;t)\,, \quad a \in [0, a^\dagger]\,,\ t \geq 0\,. \tag{6.48}$$

As in Sect. 6.1, each individual is subject to an age-dependent death rate $\mu(a)$; new individuals are produced at an age dependent birth rate $\beta(a)$, $0 \leq a \leq a^\dagger$.

The evolution of the total density $p(a;t)$ is then governed by system (6.2) for $a \in [0, a^\dagger]$, $t \geq 0$, subject to the boundary condition

$$p(0;t) = \int_0^{a^\dagger} \beta(a)\, p(a;t)\ da\ , \qquad t \geq 0 \tag{6.49}$$

and to an initial condition (6.4).

Assumptions (H1)-(H2) are kept but (H3) now is

$$\int_0^{a^\dagger} \exp\left(-\int_0^a \mu(\sigma)\, d\sigma\right)\, da\ < +\infty \tag{$H3'$}$$

and that

$$\int_0^{a^\dagger} \mu(\sigma)\, d\sigma = +\infty\ . \tag{6.50}$$

System (6.2), (6.49), (6.4) has a steady state solution given by

$$p_\infty(a) = b_o\, \exp\left(-\int_0^a \mu(\sigma)\, d\sigma\right)\,, \qquad a \in [0, a^\dagger] \tag{6.51}$$

iff the net population reproduction rate R equals one,

$$R = \int_0^{a^\dagger} \beta(a)\, \exp\left(-\int_0^a \mu(\sigma)\ d\sigma\right)\, da = 1\ . \tag{6.52}$$

We shall assume, as in Sect. 6.1, that the total population has reached its stationary demographic state (6.51).

In such a population we introduce an SIR epidemic, so that (6.48) has to be rewritten as

$$p_\infty(a) = s(a;t) + i(a;t) + r(a;t)\ , \quad a \in \left[0, a^\dagger\right],\ t \geq 0\,. \tag{6.53}$$

If we consider for the force of infection the same form as in (6.8), and a removal rate $\gamma > 0$, which now is assumed age independent, the evolution equations for the epidemic system are

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) s(a;t) = -g(i(\cdot,t))(a)\, s(a;t) - \mu(a)\, s(a;t) \tag{6.54a}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) i(a;t) = -g(i(\cdot,t))(a)\, s(a;t) - (\mu(a) + \gamma)\, i(a;t) \tag{6.54b}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) r(a;t) = \gamma\, i(a;t) - \mu(a)\, r(a;t) \tag{6.54c}$$

for $a \in \left]0, a^\dagger\right[,\ \ t > 0$, subject to the boundary conditions

$$s(0;t) = \int_0^{a^\dagger} \beta(a)\, p(a;t)\ da \tag{6.55a}$$

$$i(0;t) = 0 \tag{6.55b}$$

$$r(0;t) = 0 \tag{6.55c}$$

Under the assumption of demographic equilibrium

$$\int_0^{a^\dagger} \beta(a)\, p(a;t)\ da = \int_0^{a^\dagger} \beta(a)\, p_\infty(a)\ da = \\ = b_o \int_0^{a^\dagger} \beta(a)\ \exp\left(-\int_0^a \mu(\sigma)\, d\sigma\right)\ da = b_o R = b_o\ ,$$

so that the boundary condition (6.55a) can be rewritten as

$$s(0;t) = b_o\ . \tag{6.55a'}$$

We may renormalize all densities so that

$$s(a;t) + i(a;t) + r(a;t) = 1\ , \qquad a \in \left[0, a^\dagger\right],\ \ t \geq 0\ \ , \tag{6.56}$$

in which case

$$(6.55a'') \qquad s(0;t) = 1 \ , \qquad t \geq 0 \ .$$

The force of infection, under condition (6.56) will be given by

$$(6.57) \qquad g(i(\cdot;t))(a) = \int_0^{a^\dagger} k(a,a')\, p_\infty(a')\, i(a';t)\ da'$$

with $p_\infty(a)$ given by (6.51).

System (6.54), (6.55) has to be supplemented by suitable initial conditions

$$(6.58) \quad s(a;0) = s^o(a); \quad i(a;0) = i^o(a); \quad r(a;0) = r^o(a), \qquad a \in \left[0, a^\dagger\right] \ .$$

We shall assume in the sequel that

$$(H4') \qquad k(\cdot,\cdot) \in L^\infty\left(\left[0,a^\dagger\right] \times \left[0,a^\dagger\right]\right) \ .$$

We note that due to condition (6.56) it suffices to analyze system (6.54a), (6.54b) with the boundary conditions (6.55a”), (6.55b) subject to the initial conditions (6.58).

