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ON THE GENERAL STRUCTURE OF EPIDEMIC SYSTEMS. GLOBAL ASYMPTOTIC STABILITY

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Abstract—In this paper a general ODE model is proposed to describe epidemic systems. The mathematical structure of such a model is so general that it includes many epidemic systems already analyzed via different methods by various authors. The asymptotic analysis of the general system is carried out with applications to several models.

1. INTRODUCTION

The history of mathematical models for infectious diseases is quite long now and may be traced back at least up to the 1920s when the work by Kermack and McKendrick appeared[8].

Since then many attempts of generalization have been made to introduce more realistic situations (see [2] for a rather extensive account).

Some authors have also tried general approaches to the analysis of the asymptotic behaviour of such systems (see, e.g. [7]).

To the authors' knowledge anyway no attempt has been made up to now to analyze the general structure of epidemic systems. In this paper the authors propose a general ODE system which actually includes many of the models proposed up to now, by different authors.

This general model allows an asymptotic analysis based on the structure of the ODE system by means of the Lyapunov functional proposed by Goh[4,5] for a generalized Lotka–Volterra system.

The main result of the paper, based on a previous paper by Solimano and Beretta[14], gives sufficient conditions for the global asymptotic stability (hence uniqueness) of the nontrivial equilibrium solution of the system, whenever it exists.

An existence result regarding the positive equilibrium solution is also given. It has to be pointed out that also the case in which the total population is a dynamic variable is analyzed.

In Sec. 3 the results are applied to several models.

Sec. 4 is devoted to analyze the cases in which the total population of the epidemic system is a dynamical variable. In this case we are able, by our methods, to give global stability results of the feasible or partially feasible equilibrium.

2. THE GENERAL MODEL

Let us consider first a so-called SIR model with vital dynamics[7]. If one denotes as usual by S the susceptible population, by I the infective population and by R the removed population, this model may be written as follows:

$$\begin{aligned} dS/dt &= -kSI - \mu S + \mu, \\ dI/dt &= kSI - \mu I - \lambda I, \\ dR/dt &= \lambda I - \mu R, \end{aligned} \tag{2.1}$$

for $t > 0$, subject to suitable initial conditions.

Since clearly (2.1) implies that $S(t) + I(t) + R(t) = 1$, we may ignore the last equation in (2.1) and consider the reduced system

$$
\begin{aligned}
\mathrm{d}S/\mathrm{d}t &= -kSI - \mu S + \mu, \\
\mathrm{d}I/\mathrm{d}t &= kSI - (\mu + \lambda)I,
\end{aligned}
\tag{2.2}
$$

for $t > 0$, subject to initial conditions such that $0 < S(0) + I(0) < 1$.

We may write system (2.2) in a more general form if we introduce matrix notations as follows; we set

$$
\mathbf{A} = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \quad e = \begin{pmatrix} -\mu \\ -(\mu + \lambda) \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} \mu \\ 0 \end{pmatrix},
\tag{2.3}
$$

and system (2.2) becomes

$$
\mathrm{d}z/\mathrm{d}t = \mathrm{diag}(z)(e + \mathbf{A}z) + \mathbf{c}, \quad t > 0,
\tag{2.4}
$$

where we have also set $z = (S, I)^T$.

Consider now the simple gonorrhea model proposed in [3] and [15]. It can be seen as an SIS model for two interacting populations; if we denote by S_1, I_1 and by S_2, I_2 the susceptible and the infective populations for the two groups (males and females), we have

$$
\left.
\begin{aligned}
\mathrm{d}S_1/\mathrm{d}t &= -k_{12}S_1I_2 + \alpha_1 I_1, \\
\mathrm{d}I_1/\mathrm{d}t &= k_{12}S_1I_2 - \alpha_1 I_1, \\
\mathrm{d}S_2/\mathrm{d}t &= -k_{21}S_2I_1 + \alpha_2 I_2, \\
\mathrm{d}I_2/\mathrm{d}t &= k_{21}S_2I_1 - \alpha_2 I_2,
\end{aligned}
\right\} \quad t > 0.
\tag{2.5}
$$

Again, since clearly $S_1 + I_1 = c_1$ (const) and $S_2 + I_2 = c_2$ (const), we may analyze only the system (let $k_{12} = k_{21} = 1$, for simplicity)

$$
\left.
\begin{aligned}
\mathrm{d}I_1/\mathrm{d}t &= -I_1I_2 - \alpha_1 I_1 + c_1 I_2, \\
\mathrm{d}I_2/\mathrm{d}t &= -I_1I_2 - \alpha_2 I_2 + c_2 I_1,
\end{aligned}
\right\} \quad t > 0,
\tag{2.6}
$$

which can be now written in the general form

$$
\mathrm{d}z/\mathrm{d}t = \mathrm{diag}(z)(e + \mathbf{A}z) + \mathbf{B}z, \quad t > 0,
\tag{2.7}
$$

if we set $z = (I_1, I_2)^T$, and

$$
\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} -\alpha_1 \\ -\alpha_2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & c_1 \\ c_2 & 0 \end{pmatrix}.
\tag{2.8}
$$

Altogether, we may state that both models (2.1) and (2.5) may be rewritten in the general form

$$
\mathrm{d}z/\mathrm{d}t = \mathrm{diag}(z)(e + \mathbf{A}z) + \mathbf{b}(z), \quad t > 0,
\tag{2.9}
$$

where

$$
\mathbf{b}(z) = \mathbf{c} + \mathbf{B}z, \quad z \in \mathbf{R}^n,
$$

where $n \in \mathbf{N} - \{0\}$ for sake of generality.

In Sec. 3 we have analyzed many other examples of epidemic systems, already studied with different tools by various authors; all of them may be considered, along with models (2.1) and (2.5), as particular cases of the following general ODE model:

$$dz/dt = \text{diag}(z)(e + \mathbf{A}z) + \mathbf{b}(z), \quad t > 0, \tag{2.9}$$

in $\mathbf{R}^n$, $n \in \mathbf{N} - \{0\}$, or better in

$$\mathbf{R}^n_+ := \{z \in \mathbf{R}^n \mid z_i \geq 0, \quad i = 1, \ldots, n\}$$

(the non-negative orthant of $\mathbf{R}^n$), if we take into account that our systems are positivity preserving, in accordance with the meaning of z.

In (2.9) we usually have

(i) $e \in \mathbf{R}^n$, a constant vector;

(ii) $\mathbf{A} = (a_{ij})_{i,j=1,\ldots,n}$, a real constant matrix;

(iii) $\mathbf{b}(z) = \mathbf{c} + \mathbf{B}z$, a non-negative vector function defined for $z \in \mathbf{R}^n_+$; here $c \in \mathbf{R}^n_+$ is a constant non-negative vector, and $\mathbf{B} = (b_{ij})_{i,j=1,\ldots,n}$ is a real constant matrix such that $b_{ij} \geq 0$ for any $i, j = 1, \ldots, n$, and $b_{ii} = 0$ for any $i = 1, \ldots, n$.

Hence it is worth analyzing the general qualitative properties of system (2.9) under the above assumptions (i)–(iii), and applying the results to the various different models.

Let us remark that usually epidemic systems are considered to have constant total populations, as in the above examples; this implies in particular that in (2.2) $0 \leq S(t) + I(t) \leq 1$ for any $t \geq 0$; while in (2.6) $0 \leq I_1(t) \leq c_1$, and $0 \leq I_2(t) \leq c_2$ for any $t \geq 0$.

Referring to our general model (2.9), we shall assume at first that either

$$\Omega^n_1 := \left\{ z \in \mathbf{R}^n_+ \mid \sum_{i=1}^{n} z_i \leq 1 \right\} \tag{2.10}$$

or

$$\Omega^n_2 := \{z \in \mathbf{R}^n_+ \mid z_i \leq 1, \quad i = 1, \ldots, n\} \tag{2.11}$$

is positively invariant, as well as their interiors $\mathring{\Omega}^n_1$ or $\mathring{\Omega}^n_2$, respectively. From now on we shall denote both Ω^n_1 and Ω^n_2 by Ω^n.

We shall make then the assumption

(iv) Ω^n is positively invariant.

Remark. It may be pointed out that in concrete models assumption (iv) is always satisfied due to the particular structure of the ODE systems which describe epidemic models.

Because of the structure of

$$F(z) := \text{diag}(z)(e + \mathbf{A}z) + \mathbf{b}(z), \tag{2.12}$$

it is clear that $F \in C^1(\mathbf{R}^n_+)$, and therefore $F \in C^1(\Omega^n)$.

We shall denote by D_i the hyperplane of $\mathbf{R}^n$: $D_i = \{z \in \mathbf{R}^n \mid z_i = 0\}$.

