Methods of Hilbert Spaces

Problem Sheet 1

1. Check, which functions below are scalar products on $\mathbb{R}^2$:
   $$((x_1, x_2), (y_1, y_2)) \mapsto x_1 + x_2 + y_1 + y_2,$$
   $$((x_1, x_2), (y_1, y_2)) \mapsto x_1 y_1 + 3x_2 y_2,$$
   $$((x_1, x_2), (y_1, y_2)) \mapsto 3x_1 y_1 - x_1 y_2 - x_2 y_1 + 3x_2 y_2.$$

2. Check, which functions below are scalar products on $\mathbb{C}^2$:
   $$((z_1, z_2), (w_1, w_2)) \mapsto \Re(z_1 w_1) + \Im(z_2 w_2)$$
   $$((x_1, x_2), (y_1, y_2)) \mapsto z_1 w_1 + 5z_2 w_2$$
   $$((x_1, x_2), (y_1, y_2)) \mapsto 2z_1 \bar{w_1} - z_1 \bar{w_2} - z_2 \bar{w_1} + z_2 \bar{w_2}$$

3. Let $H = \mathbb{C}^2$ and $x = (\xi_1, \xi_2)$. Can the norm
   $$\|x\| = |\xi_1| + |\xi_2|$$
   be obtained from an inner product?

4. Let z_1, z_2 be complex numbers. Show that $\langle z_1, z_2 \rangle = z_1 \overline{z_2}$ defines an inner product on $\mathbb{C}$ which yields the usual metric. Explain under what conditions z_1 is orthogonal to z_2.

5. Give an example that in a complex unitary space the Pythagoras theorem does not imply orthogonality of elements.

6. Show that in a complex unitary space $x \perp y$ if and only if
   $$\|x + y\|^2 = \|x + iy\|^2 = \|x\|^2 + \|y\|^2.$$

7. Let H be a real inner product space. Show that $\|x\| = \|y\|$ implies $\langle x + y, x - y \rangle = 0$. Provide a geometrical interpretation for $H = \mathbb{R}^2$.

8. Let $H \ni x, y \neq 0$. Show that if $x \perp y$, then x and y are linearly independent. Extend the result to mutually orthogonal vectors.

9. Show that if $\langle x, y \rangle = \langle u, y \rangle$ for all $y \in H$, then $x = u$.

10. Let M_1, M_2 be non-empty subsets of a unitary space. Show that $(M_1 \cup M_2)^\perp = M_1^\perp \cap M_2^\perp$.

11. Prove from the definition that

$$\begin{aligned} \Re\langle x, y\rangle &= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) \\ \Im\langle x, y\rangle &= -\frac{1}{4}\left(\|ix+y\|^2 - \|ix-y\|^2\right). \end{aligned}$$

12. Let H be a complex (real) unitary space. Show that for any $x, y \in H$ the following conditions are equivalent:

    i) $x \perp y$,

    ii) $\|x\| \leq \|x + ty\|$ for any $t \in \mathbb{C}$ ($\mathbb{R}$),

    iii) $\|x + ty\| = \|x - ty\|$ for any $t \in \mathbb{C}$ ($\mathbb{R}$). Give interpretation of these conditions if $H = \mathbb{R}^2$.