Under these assumptions (H1), (H2), (H3’), (H4’) it is possible to show, by semigroup theoretical methods that this problem is well posed in the Banach space $X := L^1\left(0, a^\dagger; \mathbb{R}^2_+\right)$.

In fact, if we define

$$(6.59) \qquad \Omega_o := \left\{ f = (f_1, f_2)^T \in X \mid 0 \leq f_i \leq 1, \ \text{a.e.} \ \ i = 1,2 \right\}$$

the following theorem holds [123]

Theorem 6.7. *Let the initial condition $u^o := (s^o(\cdot), i^o(\cdot)) \in \Omega_o$. Then problem (6.54 a,b), (6.55 a”,b), (6.58) has a unique mild solution [26, 179] in Ω_o. If we further assume that the initial conditions are absolutely continuous in $\left[0, a^\dagger\right]$, then the initial value problem admits a unique global classical solution in Ω_o.*

Let us now look for steady states $u^* := (s^*(\cdot), i^*(\cdot))^T$ of our system. It is not difficult to show that it must satisfy the following

$$(6.60a) \qquad s^*(a) = \exp\left(-\int_0^a g(i^*)(\sigma)\, d\sigma\right)$$

$$(6.60b) \quad i^*(a) = \int_0^a \exp(-\gamma(a-\sigma))\, g(i^*)(\sigma) \exp\left(-\int_0^\sigma g(i^*)(\eta)\, d\eta\right) d\sigma$$

$$(6.60c) \qquad \lambda^*(a) := g(i^*)(a) = \int_0^{a^\dagger} k(a,\sigma)\, p_\infty(\sigma)\, i^*(\sigma)\ d\sigma \ .$$

From (6.60b) and (6.60c) we obtain an equation for $\lambda^*(a)$:

$$\lambda^*(a) = \int_0^{a^\dagger} \phi(a,\sigma)\,\lambda^*(\sigma)\,\exp\left(\left(-\int_0^\sigma \lambda^*(\eta)\,d\eta\right)\right) d\sigma \ , \tag{6.61}$$

where

$$\phi(a,\sigma) := \int_\sigma^{a^\dagger} k(a,\tau)\,p_\infty(\tau)\,\exp(-\gamma(\tau-\sigma))\,d\tau \ .$$

It is clear that (6.61) always admits the trivial solution $\lambda^*(a) \equiv 0$ in $[0,a^\dagger]$, which corresponds to a disease-free steady state $u^* := (1,0)^T$. So we look for nontrivial solutions of (6.61).

If we denote by Φ the nonlinear operator

$$x \in E \longrightarrow \Phi(x)(a) := \int_0^{a^\dagger} \phi(a,\sigma)\,x(\sigma)\,\exp\left(-\int_0^\sigma x(\eta)\,d\eta\right) d\sigma \tag{6.62}$$

with $a \in [0,a^\dagger]$, and defined in the Banach space $E := L^1\left(0,a^\dagger\right)$, equation (6.61) is equivalent to a fixed point problem for Φ:

$$\lambda^* = \Phi(\lambda^*)$$

(Note that $L^\infty\left(0,a^\dagger\right) \subset E$).

The nonlinear operator Φ has a linear majorant $T:\ E \longrightarrow E$ defined by

$$x \in E \longrightarrow T(x)(a) := \int_0^{a^\dagger} \phi(a,\sigma)\,x(\sigma)\ d\sigma \ , \qquad a \in [0,a^\dagger] \tag{6.63}$$

which is positive with respect to the cone $E_+ := \{x \in E \,|\, 0 \le x\}$, i.e.

$$T(E_+) \subset E_+ \ .$$

In addition to (H4') let us further assume that

(H5) a) $\displaystyle\lim_{h\to 0} \int_0^{a^\dagger} |k(a+h,\sigma) - k(a,\sigma)|\,da = 0$ uniformly in $\sigma \in \mathbb{R}$.
(k is extended by $k(a,\sigma) = 0$ for $a,\sigma \in (-\infty,0) \cup (a^\dagger,+\infty)$).

b) There exist numbers $\alpha > 0$ and $\epsilon > 0$ with $0 < \alpha < a^\dagger$, such that $k(a,\sigma) \ge \epsilon$ for a.e. $(a,\sigma) \in \left(0,a^\dagger\right) \times \left(a^\dagger - \alpha, a^\dagger\right)$.