Clearly, $D_i \cap \Omega^n$ will be positively invariant if $b_{i|D_i} = 0$, while $D_i \cap \Omega^n$ will be a repulsive set whenever $F(z)$ is pointing inside Ω^n on D_i, i.e. $b_{i|D_i} > 0$.

Because of the invariance of Ω^n and the fact that $F \in C^1(\Omega^n)$, fixed point theorems[12] assure the existence of at least one equilibrium solution of (2.9), within Ω^n.

As stated in the introduction, our aim here is to show that, under suitable conditions, the general system (2.9) is such that, whenever a strictly positive equilibrium z^* exists in $\Omega^n_+ := \{z \in \Omega^n \mid z_i > 0, i = 1, \ldots, n\}$, then it is globally asymptotically stable with respect to Ω^n_+. Uniqueness of the equilibrium within Ω^n_+ clearly follows.

Let $z^* = (z_1^*, \ldots, z_n^*)$ be a strictly positive equilibrium of (2.9); then

$$e = -\mathbf{A}z^* - \operatorname{diag}(z^{*-1})\mathbf{b}(z^*), \qquad (2.13)$$

where we have denoted $z^{*-1} := (1/z_1^*, \ldots, 1/z_n^*)^T$.

By substitution in (2.9), we have

$$dz/dt = \operatorname{diag}(z)[\mathbf{A} + \operatorname{diag}(z^{*-1})\mathbf{B}](z - z^*) - \operatorname{diag}(z - z^*)\operatorname{diag}(z^{*-1})\mathbf{b}(z). \qquad (2.14)$$

Since, without the vector $\mathbf{b}(z)$, system (2.9) is a generalized Volterra system, in order to study the global asymptotic stability of z^* we shall make use of the Lyapunov function $V:\mathrm{R}_+^{n*} \to \mathrm{R}_+$ proposed by Goh[4,5] (here $\mathrm{R}_+^{n*} := \{z \in \mathrm{R}^n \mid z_i > 0, i = 1, \ldots, n\}$),

$$V(z) := \sum_{i=1}^{n} W_i \left(z_i - z_i^* - z_i^* \ln \frac{z_i}{z_i^*} \right), \qquad (2.15)$$

where $W_i > 0$ are real constants.

We introduce now some definitions.

DEFINITION 2.1

Let $\mathbf{B}$ be a real $n \times n$ matrix. We say that $\mathbf{B} \in S_W$ (resp. $\mathbf{B} \in \tilde{S}_W$) iff there exists a positive diagonal real matrix $\mathbf{W}$ such that $\mathbf{WB} + \mathbf{B}^T\mathbf{W}$ is positive definite (resp. non-negative definite).

DEFINITION 2.2

$\mathbf{B}$ is $\mathbf{W}$-skew symmetrizable (resp. $\mathbf{W}$-symmetrizable) iff there exists a positive diagonal real matrix $\mathbf{W}$ such that $\mathbf{WB}$ is skew-symmetric (resp. symmetric).

The structure of the ODE system (2.14) stimulates the analysis of the matrix $\tilde{\mathbf{A}} := \mathbf{A} + \operatorname{diag}(z^{*-1})\mathbf{B}$.

As we shall see in Sec. 3, many epidemic systems are such that either

(v) $\tilde{\mathbf{A}}$ is $\mathbf{W}$-skew symmetrizable

or

$$(\mathrm{v}') \quad \left[\tilde{\mathbf{A}} + \operatorname{diag}\left(-\frac{b_1(z)}{z_1 z_1^*}, \ldots, -\frac{b_n(z)}{z_n z_n^*} \right) \right] \in S_W,$$

where $b_i(z)$, $i = 1, \ldots, n$ are the components of the vector function $\mathbf{b}(z)$, defined in (iii).

For example model (2.2) belongs to case (v) while model (2.6) belongs to case (v').

Hence we shall analyze these two cases in detail.

Consider first the case (v), and let R be the following subset of Ω_+^n,

$$R := \{z \in \Omega_+^n \mid z_i = z_i^*, \quad \text{for any } i = 1, \ldots, n \text{ s.t. } b_i(z) > 0\}. \qquad (2.16)$$

On R, system (2.14) becomes

$$dz/dt\,|_R = \operatorname{diag}(z)\tilde{\mathbf{A}}(z - z^*), \qquad (2.17)$$

which has the same structure as predator–prey Volterra systems when looking for the largest invariant set for such systems, where the diagonal elements of the community matrix in the Volterra system play the same role as the positive components of $\mathbf{b}(z)$ in our system. We may then state that the sufficient conditions given in [13] and [14] in order that the largest invariant set of predator–prey Volterra systems reduces to the strictly positive equilibrium $\{z^*\}$, give us sufficient conditions for the largest invariant subset of (2.9) in R reduce to $\{z^*\}$.

In order to state these conditions we need to introduce some graph nomenclature.

Under the assumption (v), the elements of $\tilde{\mathbf{A}}$ have a skew-symmetric sign distribution. Then we associate a graph with $\tilde{\mathbf{A}}$ by the following rules:

(A) each component of z, say the ith, is represented by the labeled knot $\overset{\bullet}{i}$ if $b_i(z) \equiv 0$; by i otherwise.

(B) each pair of elements $\tilde{a}_{ij}\tilde{a}_{ji} < 0$ is represented by an arc connecting knots "i" and "j".

The following result holds[3,4]:

LEMMA 2.1

Assume that $\tilde{\mathbf{A}}$ is **W**-skew symmetrizable. If the associated graph is either (a) a tree and $\rho - 1$ of the ρ terminal knots are $\bullet$, or (b) a chain and two consecutive internal knots are $\bullet$ or (c) a cycle and two consecutive knots are $\bullet$, then $M \equiv \{z^*\}$ within R.

Hence we can prove the following:

THEOREM 2.1

If system (2.9) has a positive equilibrium $z^* \in \Omega^n_+$ and case (v) holds true under one of the hypotheses of Lemma 2.1 or, otherwise, case (v′) holds true, then the positive equilibrium z^* is globally asymptotically stable within Ω^n_+. The uniqueness of z^* within Ω^n_+ follows from its global asymptotic stability.

Proof. Consider the scalar function (2.15). It has the property of positive definiteness of a Lyapunov function: $V(z) \geq 0$ and $V(z) = 0$ if and only if $z = z^*$. Moreover, $V(z) \to +\infty$ when $z_i \to +\infty$ or $z_i \to 0^+$ for some i. Its time derivative along the trajectories of (2.9) is

$$\dot{V}(z) = (z - z^*)^T \mathbf{W}\,\mathrm{diag}(z^{-1})\dot{z}, \tag{2.18}$$

where $\mathbf{W} = \mathrm{diag}(W_1, \ldots, W_n)$, W_i being the positive real numbers occurring in definition (2.15) of $V(z)$. On account of (2.14), (2.18) becomes

$$\dot{V}(z) = (z - z^*)^T \mathbf{W}\tilde{\mathbf{A}}(z - z^*) - \sum_{i=1}^{n} \frac{W_i b_i(z)}{z_i z_i^*}(z_i - z_i^*)^2,$$

where within Ω^n_+ we have $z_i > 0$, $b_i(z) \geq 0$ for all i. Let us write

$$\dot{V}(z) = (z - z^*)^T \mathbf{W}\left[\tilde{\mathbf{A}} + \mathrm{diag}\left(-\frac{b_1(z)}{z_1 z_1^*}, \ldots, -\frac{b_n(z)}{z_n z_n^*}\right)\right](z - z^*). \tag{2.19}$$

In case (v′), from (2.19) it follows that $\dot{V}(z) \leq 0$ and the equality applies if and only if $z = z^*$. Hence the global asymptotic stability of z^* within Ω^n_+ follows. In case (v), since $\mathbf{W}\tilde{\mathbf{A}}$ is skew-symmetric, (2.18) reads

$$\dot{V}(z) = -\sum_{i=1}^{n} \frac{W_i b_i(z)}{z_i z_i^*}(z_i - z_i^*)^2, \tag{2.20}$$

where the nonpositive definiteness of $\dot{V}(z)$ within Ω^n_+ follows from the assumption that $b_i(z) \geq 0$ for all i. Let R be the set of all points within Ω^n_+ where $\dot{V}(z) = 0$, i.e. the set (2.16), and M be the largest invariant set in R. Owing to Lemma 2.1 we have $M \equiv \{z^*\}$.

