Chapter 5

Comments on the Stable Manifold Theorem

In Theorem 5.4 (ii) we stated that if at least one eigenvalue of the linearized system has positive real part, then the corresponding nonlinear system is unstable. In this subsection we shall concentrate on planar systems though the main theorem can be proved in an arbitrary dimension; the proof is practically the same as in two dimensions provided we have sufficient background in differential geometry to operate with manifolds rather than with curves.

In two dimensions the only possible situation described in Theorem 5.4 (ii) is that $\lambda_1 < 0$ and $\lambda_2 > 0$ (recall that the case of zero eigenvalue has been ruled out by the non-degeneracy assumption on the matrix $\mathcal{A}$. To simplify notation let us denote $\lambda_1 = -\mu$, $\lambda_2 = \lambda$, $\mu, \lambda > 0$. For the linear system this case corresponds to the origin $\mathbf{0}$ being a saddle, that is, solutions tend to zero as $t \to \pm\infty$ if and only if the initial conditions are on one of the two half-lines determined by the eigenvectors corresponding to the negative (resp. positive) eigenvalue. For any other initial conditions the solutions become unbounded for both $t \to \pm\infty$.

Our aim here is to show that for a nonlinear system for which the origin is a saddle of its linearization, the phase portrait close to the origin is similar to a saddle in the sense that there exist two curves defined in some neighbourhood of $\mathbf{0}$ such that the solutions with initial conditions at $t = 0$ on one of these curves will remain on it for all $t \geq 0$ and with the initial condition on the other will be on it for all $t \leq 0$ and in both cases the solution will converge to $\mathbf{0}$ as $t \to \infty$ (resp. $t \to -\infty$). For any other initial condition, however small but non-zero, from this neighbourhood, the solution will become unbounded for both $t \to \pm\infty$. Let us first illustrate this idea on an example.

Example 0.1 Consider

$$\begin{aligned} x' &= -x, \\ y' &= y + x^2 \end{aligned}$$

Solving the first equation and inserting the solution to the second we obtain the solution in the form

$$\begin{aligned} x(t) &= x_0 e^{-t}, \\ y(t) &= \left(y_0 + \frac{x_0^2}{3}\right) e^t - \frac{x_0^2}{3} e^{-2t}. \end{aligned}$$

From this solution we clearly see that only the initial conditions satisfying

$$y_0 = -\frac{x_0^2}{3}$$

yield the solution tending to zero as $t \to \infty$. Any solution of this type will have the form

$$\begin{aligned} x(t) &= x_0 e^{-t}, \\ y(t) &= -\frac{x_0^2}{3} e^{-2t}. \end{aligned}$$

so that the orbit will be on the parabola $y = -\frac{1}{3}x^2$ (this parabola will consists of three orbits: the right-hand branch for $x_0 > 0$, the left-hand branch for $x_0 < 0$ and the stationary point $\mathbf{0}$. Thus, as in the linear case, the curve consisting of the initial conditions giving solutions converging to zero consists itself of solutions.

On the other hand, only the initial conditions $x_0 = 0$ with arbitrary y_0 produce solutions tending to zero as $t \to -\infty$ and of course, again, this curve (straight line) consists of orbits of the solutions.

Example 0.2 To prepare the ground for the proof of the main result, let us consider the following system

$$\begin{aligned} x' &= -\mu x + f(t), \\ y' &= \lambda y + g(t) \end{aligned}$$

where f and g are known bounded continuous functions. The question is: can we find initial condition for this system so that the solution will be bounded. We can immediately find the general solution to this system:

$$\begin{aligned} x(t) &= e^{-\mu t} x_0 + e^{-\mu t} \int\limits_0^t e^{\mu s} f(s) ds, \\ y(t) &= e^{\lambda t} y_0 + e^{\lambda t} \int\limits_0^t e^{-\lambda s} g(s) ds, \end{aligned}$$

with $x(0) = x_0, y(0) = y_0$. Clearly, $x(t)$ is bounded if $f(t)$ is bounded as $t \to \infty$ (why?) but $y(t)$ in general is unbounded. Let us then write $y(t)$ in the following form

$$y(t) = e^{\lambda t} y_0 + e^{\lambda t} \int\limits_0^\infty e^{-\lambda s} g(s) ds - e^{\lambda t} \int\limits_t^\infty e^{-\lambda s} g(s) ds.$$

The last term is well-defined as

$$\left| \int\limits_t^\infty e^{-\lambda s} g(s) ds \right| \leq \max |g(t)| \lambda^{-1} e^{-\lambda t}$$

and bounded as $t \to \infty$. Thus, if $y_0 = -\int\limits_0^\infty e^{-\lambda s} g(s) ds$, then the first term will vanish and the solution $y(t)$ will be also bounded. Hence, we found that the initial conditions lying on the straight line $(x_0, -\int\limits_0^\infty e^{-\lambda s} g(s) ds)$ are the only initial conditions producing bounded solutions that can be written in the form

$$\begin{aligned} x(t) &= e^{-\mu t} x_0 + e^{-\mu t} \int\limits_0^t e^{\mu s} f(s) ds, \\ y(t) &= -e^{\lambda t} \int\limits_t^\infty e^{-\lambda s} g(s) ds. \end{aligned}$$

