Methods of Hilbert Spaces

Problem Sheet 2

1. Show that for any sequence $(x_n)_{n\in\mathbb{N}}$ in an inner product space H the conditions $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle$ imply $x_n \to x$.

2. Let $T : H \to H$ be a bounded linear operator on a complex inner product space H. Show that if $\langle Tx, x, =\rangle 0$ for any $x \in H$, then $Tx = 0$ for any $x \in H$ (that is, T is the zero operator). Show that this statement fails in a real inner product spaces. *Hint. Consider a rotation in* $\mathbb{R}^2$.

3. Which functionals below are scalar products on the real space of continuous functions on $\mathcal{C}([0, 2])$:

   $S(x, y) = \int\limits_0^2 |x(t)y(t)|$,

   $S(x, y) = \int\limits_0^2 w(t)x(t)y(t)|$, where w is a continuous strictly positive function on $[0, 1]$;

   $S(x, y) = \int\limits_0^1 x(t)y(t)$.

4. Show that on the space $\mathcal{C}([0, 1])$ the norm $\|x\|_\infty = \sup_{t\in[0,1]}$ of $C([0, 1])$ is stronger than the norm $\|\cdot\|_2$ induced by the inner product $\int\limits_0^1 x(t)\overline{y(t)}dt$.

5. Let w be a continuous function on $[-1, 1]$ satisfying $w_0 \le w(t) \le w_1$ for $t \in [-1, 1]$ and some positive constants w_0, w_1. Show that $\langle x, y\rangle_w = \int\limits_{-1}^1 w(t)x(t)\overline{y(t)}dt$ generates norm equivalent to $\|\cdot\|_2$ on $\mathcal{C}([-1, 1])$.

6. Which of the given sets B are orthogonal in respective inner spaces H

   $B = \{x^n\}_{n\in\mathbb{N}\cup\{0\}}$ in real $H = L_2([-\pi, \pi])$;

   $B = \{x_n\}_{n\in\mathbb{N}}$ in real $H = l_2$ where $x_{2k-1} = \{0, \ldots, 0, 1, 1, 0, 0, \ldots\}$ $x_{2k} = \{0, \ldots, 0, 1, -1, 0, 0, \ldots\}$ and we have $2k - 1$ zeros at the beginning;

   $B = \{e^{int}\}_{n\in\mathbb{N}\cup\{0\}}$ in complex $H = L_2([-\pi, \pi])$

7. Let $M \subset H$. Show that $M^\perp = \overline{\mathcal{L}in M}^\perp$. Hence find the orthogonal complement to $\overline{\mathcal{L}in\{(\delta_{2,k})_{k\in\mathbb{N}}, (\delta_{3,k})_{k\in\mathbb{N}}\}}$.