As a consequence we may state, that every solution $z(t)$ in Ω^n_+ tends to z^* as $t \to +\infty$. This is due to the following extension of Lyapunov's stability theorem: If dV/dt is negative semidefinite, then every solution tends to the largest invariant subset of the set of all points in R^n for which $dV/dt = 0$[10] (see also [9]). Hence the global asymptotic stability of z^* within Ω^n_+ is proven for both cases (v) and (v′).

COROLLARY 2.1

If the vector $\mathbf{c}$ in (iii) is positive definite, then system (2.9) has a positive equilibrium $z^* \in \Omega^n_+$. In case (v) or (v′), the positive equilibrium z^* is globally asymptotically stable (and therefore unique) with respect to Ω^n_+.

Proof. Concerning the existence of a positive equilibrium, we can observe that, by arguments of positive invariance of Ω^n, at least one equilibrium belonging to Ω^n exists. However, if $\mathbf{c}$ is positive definite, system (2.9) cannot have equilibria with some vanishing components. Hence a positive equilibrium $z^* \in \Omega^n_+$ exists. In case (v), since $\tilde{\mathbf{A}}$ is $\mathbf{W}$-skew symmetrizable and $b(z)$ is positive definite, by (2.20) $\dot{V}(z) \leqslant 0$ and $\dot{V}(z) = 0$ if, and only if, $z = z^*$. This gives the global asymptotic stability (and uniqueness) of z^* within Ω^n_+. In case (v′) the asymptotic stability (and uniqueness) of z^* within Ω^n_+ directly follows from (2.19).

Some epidemic models, described by the ODE system (2.9), differ from usual epidemic models in that $n(t) = \Sigma^n_{i=1} z_i(t)$ is a dynamical variable rather than a specified constant (see, e.g. [11]).

Accordingly, we must drop assumption (iv). For these models, the accessible space is the whole non-negative orthant R^n_+ of the Euclidean space, and Theorem 2.1 can be reformulated by substituting the bounded set Ω^n_+ with the positive orthant R^{n*}_+. Furthermore, it has to be noticed that we cannot apply fixed point theorems even when vector $\mathbf{c}$ is positive definite. Hence Corollary 2.1 cannot be applied to these models. However, the structure of system (2.9) is such that the positive invariance of the non-negative orthant R^n_+ is assured. When vector $\mathbf{c}$ in (iii) has some identically vanishing components, system (2.9) may have equilibria z^* with some vanishing components. Let $\mathcal{N}$ be the set of indices $\mathcal{N} = \{1, \ldots, n\}$ and I the subset of $\mathcal{N}$ such that $z^*_i = 0$ when $i \in I$. According to Goh[4] if $I \neq \emptyset$, we say that z^* is partially feasible and we can study the sectorial stability of z^* with respect to

$$\mathrm{R}^n_I = \{z \in \mathrm{R}^n_+ \mid z_i > 0, \quad i \in \mathcal{N} - I; \quad z_i \geqslant 0, \quad i \in I\}. \tag{2.21}$$

The definition of sectorial stability can be found in [4]. By sectorial stability we mean that z^* is globally asymptotically stable with respect to R^n_I. If $I = \emptyset$, then z^* is feasible and we can study its global asymptotic stability with respect to R^n_+ by Theorem 2.1.

Assume that z^* is a partially feasible equilibrium of (2.9), i.e. $z^*_i = 0$, $i \in I$, $I \neq \emptyset$.

Define the matrix $\tilde{\mathbf{A}} = (\tilde{\mathbf{a}}_{ij})_{i,j=1,\ldots,n}$ as

$$\begin{aligned} \tilde{\mathbf{a}}_{ij} &= a_{ij} + b_{ij}/z^*_i, \quad \text{for all } i \in \mathcal{N} - I, \quad j \in \mathcal{N}, \\ \tilde{\mathbf{a}}_{ij} &= a_{ij}, \quad \text{otherwise}, \end{aligned} \tag{2.22}$$

where a_{ij} are the elements of the matrix $\mathbf{A}$ in (2.9) and b_{ij} are the elements of $\mathbf{B}$ defined in (iii). Let R be the subset of R^n_I such that

$$R = \{z \in \mathrm{R}^n_I \mid z_i = 0, \quad \text{for all } i \in I, \quad z_i = z^*_i, \quad \text{for all } i \in \mathcal{N} - I \text{ s.t. } b_i(z) > 0\} \tag{2.23}$$

and let M be the largest invariant set within R. By using the scalar function suggested by Goh in studying sectorial stability

$$V(z) = \sum_{i \in \mathcal{N} - I} W_i \left[z_i - z^*_i - z^*_i \ln \frac{z_i}{z^*_i} \right] + \sum_{i \in I} W_i z_i, \quad W_i > 0, \tag{2.24}$$

we prove the following:

THEOREM 2.2

Let z^* be a partially feasible equilibrium of (2.9) and assume that $\tilde{\mathbf{A}}$ is $\mathbf{W}$-skew symmetrizable. If (a) $e_i + \Sigma_{j \in \mathcal{N}} a_{ij} z^*_j \leqslant 0$ for all $i \in I$, (b) $b_i(z) \equiv 0$ for all $i \in I$, (c) $M \equiv \{z^*\}$, then z^* is globally asymptotically stable with respect to R^n_I.

Proof. If z^* is a partially feasible equilibrium of (2.9) with $z^*_i = 0$ for all $i \in I$, by hypothesis (b) the equations of (2.9) read

$$\begin{aligned} \dot{z}_i &= z_i \sum_{j\in \mathcal{N}} \tilde{a}_{ij}(z_j - z_j^*) - \left(\frac{z_i - z_i^*}{z_i^*}\right) b_i(z), \quad \text{for all } i \in \mathcal{N} - I, \\ \dot{z}_i &= z_i \left(e_i + \sum_{j\in \mathcal{N}} a_{ij} z_j\right), \quad \text{for all } i \in I. \end{aligned} \tag{2.25}$$

Consider the function (2.24). $V(z) \in C^1(\mathrm{R}_I^n)$, and is a non-negative definite function with a single minimum at $z = z^*$ where $V(z^*) = 0$.

On account of (2.25), the time derivative of $V(z)$ along the trajectories of (2.9) is

$$\begin{aligned} \dot{V}(z) = &\sum_{i\in \mathcal{N}-I} W_i \frac{(z_i - z_i^*)}{z_i} \left\{ z_i \sum_{j\in \mathcal{N}} \tilde{a}_{ij}(z_j - z_j^*) - \frac{(z_i - z_i^*)}{z_i^*} b_i(z) \right\} \\ &+ \sum_{i\in I} W_i z_i \left(e_i + \sum_{j\in \mathcal{N}} a_{ij} z_j\right). \end{aligned} \tag{2.26}$$

By the definition of the matrix $\tilde{\mathbf{A}}$, from (2.22) we get

$$\begin{aligned} \dot{V}(z) = &(z - z^*)^T \mathbf{W}\tilde{\mathbf{A}}(z - z^*) - \sum_{i\in \mathcal{N}-I} \frac{W_i b_i(z)}{z_i z_i^*}(z_i - z_i^*)^2 \\ &+ \sum_{i\in I} W_i z_i \left(e_i + \sum_{j\in \mathcal{N}} a_{ij} z_j^*\right). \end{aligned} \tag{2.27}$$

Since $\tilde{\mathbf{A}}$ is $\mathbf{W}$-skew symmetrizable, the first term in (2.27) vanishes. By hypothesis (a) we have that within R_I^n, $\dot{V}(z) \leqslant 0$. Now we are in position to apply the already quoted extension of Lyapunov's stability theorem [10, Theorem VI Sec. 13].

The set R of all points within R_I^n where $\dot{V}(z) = 0$ is given by (2.23). Since, by hypothesis (c), the largest invariant set within R is z^*, then every solution $z(t)$ with initial conditions in R_I^n tends to z^* as $t \to +\infty$.

When z^* is feasible, $I = \emptyset$, $R = \{z \in {}^{n*}_{+} \mid z_i = z_i^* \text{ for all } i : b_i(z) > 0\}$. From Theorem 2.2 we obtain:

COROLLARY 2.2

Let z^* a feasible equilibrium of (2.9) and assume that $\tilde{\mathbf{A}}$ is $\mathbf{W}$-skew symmetrizable. If $M \equiv \{z^*\}$, then z^* is globally asymptotically stable within R^{n*}_{+}.

Corollary 2.2 can be seen as a new formulation of Theorem 2.1 case (v), for epidemic models for which assumption (iv) is dropped. Concerning Corollary 2.2, we may observe that, if the graph associated with $\tilde{\mathbf{A}}$ by the rules (vii), (viii), satisfies one of the hypotheses of Lemma 2.1, then within R we have $M \equiv \{z^*\}$.