Under the above assumptions [123] (see [160] for the terminology):

Lemma 6.8. *The linear operator T is nonsupporting and compact.*

As a consequence [160] the spectral radius $r(T)$ of the operator T is a positive eigenvalue of T, and it is the only positive eigenvalue with a positive eigenvector.

The following threshold theorem holds [123].

Theorem 6.9. *Let $r(T)$ be the spectral radius of the operator T defined in (6.63).*

a) If $r(T) \leq 1$, then the only nonnegative fixed point x of Φ is the trivial solution $x \equiv 0$.
b) If $r(T) > 1$, then Φ has at least one nontrivial positive solution.

In order to have uniqueness of the nontrivial solution, a further assumption is made.

(H6) For all $(a,\sigma) \in [0, a^\dagger] \times [0, a^\dagger]$, the following inequality holds:

$$k(a,\sigma)\, p_\infty(\sigma) - \gamma \int_\sigma^{a^\dagger} k(a,\tau)\, p_\infty(\tau)\, \exp(-\gamma(\tau-\sigma))\, d\tau \geq 0\ .$$

Now observe that, from the definition of Φ, it follows that

$$\begin{aligned}\Phi(x)\,(a) &= \int_0^{a^\dagger} \phi(a,\sigma) \left(-\frac{d}{d\sigma}\right) \exp\left(-\int_0^\sigma x(\eta)\, d\eta\right)\, d\sigma \\ &= \phi(a,0) - \int_0^{a^\dagger} [k(a,\sigma)\, p_\infty(\sigma) - \gamma\phi(a,\sigma)]\, \exp\left(-\int_0^\sigma x(\eta)\, d\eta\right)\, d\sigma\end{aligned}$$

so that Φ is monotone increasing with respect to the cone E_+ .

Furthermore it can be shown [123], under the same assumption, that Φ is strongly concave in the sense of Krasnoselskii (see Appendix A, Section A.4.2).

As a consequence of positivity, monotonicity and strong concavity we know that (Theorem A.36 in Appendix A) the operator Φ has at most one positive fixed point. Thus

Theorem 6.10. *Under the assumption (H6), if $r(T) > 1$, then Φ admits a unique nontrivial positive solution.*

Remark. A sufficient condition for assumption (H6) to be verified is

$(H6')$ $\qquad k(a,\sigma)\, p_\infty(\sigma)$ is nonincreasing with respect to σ

.

A particular case of (H6') is $k(a, \sigma) \equiv k(a)$ independent of the age σ of the infectives, since $p_\infty(\sigma)$ is a decreasing function of σ.

Remark. Independently of the fact that (H6) holds, it can be easily shown (see e.g. [99]) that if $k(a, \sigma) = k_1(a)\, k_2(\sigma)$ (proportionate mixing assumption [84]), then there exists a unique nontrivial steady state under the condition

$$r(T) = \int_0^{a^\dagger} \phi(\sigma, \sigma)\, d\sigma \ > 1$$

.

In this case $f(\cdot)$ is the eigenvector of the operator T corresponding to the spectral radius $r(T)$.

We are left now with the problem of stability of the steady states.

Since the analysis in [123] contains technicalities which go beyond the interests of this presentation, we shall limit ourselves to reporting the main results.

Theorem 6.11.

a) *If $r(T) < 1$, then the trivial equilibrium is GAS for system (6.54), (6.55) with respect to positive initial conditions .*

b) *If $r(T) > 1$, then the trivial solution of system (6.54), (6.55) is unstable. If furthermore the nontrivial endemic state is such that*

(H7) $$i^*(a^\dagger) < e^{-\gamma a^\dagger}$$

then it is LAS.

Condition (H7), being an a priori condition on the steady state, is not very convenient in the applications.

In [123] it is shown that (H7) holds when $k(a, \sigma) \equiv k$, a constant.

We conclude this section, by remarking that again for SIR models with structures the problem of GAS is more difficult and in fact it left open for many relevant cases.