3. EXAMPLES

In this section we review a list of epidemic systems which exhibit the structure of general model (2.9). Their asymptotic behaviour will be obtained then by applying the theorems proved in the previous section.

We shall distinguish three main cases: In the first one condition (v) is satisfied; in the second one condition (v′) is satisfied; finally in the third case we shall consider epidemic models for which the total population $n(t)$ is a dynamical variable.

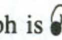
3.1 *Epidemic systems for which the matrix $\tilde{\mathbf{A}}$ is $\mathbf{W}$-skew symmetrizable*

In the following it is assumed that all the parameters are non-negative real numbers.

3.1.1 *SIR model with vital dynamics*[7]. This model has already been presented in (2.2).

Since in this case $B = 0$, we have

$$\tilde{\mathbf{A}} = \mathbf{A} = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \quad \text{and } \mathbf{b}(z) = \mathbf{c} = \begin{pmatrix} \mu \\ 0 \end{pmatrix}. \tag{3.1}$$

Since $\tilde{\mathbf{A}}$ is skew-symmetric and the associated graph is , Theorem 2.1 applies.

3.1.2 *SIRS model with temporary immunity*[7].

$$\begin{aligned} \mathrm{d}S/\mathrm{d}t &= -\lambda IS + (\delta + \alpha) - (\delta + \alpha)S - \alpha I, \\ \mathrm{d}I/\mathrm{d}t &= \lambda IS - (\gamma + \delta)I, \end{aligned} \tag{3.2}$$

with $S + I \leqslant 1$.

We change the variables (S, I) into $(\tilde{S}, I)$ such that $\tilde{S} = S + \alpha/\lambda$, so that system (3.2) becomes

$$\begin{aligned} \mathrm{d}\tilde{S}/\mathrm{d}t &= -(\delta + \alpha)\tilde{S} - \lambda\tilde{S}I + (\delta + \alpha)(1 + \alpha/\lambda), \\ \mathrm{d}I/\mathrm{d}t &= -(\gamma + \delta + \alpha)I + \lambda\tilde{S}I, \end{aligned} \tag{3.3}$$

which can now be put in the form (2.9) by introducing

$$\mathbf{A} = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}, \quad e = \begin{pmatrix} -(\delta + \alpha) \\ -(\gamma + \delta + \alpha) \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} (\delta + \alpha)(1 + \alpha/\lambda) \\ 0 \end{pmatrix}, \quad \mathbf{B} = 0. \tag{3.4}$$

Again $\tilde{\mathbf{A}} = \mathbf{A}$ and $\mathbf{b}(z) = \mathbf{c}$, with $\tilde{\mathbf{A}}$ skew-symmetric. Also in this case the associated graph is , and Theorem 2.1 applies.

3.1.3 *SIR model with carriers*[7].

$$\begin{aligned} \mathrm{d}S/\mathrm{d}t &= -\lambda(I + C)S + \delta - \delta S, \\ \mathrm{d}I/\mathrm{d}t &= \lambda(I + C)S - \gamma I - \delta I, \end{aligned} \tag{3.5}$$

where $S + I \leqslant 1$. By the change of variables $(S, I) \to (S, \bar{I})$, $\bar{I} = I + C$, we have

$$\begin{aligned} \mathrm{d}S/\mathrm{d}t &= -\delta S - \lambda\bar{I}S + \delta, \\ \mathrm{d}\bar{I}/\mathrm{d}t &= -(\gamma + \delta)\bar{I} + \lambda\bar{I}S + (\gamma + \delta)C, \end{aligned} \tag{3.6}$$

which can be put in the form (2.9), where

$$e = \begin{pmatrix} -\delta \\ -(\gamma + \delta) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} \delta \\ (\gamma + \delta)C \end{pmatrix}, \quad \mathbf{B} = 0, \tag{3.7}$$

and

$$\mathbf{b}(z) \equiv \mathbf{c}, \quad \tilde{\mathbf{A}} \equiv \mathbf{A}. \tag{3.8}$$

Since $\mathbf{c}$ is positive definite and $\tilde{\mathbf{A}}$ is skew-symmetric, by Corollary 2.1 a unique positive equilibrium z^* exists, which is globally asymptotically stable with respect to Ω^2_+, i.e. the interior of $\Omega^2 = \{z : S + \bar{I} \leqslant 1 + C\}$.

3.2 *Epidemic models for which* $-[\tilde{\mathbf{A}} + diag(-b_1(z)/z_1z_1^*, \ldots, -b_n(z)/z_nz_n^*)] \in S_W$

3.2.1 *SIS model in two communities with migration*[7].

$$\begin{aligned} \mathrm{d}I_1/\mathrm{d}t &= \lambda_1 I_1(1 - I_1) - \gamma_1 I_1 - \delta_1 I_1 + \theta(I_2 - I_1)/\bar{N}_1, \quad I_1 + S_1 = 1, \\ \mathrm{d}I_2/\mathrm{d}t &= \lambda_2 I_2(1 - I_2) - \gamma_2 I_2 - \delta_2 I_2 + \theta(I_1 - I_2)/\bar{N}_2, \quad I_2 + S_2 = 1. \end{aligned} \tag{3.9}$$

In (3.9), putting $\theta_1 = \theta/\overline{N}_1$, $\theta_2 = \theta/\overline{N}_2$, we obtain

$$\begin{aligned} dI_1/dt &= (\lambda_1 - \gamma_1 - \delta_1 - \theta_1)I_1 - \lambda_1 I_1^2 + \theta_1 I_2, \\ dI_2/dt &= (\lambda_2 - \gamma_2 - \delta_2 - \theta_2)I_2 - \lambda_2 I_1^2 + \theta_2 I_1, \end{aligned} \tag{3.10}$$

with $I_i \leq 1$, $i = 1, 2$. System (3.10) can be set in the form (2.9), where

$$e = \begin{pmatrix} \lambda_1 - \gamma_1 - \delta_1 - \theta_1 \\ \lambda_2 - \gamma_2 - \delta_2 - \theta_2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \quad \mathbf{c} = 0, \quad \mathbf{B} = \begin{pmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{pmatrix} \tag{3.11}$$

and

$$\mathbf{b}(z) \equiv \mathbf{B}z = (\theta_1 I_2, \theta_2 I_1)^T, \quad \bar{\mathbf{A}} = \begin{pmatrix} -\lambda_1 & \dfrac{\theta_1}{I_1^*} \\ \dfrac{\theta_2}{I_2^*} & -\lambda_2 \end{pmatrix}. \tag{3.12}$$

Let Ω^2 be the set $\Omega^2 = \{z \in \mathbb{R}_+^{2*} \mid I_i \leq 1, i = 1, 2\}$. By our approach, the sufficient condition for asymptotic stability of a positive equilibrium z^*, with respect to Ω_+^2 is $-[\bar{\mathbf{A}} + \operatorname{diag}(-\theta_1 I_2/I_1^* I_1, -\theta_2 I_1/I_2^* I_2)] \in S_W$. We can observe that

$$\mathbf{W}\bar{\mathbf{A}} + \operatorname{diag}\left(-\frac{\theta_1 I_2}{I_1^* I_1} W_1, -\frac{\theta_2 I_2}{I_2^* I_2} W_2\right) = \begin{pmatrix} -\dfrac{\theta_1 I_2}{I_1^* I_1} W_1 & \dfrac{\theta_1}{I_1^*} W_1 \\ \dfrac{\theta_2}{I_2^*} W_2 & -\dfrac{\theta_2 I_1}{I_2^* I_2} W_2 \end{pmatrix} + \operatorname{diag}(-\lambda_1 W_1, -\lambda_2 W_2). \tag{3.13}$$

The first matrix on the right-hand side of (3.13) is symmetric if we choose $W_1 > 0$, $W_2 = (\theta_1/\theta_2)(I_2^*/I_1^*)W_1$. This matrix is negative semidefinite since

$$\left(\frac{\theta_1 I_2}{I_1^* I_1}\frac{\theta_2 I_1}{I_2^* I_2} - \frac{\theta_1}{I_1^*}\frac{\theta_2}{I_2^*}\right) W_1 W_2 = 0. \tag{3.14}$$

Because of the presence of the diagonal negative matrix on the right-hand side of (3.13), the sufficient condition of Theorem 2.1 holds true provided that $\lambda_1 > 0$, $\lambda_2 > 0$. Under this assumption, if a positive equilibrium z^* exists, then it is globally asymptotically stable within Ω_+^2.

3.2.2 *SIS model for two dissimilar groups*[7].

$$\begin{aligned} dI_1/dt &= [\lambda_{11}I_1 + \lambda_{12}(\overline{N}_2/\overline{N}_1)I_2](1 - I_1) - \gamma_1 I_1 - \delta_1 I_1, \quad I_1 + S_1 = 1, \\ dI_2/dt &= [\lambda_{22}I_2 + \lambda_{21}(\overline{N}_1/\overline{N}_2)I_1](1 - I_2) - \gamma_2 I_2 - \delta_2 I_2, \quad I_2 + S_2 = 1. \end{aligned} \tag{3.15}$$

In (3.15), by setting $\beta_{12} = \lambda_{12}(\overline{N}_2/\overline{N}_1)$, $\beta_{21} = \lambda_{21}(\overline{N}_1/\overline{N}_2)$, we get

$$\begin{aligned} dI_1/dt &= (\lambda_{11} - \gamma_1 - \delta_1)I_1 - \lambda_{11}I_1^2 - \beta_{12}I_1I_2 + \beta_{12}I_2, \\ dI_2/dt &= (\lambda_{22} - \gamma_2 - \delta_2)I_2 - \lambda_{22}I_2^2 - \beta_{21}I_2I_1 + \beta_{21}I_1, \end{aligned} \tag{3.16}$$

with $I_i \leq 1$, $i = 1, 2$. System (3.16) can be set in the form (2.9), where

$$e = \begin{pmatrix} \lambda_{11} - \gamma_1 - \delta_1 \\ \lambda_{22} - \gamma_2 - \delta_2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -\lambda_{11} & -\beta_{12} \\ -\beta_{21} & -\lambda_{22} \end{pmatrix}, \quad \mathbf{c} = 0, \quad \mathbf{B} = \begin{pmatrix} 0 & \beta_{12} \\ \beta_{21} & 0 \end{pmatrix}, \tag{3.17}$$

and $z = (I_1, I_2)^T$,

$$\mathbf{b}(z) \equiv \mathbf{B}z = (\beta_{12}I_2, \beta_{21}I_1)^T, \quad \tilde{\mathbf{A}} = \begin{pmatrix} -\lambda_{11} & \dfrac{\beta_{12}}{I_1^*}(1 - I_1^*) \\ \dfrac{\beta_{21}}{I_2^*}(1 - I_2^*) & -\lambda_{22} \end{pmatrix}. \tag{3.18}$$

Consider now $\mathbf{W}[\tilde{\mathbf{A}} + \mathrm{diag}(-\beta_{12}I_2/I_1^*I_1, -\beta_{21}I_1/I_2^*I_1)]$:

$$\mathbf{W}\left[\tilde{\mathbf{A}} + \mathrm{diag}\left(-\frac{\beta_{12}I_2}{I_1^*I_1}, -\frac{\beta_{21}I_1}{I_2^*I_2}\right)\right] = \begin{pmatrix} -\dfrac{\beta_{12}I_2}{I_1^*I_1}W_1 & \dfrac{\beta_{12}}{I_1^*}(1 - I_1^*)W_1 \\ \dfrac{\beta_{21}}{I_2^*}(1 - I_2^*)W_2 & -\dfrac{\beta_{21}I_1}{I_2^*I_2}W_2 \end{pmatrix} + \mathrm{diag}(-\lambda_{11}W_1, -\lambda_{22}W_2), \tag{3.19}$$

where the first matrix on the right of (3.19) is symmetric when choosing $\mathbf{W} > 0$ and W_2 such that $(\beta_{21}/I_2^*)(1 - I_2^*)W_2 = (\beta_{12}/I_1^*)(1 - I_1^*)W_1$. Moreover, since $0 < I_i^* < 1$, $i = 1, 2$, this matrix is negative definite. In fact,

$$\left(\frac{\beta_{12}I_2}{I_1^*I_1} \cdot \frac{\beta_{21}I_1}{I_2^*I_2} - \frac{\beta_{12}}{I_1^*}(1 - I_1^*)\frac{\beta_{21}}{I_2^*}(1 - I_2^*)\right)W_1W_2 > 0. \tag{3.20}$$

Hence, provided that $\lambda_{11} \geqslant 0$, $\lambda_{22} \geqslant 0$, $-[\tilde{\mathbf{A}} + \mathrm{diag}(-\beta_{12}I_2/I_1^*I_1, -\beta_{21}I_1/I_2^*I_2)] \in S_W$, and Theorem 2.1 assures the asymptotic stability of the positive equilibrium z^* with respect to Ω_+^2, $\Omega^2 = \{z \in \mathbb{R}_+^2 \mid I_i \leqslant 1, i = 1, 2\}$.

3.2.3 *Gonorrhea model*[3,15].

$$\begin{aligned} dI_1/dt &= -I_1I_2 - \alpha_1I_1 + c_1I_2, \quad I_1 + S_1 = c_1, \\ dI_2/dt &= -I_2I_1 - \alpha_2I_2 + c_2I_1, \quad I_2 + S_2 = c_2, \end{aligned} \tag{3.21}$$

which can be put in the form (2.9), where

$$e = \begin{pmatrix} -\alpha_1 \\ -\alpha_2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{c} = 0, \quad \mathbf{B} = \begin{pmatrix} 0 & c_1 \\ c_2 & 0 \end{pmatrix}, \tag{3.22}$$

and

$$\mathbf{b}(z) \equiv \mathbf{B}z = (c_1I_2, c_2I_1)^T, \quad \tilde{\mathbf{A}} = \begin{pmatrix} 0 & \dfrac{c_1 - I_1^*}{I_1^*} \\ \dfrac{c_2 - I_2^*}{I_2^*} & 0 \end{pmatrix}. \tag{3.23}$$

Consider

$$\mathbf{W}\left[\tilde{\mathbf{A}} + \mathrm{diag}\left(-\frac{c_1I_2}{I_1^*I_1}, -\frac{c_2I_1}{I_2^*I_2}\right)\right] = \begin{pmatrix} -\dfrac{c_1I_2}{I_1^*I_1}W_1 & W_1\dfrac{S_1^*}{I_1^*} \\ W_2\dfrac{S_2^*}{I_2^*} & -\dfrac{c_2I_1}{I_2^*I_2}W_2 \end{pmatrix}, \tag{3.24}$$

which is a symmetric matrix if we choose $W_1 > 0$, W_2 such that $W_2(S_2^*/I_2^*) = W_1(S_1^*/I_1^*)$. The symmetric matrix (3.24) is negative definite.

In fact, the diagonal elements are negative and

$$\left(\frac{c_1 I_2}{I_1^* I_1}\frac{c_2 I_1}{I_1^* I_2} - \frac{S_1^* S_2^*}{I_1^* I_2^*}\right) W_1 W_2 = \frac{W_1 W_2}{I_1^* I_2^*}(c_1 c_2 - S_1^* S_2^*) > 0, \tag{3.25}$$

where the fact that $0 < S_i^* < c_i$, is taken into account since z^* is a positive equilibrium.

3.2.4 *SIS model with vectors*[7]. This model is obtained from "SIS model for two dissimilar groups" when $\lambda_{11} = \lambda_{22} = 0$:

$$\begin{aligned} dI_1/dt &= -(\gamma_1 + \delta_1)I_1 - \beta_{12}I_1I_2 + \beta_{12}I_2, \\ dI_2/dt &= -(\gamma_2 + \delta_2)I_2 - \beta_{21}I_2I_1 + \beta_{21}I_1, \end{aligned} \tag{3.26}$$

where $I_i \leq 1$, $i = 1, 2$. The asymptotic stability of the positive equilibrium z^*, with respect to Ω^2_+, follows as a particular case of the SIS model for two dissimilar groups (Sec. 3.2.2.).

3.2.5 *Host-vector-host model*[7].

$$\begin{aligned} dI_1/dt &= \lambda_{12}(\overline{N}_2/\overline{N}_1)I_2(1 - I_1) - (\gamma_1 + \delta_1)I_1, \quad S_1 + I_1 = 1, \\ dI_2/dt &= [\lambda_{21}(\overline{N}_1/\overline{N}_2)I_1 + \lambda_{23}(\overline{N}_3/\overline{N}_2)I_3](1 - I_2) - (\gamma_2 + \delta_2)I_2, \quad S_2 + I_2 = 1, \\ dI_3/dt &= \lambda_{32}(\overline{N}_2/\overline{N}_3)I_2(1 - I_3) - (\gamma_3 + \delta_3)I_3, \quad S_3 + I_3 = 1. \end{aligned} \tag{3.27}$$

If we introduce the new parameters $\beta_{12} = \lambda_{12}(\overline{N}_2/\overline{N}_1)$, $\beta_{21} = \lambda_{21}(\overline{N}_1/\overline{N}_2)$, $\beta_{23} = \lambda_{23}(\overline{N}_3/\overline{N}_2)$, $\beta_{32} = \lambda_{32}(\overline{N}_2/\overline{N}_3)$, and the new variables $S_i = 1 - I_i$, $i = 1, 2, 3$, system (3.27) reads

$$\begin{aligned} dS_1/dt &= [-\beta_{12} - (\gamma_1 + \delta_1)]S_1 + \beta_{12}S_1S_2 + (\gamma_1 + \delta_1), \\ dS_2/dt &= [-\beta_{21} - \beta_{23} - (\gamma_2 + \delta_2)]S_2 + \beta_{21}S_2S_1 + \beta_{23}S_2S_3 + (\gamma_2 + \delta_2), \\ dS_3/dt &= [-\beta_{32} - (\gamma_3 + \delta_3)]S_3 + \beta_{32}S_3S_2 + (\gamma_3 + \delta_3). \end{aligned} \tag{3.28}$$

Let Ω^3 be the set $\Omega^3 = \{z \in \mathbb{R}^3_+ \mid S_i \leq 1,\ i = 1, 2, 3\}$. System (3.28) can be put in the form (2.9), where

$$e = \begin{pmatrix} -\beta_{12} - (\gamma_1 + \delta_1) \\ -\beta_{21} - \beta_{23} - (\gamma_2 + \delta_2) \\ -\beta_{32} - (\gamma_3 + \delta_3) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & \beta_{12} & 0 \\ \beta_{21} & 0 & \beta_{23} \\ 0 & \beta_{32} & 0 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} \gamma_1 + \delta_1 \\ \gamma_2 + \delta_2 \\ \gamma_3 + \delta_3 \end{pmatrix}, \quad \mathbf{B} = 0 \tag{3.29}$$

and

$$\mathbf{b}(z) \equiv \mathbf{c}, \quad \tilde{\mathbf{A}} \equiv \mathbf{A}. \tag{3.30}$$

By Corollary 2.1, since $\mathbf{c}$ is a positive definite vector, one positive equilibrium $z^* \in \Omega^3_+$ exists. $\tilde{\mathbf{A}}$ has a symmetric sign structure. Hence, by Corollary 2.1, if $-[\mathbf{A} + \operatorname{diag}(-(\gamma_1 + \delta_1)/S_1S_1^*, -(\gamma_2 + \delta_2)/S_2S_2^*, -(\gamma_3 + \delta_3)/S_3S_3^*)] \in S_W$, then z^* is asymptotically stable within Ω^3_+. If we take into account that $S_i \leq 1$, $i = 1, 2, 3$ from (2.19) we see that a sufficient condition for the asymptotic stability of z^* is $-[\mathbf{A} + \operatorname{diag}(-(\gamma_1 + \delta_1), -(\gamma_2 + \delta_2), -(\gamma_3 + \delta_3))] \in S_W$. Accordingly, let us take

$$\mathbf{W}[\mathbf{A} + \operatorname{diag}(-(\gamma_1 + \delta_1), -(\gamma_2 + \delta_2), -(\gamma_3 + \delta_3))] = \begin{pmatrix} -(\gamma_1 + \delta_1)W_1 & \beta_{12}W_1 & 0 \\ \beta_{21}W_2 & -(\gamma_2 + \delta_2)W_2 & \beta_{23}W_2 \\ 0 & \beta_{32}W_3 & -(\gamma_3 + \delta_3)W_3 \end{pmatrix}. \tag{3.31}$$

This matrix is symmetric if we choose $W_1 > 0$, $W_2 = (\beta_{12}/\beta_{21})W_1$, $W_3 = (\beta_{23}/\beta_{32})(\beta_{12}/\beta_{21})W_1$. It is negative definite if

$$[(\gamma_1 + \delta_1)(\gamma_2 + \delta_2) - \beta_{12}\beta_{21}]W_1W_2 > 0,$$
$$-[(\gamma_1 + \delta_1)(\gamma_2 + \delta_2)(\gamma_3 + \delta_3) - (\gamma_3 + \delta_3)\beta_{12}\beta_{21} + - (\gamma_1 + \delta_1)\beta_{23}\beta_{32}]W_1W_2W_3 < 0. \quad (3.32)$$

We can observe that, if inequalities in (3.32) hold true, then

$$[(\gamma_2 + \delta_2)(\gamma_3 + \delta_3) - \beta_{23}\beta_{32}]W_2W_3 > 0. \quad (3.33)$$

Hence (3.32) is the sufficient condition for the asymptotic stability (and uniqueness) of the positive equilibrium z^* within Ω^3_+.

From (3.28), the positive equilibrium z^* has the following components:

$$S_1^* = \frac{\gamma_1 + \delta_1}{\beta_{12}(1 - S_2^*) + (\gamma_1 + \delta_1)}, \quad S_3^* = \frac{\gamma_3 + \delta_3}{\beta_{32}(1 - S_2^*) + (\gamma_3 + \delta_3)}, \quad (3.34)$$

where S_2^* is a solution of

$$(1 - S_2)\{p(1 - S_2)^2 + q(1 - S_2) + r\} = 0, \quad (3.35)$$

and

$$\begin{aligned} p &= \beta_{12}\beta_{32}[(\beta_{21} + \beta_{23}) + (\gamma_2 + \delta_2)], \\ q &= \beta_{32}[(\gamma_1 + \delta_1)(\gamma_2 + \delta_2) - \beta_{12}\beta_{21}] + \beta_{12}[(\gamma_2 + \delta_2)(\gamma_3 + \delta_3) - \beta_{23}\beta_{32} \\ &\quad + \beta_{12}\beta_{21}(\gamma_3 + \delta_3) + \beta_{23}\beta_{32}(\gamma_1 + \delta_1), \\ r &= (\gamma_1 + \delta_1)(\gamma_2 + \delta_2)(\gamma_3 + \delta_3) - (\gamma_3 + \delta_3)\beta_{12}\beta_{21} - (\gamma_1 + \delta_1)\beta_{23}\beta_{32}. \end{aligned} \quad (3.36)$$

It is to be noticed that when (3.32) holds true, then $q > 0$, $r > 0$, thus assuring that the unique asymptotically stable equilibrium is such that $S_2^* = 1$, i.e. $z^* = (1, 1, 1)^T$. When (3.32) fails to hold, by (3.35) we have another positive equilibrium for which $S_2^* < 1$ and its remaining components are given by (3.34).

To study the asymptotic stability of this equilibrium we can remember that $I_i + S_i = 1$, $i = 1, 2, 3$, thus assuring to have a positive equilibrium $z^* = (I_1^*, I_2^*, I_3^*)^T$, $0 < I_i^* < 1$, $i = 1, 2, 3$ within the subset $\tilde{\Omega}^3 = \{z \in \mathbb{R}^3_+ : I_i \leqslant 0, i = 1, 2, 3\}$. In the old variables I_i, $i = 1, 2, 3$ the positive equilibrium $(1, 1, 1)^T$ becomes the origin and the ODE system (3.27) can be arranged in this form:

$$\begin{aligned} \mathrm{d}I_1/\mathrm{d}t &= -(\gamma_1 + \delta_1)I_1 - \beta_{12}I_1I_2 + \beta_{12}I_2, \\ \mathrm{d}I_2/\mathrm{d}t &= -(\gamma_2 + \delta_2)I_2 - \beta_{21}I_2I_1 - \beta_{23}I_2I_3 + (\beta_{21}I_1 + \beta_{23}I_3), \\ \mathrm{d}I_3/\mathrm{d}t &= -(\gamma_3 + \delta_3)I_3 - \beta_{32}I_2I_3 + \beta_{32}I_2, \end{aligned} \quad (3.37)$$

where, concerning (2.9), we have

$$e = \begin{pmatrix} -(\gamma_1 + \delta_1) \\ -(\gamma_2 + \delta_2) \\ -(\gamma_3 + \delta_3) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & -\beta_{12} & 0 \\ -\beta_{21} & 0 & -\beta_{23} \\ 0 & -\beta_{32} & 0 \end{pmatrix},$$
$$\mathbf{c} = 0, \quad \mathbf{B} = \begin{pmatrix} 0 & \beta_{12} & 0 \\ \beta_{21} & 0 & \beta_{23} \\ 0 & \beta_{32} & 0 \end{pmatrix}. \quad (3.38)$$

Thus

$$\mathbf{b}(z) \equiv \mathbf{B}z, \quad \bar{\mathbf{A}} = \begin{pmatrix} 0 & \frac{\beta_{12}S_1^*}{I_1^*} & 0 \\ \frac{\beta_{21}S_2^*}{I_2^*} & 0 & \frac{\beta_{23}S_2^*}{I_2^*} \\ 0 & \frac{\beta_{32}S_3^*}{I_3^*} & 0 \end{pmatrix} \tag{3.39}$$

For the asymptotic stability of $z^* = (I_1^*, I_2^*, I_3^*)^T$ within $\bar{\Omega}_+^3$ we can apply Theorem 2.1 by requiring that $-[\bar{\mathbf{A}} + \operatorname{diag}(-b_1(z)/I_1I_1^*, -b_2(z)/I_2I_2^*, -b_3(z)/I_3I_3^*)] \in S_W$.

Hence consider

$$\mathbf{W}\left[\bar{\mathbf{A}} + \operatorname{diag}\left(-\frac{b_1(z)}{I_1I_1^*}, \frac{b_2(z)}{I_2I_2^*}, \frac{b_3(z)}{I_3I_3^*}\right)\right]$$

$$= \begin{pmatrix} -\frac{\beta_{12}I_2}{I_1I_1^*}W_1 & \frac{\beta_{12}S_1^*}{I_1^*}W_1 & 0 \\ \frac{\beta_{21}S_2^*}{I_2^*}W_2 & -\frac{(\beta_{21}I_1 + \beta_{23}I_3)}{I_2I_2^*}W_2 & \frac{\beta_{23}S_2^*}{I_2^*}W_2 \\ 0 & \frac{\beta_{32}S_3^*}{I_3^*}W_3 & -\frac{\beta_{32}I_2}{I_3I_3^*}W_3 \end{pmatrix}; \tag{3.40}$$

this matrix is symmetric if we choose $W_1 > 0$, $W_2 = (\beta_{12}S_1^*/\beta_{21}S_2^*)(I_2^*/I_1^*)W_1$, $W_3 = (\beta_{23}S_2^*/\beta_{32}S_3^*)(I_3^*/I_2^*)W_2$.

To apply Theorem 2.1, we must require that the symmetric matrix (3.40) be negative definite. Since the diagonal elements are negative, the sufficient condition is

$$\left[\frac{\beta_{12}I_2}{I_1}\frac{(\beta_{21}I_1 + \beta_{23}I_3)}{I_2} - \beta_{12}S_1^*\beta_{21}S_2^*\right]\frac{W_1W_2}{I_1^*I_2^*} > 0,$$

$$\left[-\frac{\beta_{12}I_2}{I_1}\frac{(\beta_{21}I_1 + \beta_{23}I_3)}{I_2}\frac{\beta_{32}I_2}{I_3} + \frac{\beta_{32}I_2}{I_3}\beta_{12}S_1^*\beta_{21}S_2^* + \frac{\beta_{12}I_2}{I_1}\beta_{23}S_2^*\beta_{32}S_3^*\right]\frac{W_1W_2W_3}{I_1^*I_2^*I_3^*} < 0. \tag{3.41}$$

Now we observe that the sufficient condition (3.41) is always met by a positive equilibrium $z^* \in \bar{\Omega}_+^3$. In fact,

$$\frac{\beta_{12}I_2}{I_1}\frac{(\beta_{21}I_1 + \beta_{23}I_3)}{I_2} - \beta_{12}S_1^*\beta_{21}S_2^* > \frac{\beta_{12}I_2}{I_1}\cdot\frac{\beta_{21}I_1}{I_2} - \beta_{12}\beta_{21} = 0$$

and

$$\frac{-\beta_{12}I_2}{I_1}\frac{\beta_{21}I_1}{I_2}\frac{\beta_{32}I_2}{I_3} + \frac{\beta_{32}I_2}{I_3}\beta_{12}S_1^*\beta_{21}S_2^* - \frac{\beta_{12}I_2}{I_1}\frac{\beta_{23}I_3}{I_2}\frac{\beta_{32}I_2}{I_3} + \frac{\beta_{12}I_2}{I_1}\beta_{23}S_2^*\beta_{32}S_3^*$$

$$= \frac{I_2}{I_3}\beta_{32}(-\beta_{12}\beta_{21} + \beta_{12}S_1^*\beta_{21}S_2^*) + \frac{\beta_{12}I_2}{I_1}(-\beta_{23}\beta_{32} + \beta_{23}S_2^*\beta_{32}S_3^*) < 0,$$

where, proving the inequalities, we have taken into account that $S_i^* < 1$, $i = 1, 2, 3$. Hence we can conclude for the host-vector-host model that

COROLLARY 3.1

If the sufficient condition (3.32) holds true, then the origin is asymptotically stable with respect to $\bar{\Omega}^3$. Otherwise besides the origin a positive equilibrium $N^* \in \bar{\Omega}^3_+$ exists which is asymptotically stable within Ω^3_+.

4. EPIDEMIC MODELS WITH NONCONSTANT TOTAL POPULATION

In this section we deal with some epidemic models, described by the ODE system (2.9), which differ from the usual epidemic models presented in Section 3 in that $n(t) = \Sigma_{i=1}^n z_i(t)$ is a dynamical variable, rather than a specified constant. Furthermore, these models admit either a feasible or a partially feasible equilibrium (see Sec. 2). We shall consider two specific examples.

4.1 *Parasite-host system*[11]

The epidemic model is

$$\begin{aligned} dx/dt &= (r - k)x - Cxy - Cxv + ry + rv, \\ dy/dt &= -(\beta + k)y + Cxy - CSyv, \\ dv/dt &= -(\beta + k + \sigma)v + Cxv + CSyv. \end{aligned} \tag{4.1}$$

As discussed in [11] the two cases $r < k$ and $r > \beta + k + \sigma$ do not give rise to nontrivial equilibrium solutions. We shall then restrict our analysis to the case $\beta + \sigma + k > r > k$ in which there is an equilibrium at

$$x^* = \frac{r}{C}\frac{\sigma}{\sigma - S(r - k)}, \quad y^* = \frac{\beta + k + \sigma}{CS} - \frac{1}{S}x^*, \quad v^* = \frac{1}{S}x^* - \frac{\beta + k}{CS}. \tag{4.2}$$

Local stability results were already given in [11]. According to the aim of this paper, we shall study global asymptotic stability of the feasible or partially feasible equilibrium.

The equilibrium $z^* = (x^*, y^*, v^*)^T$ is feasible, i.e. its components are positive if

$$\frac{r}{\beta + k + \sigma} < 1 - \frac{S(r - k)}{\sigma} < \frac{r}{\beta + k}. \tag{4.3}$$

If $\sigma < \sigma_1$, where σ_1 is such that

$$\frac{r}{\beta + k + \sigma_1} = 1 - \frac{S(r - k)}{\sigma_1}, \tag{4.4}$$

the first inequality in (4.3) is violated and only a partially feasible equilibrium is present given by

$$x^* = \frac{\beta + k + \sigma}{C}, \quad y^* = 0, \quad v^* = \frac{r - k}{\beta + k + \sigma - r}x^*, \tag{4.5}$$

since $r < \beta + k + \sigma$. If $\sigma = \sigma_1$, then (4.2) coalesces in (4.5).

If $r < \beta + k$ and $\sigma > \sigma_2$, where σ_2 is such that

$$1 - \frac{S(r - k)}{\sigma_2} = \frac{r}{\beta + k}, \tag{4.6}$$

then the second inequality in (4.3) is violated and only a partially feasible equilibrium is present, given by

$$x^* = \frac{\beta + k}{C}, \quad y^* = \frac{r - k}{\beta + k - r}x^*, \quad v^* = 0, \tag{4.7}$$

since $r > k$. If $\sigma = \sigma_2$, then (4.2) coalesces in (4.7).

Concerning model (4.1), we can put it in the matrix form (2.9), where

$$e = \begin{pmatrix} r - k \\ -(\beta + k) \\ -(\beta + k + \sigma) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & -C & -C \\ C & 0 & -CS \\ C & CS & 0 \end{pmatrix}, \quad \mathbf{c} = 0, \quad \mathbf{B} = \begin{pmatrix} 0 & r & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.8}$$

Now consider the case in which the equilibrium (4.2) is feasible, i.e. $z^* \in \mathrm{R}^3_+$. Then

$$\mathbf{b}(z) \equiv \mathbf{B}z, \quad \tilde{\mathbf{A}} = \begin{pmatrix} 0 & -\left(C - \dfrac{r}{x^*}\right) & -\left(C - \dfrac{r}{x^*}\right) \\ C & 0 & -CS \\ C & CS & 0 \end{pmatrix}, \tag{4.9}$$

where z is a vector $z = (x, y, v)^T$ belonging to the non-negative orthant R^3_+. Since $C - r/x^* = CS(r - k)/\sigma$, provided that $r > k$, matrix $\tilde{\mathbf{A}}$ is $\mathbf{W}$-skew symmetrizable by the diagonal positive matrix $\mathbf{W} = \mathrm{diag}(W_1, W_2, W_3)$, where $W_1 = \sigma/S(r - k)$, $W_2 = W_3 = 1$. In fact, we obtain

$$\mathbf{W}\tilde{\mathbf{A}} = \begin{pmatrix} 0 & -C & -C \\ C & 0 & -CS \\ C & CS & 0 \end{pmatrix}. \tag{4.10}$$

Now we are in position to apply Corollary 2.2. Since $b(z) = (r(y + z), 0, 0)^T$, the subset of all points within R^{n*}_+ where we have $\dot{V}(z) = 0$, is

$$R = \{z \in \mathrm{R}^n_+ \mid x = x^*\}. \tag{4.11}$$

Now we look for the largest invariant subset M within R. Since $x = x^*$ for all t, $\mathrm{d}x/\mathrm{d}t|_R = 0$, and from the first of the Eqns. (4.1) we obtain

$$(y + v)|_R = \frac{r - k}{C - r/x^*} = \frac{\sigma}{CS}, \quad \text{for all } t. \tag{4.12}$$

Therefore, $\mathrm{d}(y + v)/\mathrm{d}t|_R = 0$, and by the last two Eqns. (4.1) we obtain

$$z|_R = \frac{1}{\sigma}\{[Cx^* - (\beta + k)][(y + v)]_R\} = \frac{1}{CS}[Cx^* - (\beta + k)] = \frac{x^*}{S} - \frac{\beta + k}{CS}. \tag{4.13}$$

Then, taking into account (4.2), we have $z|_R \equiv z^*$. Immediately follows

$$y|_R = \frac{\sigma}{CS} - v^* = \frac{\beta + k + \sigma}{CS} - \frac{x^*}{S}, \tag{4.14}$$

i.e. $y|_R = y^*$. Then the largest invariant set M within R is z^*. From Corollary 2.2 follows the global asymptotic stability of the feasible equilibrium (4.2) within R^{3*}_+.

It is to be noticed that the only assumptions made in this proof are $r > k$ and that equilibrium (4.2) is feasible. Under these assumptions we exclude that unbounded solutions may exist.

Suppose that $\sigma \leqslant \sigma_1$, i.e. the equilibrium (4.2) is not feasible and we get the partially feasible equilibrium (4.5) which belongs to

$$R^3_2 = \{z \in \mathrm{R}^3 \mid z_i > 0, i = 1, 3, z_i \geqslant 0, i = 2\}. \tag{4.15}$$

In order to apply Theorem 2.2 hypotheses (a) and (b) must be verified. Concerning hypothesis (a), we have

$$-(\beta + k) + cx^* - cSv^* \leqslant 0, \tag{4.16}$$

from which, taking into account (4.5), we obtain

$$1 - \frac{S(r-k)}{\sigma} \leqslant \frac{r}{\beta + k + \sigma}. \tag{4.17}$$

Inequality (4.17) is satisfied in the whole range $\sigma \leqslant \sigma_1$, within which the partially feasible equilibrium (4.5) occurs. When $\sigma = \sigma_1$, the equality applies in (4.17). Hypothesis (b) is satisfied because $\mathbf{b}(z) = (r(y+v), 0, 0)^T$ and therefore $\mathbf{b}_2(z) \equiv 0$. Concerning hypothesis (c), consider first the case $\sigma < \sigma_1$, i.e. the inequality applies in (4.16). Then the subset (2.23) is

$$R = \{z \in \mathrm{R}_2^3 \mid y = 0, \quad x = x^*\}. \tag{4.18}$$

Now we look for the largest invariant subset M within R.

Since $x = x^*$, $y = 0$ for all t, $\mathrm{d}x/\mathrm{d}t|_R = 0$, and from the first of equations (4.1) we get

$$v|_R = \frac{r-k}{C - r/x^*}, \quad \text{where } x^* = \frac{\beta + k + \sigma}{C}. \tag{4.19}$$

Therefore, we obtain $v|_R = [(r-k)/(\beta + k + \sigma - r)] \cdot x^*$, i.e. $v|_R \equiv v^*$. Thus the largest invariant set within R is

$$z^* = \left(x^* = \frac{\beta + k + \sigma}{C}, \quad y^* = 0, \quad v^* = \frac{r-k}{\beta + k + \sigma - r}\, x^*\right)^T. \tag{4.20}$$

When $\sigma = \sigma_1$, then equality applies in (4.17) and (2.23) becomes

$$R = \{z \in \mathrm{R}_2^3 \mid x = x^*\}.$$

In this case, we have already proven that $M \equiv \{z^*\}$. Hence hypothesis (c) is satisfied. Then by Theorem 2.2 the partially feasible equilibrium (4.5) is globally asymptotically stable with respect to R_2^3.

If $r < \beta + k$ and $\sigma \geqslant \sigma_2$, then the partially feasible equilibrium (4.7) occurs. This equilibrium belongs to

$$\mathrm{R}_3^3 = \{z \in \mathrm{R}_+^3 \mid z_i > 0, \quad i = 1, 2;\ z_i \geqslant 0, \quad i = 3\}. \tag{4.21}$$

Hypothesis (a) of Theorem 2.2 requires

$$-(\beta + k + \sigma) + Cx^* + CSy^* \leqslant 0, \tag{4.22}$$

from which, taking into account (4.7), we obtain

$$1 - \frac{S(r-k)}{\sigma} \geqslant \frac{r}{\beta + k}. \tag{4.23}$$

This inequality is satisfied in the whole range of existence of the equilibrium (4.7), i.e. for all $\sigma \geqslant \sigma_2$. When $\sigma = \sigma_2$, the equality applies in (4.23). Hypothesis (b) of Theorem 2.2 is obviously satisfied. Concerning hypothesis (c), at first we consider the case in which $\sigma > \sigma_2$. Therefore, the inequality applies in (4.22) and the subset (2.23) of R_3^3 is

$$R = \{z \in \mathrm{R}_3^3 \mid v = 0, \quad x = x^*\}. \tag{4.24}$$

From (4.7), we are ready to prove that $M \equiv \{z^*\}$. When $\sigma = \sigma_2$, R becomes

$$R = \{z \in \mathrm{R}_3^3 \mid x = x^*\},$$

and we have already proven that $M \equiv \{z^*\}$. Hypothesis (c) is satisfied. Also, in this case Theorem 2.2 assures the global asymptotic stability of the partially feasible equilibrium (4.7) with respect to $\mathbb{R}^3_3$.

4.2 *SIS model with vital dynamics*[1]

$$\begin{aligned} dS/dt &= (r - b)S - \rho SI + (\mu + r)I, \\ dI/dt &= -(\theta + b + \mu)I + \rho SI, \end{aligned} \tag{4.25}$$

where, denoting by $n = S + I$, we have

$$dn/dt = (r - b)n - \theta I. \tag{4.26}$$

Provided that $r > b$, $\theta > r - b$, system (4.25) has the feasible equilibrium $z^* \in \mathbb{R}^{2*}_+$:

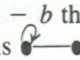
$$S^* = \frac{\theta + b + \mu}{\rho}, \quad I^* = \frac{r - b}{\theta + b - r} S^*. \tag{4.27}$$

When $r \leqslant b$, or $r > \theta + b$, the equilibrium (4.27) is not feasible and the only equilibrium of (4.25) is the origin. System (4.25) may be put in the form (2.9), where

$$e = \begin{pmatrix} r - b \\ -(\theta + b + \mu) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & -\rho \\ \rho & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & \mu + r \\ 0 & 0 \end{pmatrix}, \quad c = 0, \tag{4.28}$$

and $\mathbf{b}(z) \equiv \mathbf{B}z = ((\mu + r)I, 0)^T$. When z^* is a feasible equilibrium the matrix $\tilde{\mathbf{A}} = \mathbf{A} + \operatorname{diag}(z^{*-1})\mathbf{B}$ is given by

$$\tilde{\mathbf{A}} = \begin{pmatrix} 0 & -\left[\rho - \dfrac{(\mu + r)}{S^*}\right] \\ \rho & 0 \end{pmatrix}. \tag{4.29}$$

Since $S^* = (\theta + b + \mu)/\rho$, provided that $\theta > r - b$ the matrix $\tilde{\mathbf{A}}$ is skew-symmetrizable. Because $b_1(z) \geqslant 0$, the graph associated with $\tilde{\mathbf{A}}$ is  and by Corollary 2.2 the global asymptotic stability of z^* with respect to $\mathbb{R}^2_+$, follows.

When $r \leqslant b$, $r > \theta + b$ Theorem 2.2 cannot be applied to study attractivity of the origin because hypothesis (b) is violated.

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