After these preliminaries, we are ready to formulate the theorem for nonlinear systems. As before, we consider the system in the form

$$\mathbf{x}' = \mathcal{A}\mathbf{x} + \mathbf{g}(\mathbf{x}), \tag{5.0.1}$$

where we assume that $-\mu$ and λ, $\mu, \lambda > 0$, are eigenvalues of the matrix $\mathcal{A}$. We can assume then that

$$\mathcal{A} = \begin{pmatrix} -\mu & 0 \\ 0 & \lambda \end{pmatrix},$$

otherwise any matrix with two distinct real eigenvalues can be reduced to such a form by a linear change of variables called diagonalization and a linear change of variables does not alter the properties of $\mathbf{g}$ that are relevant in the proof below.

Theorem 0.1 *Let $\mathbf{g}$ be continuously differentiable for $\|\mathbf{x}\| < k$ for some constant $k > 0$, with $\mathbf{g}(\mathbf{0}) = \mathbf{0}$ and*

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{\partial \mathbf{g}}{\partial y_j}(\mathbf{y}) = \mathbf{0}.$$

If the eigenvalues of $\mathcal{A}$ are $\lambda, -\mu$ with $\lambda, \mu > 0$, then there exists in the space $\mathbf{x}$ a curve C: $x_2 = \phi(x_1)$, passing through the origin such that if $\mathbf{x}(t)$ is any solution with $\mathbf{x}(0)$ on C and $\|\mathbf{x}(0)\|$ sufficiently small, then $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$. Similarly, there is a curve C' such that if $\mathbf{x}(t)$ is any solution with $\mathbf{x}(0)$ on C' and $\|\mathbf{x}(0)\|$ sufficiently small, then $\mathbf{x}(t) \to \mathbf{0}$ as $t \to -\infty$. Moreover, no solution $\mathbf{x}(t)$ with $\mathbf{x}(0)$ small enough, but not on C (resp. C') can remain bounded as $t \to \infty$ (resp. $t \to -\infty$).

Proof. Let us denote

$$\mathcal{U}(t) = \begin{pmatrix} -\mu & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{V}(t) = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} \tag{5.0.2}$$

so that

$$e^{t\mathcal{A}} = \mathcal{U}(t) + \mathcal{V}(t).$$

Then, for any $\mathbf{a} \in \mathbb{R}^2$

$$\|\mathcal{U}(t)\mathbf{a}\| \leq e^{-\mu t}\|\mathbf{a}\|, \qquad \|\mathcal{V}(t)\mathbf{a}\| \leq e^{\lambda t}\|\mathbf{a}\|.$$

In the proof it will be convenient to write these estimates as

$$\begin{aligned} \|\mathcal{U}(t)\mathbf{a}\| &\leq e^{-(\alpha+\sigma)t}\|\mathbf{a}\|, \quad t \geq 0 \\ \|\mathcal{V}(t)\mathbf{a}\| &\leq e^{\sigma t}\|\mathbf{a}\|, \quad t \leq 0, \end{aligned}$$

where $\alpha, \sigma > 0$ are chosen so that $\sigma \leq \lambda$ and $\sigma + \alpha \leq \mu$.

The next preliminary step is the observation that due to the assumptions on $\mathbf{g}$ (smallness of the partial derivatives in a neighbourhood of $\mathbf{0}$), for any $\epsilon > 0$ there is $\delta > 0$ such that if $\|\mathbf{x}\|, \|\mathbf{x}^*\| < \delta$, then

$$\|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)\| \leq \epsilon\|\mathbf{y} - \mathbf{y}^*\|.$$

Passing to the main part of the proof, we use the considerations of Example 0.2 and will consider, for $\mathbf{a} = (a_1, 0)$ the integral equation

$$\mathbf{x}(t,\mathbf{a}) = \mathcal{U}(t)\mathbf{a} + \int_0^t \mathcal{U}(t-s)\mathbf{g}(\mathbf{x}(s,\mathbf{a}))ds - \int_t^\infty \mathcal{V}(t-s)\mathbf{g}(\mathbf{x}(s,\mathbf{a}))ds, \tag{5.0.3}$$

or in the expanded form

$$\begin{aligned} x_1(t,a_1) &= e^{-\mu t}a_1 + e^{-\mu t}\int_0^t e^{\mu s} g_1(x_1(s,a_1), x_2(s,a_1))ds, \\ x_2(t,a_1) &= -e^{\lambda t}\int_t^\infty e^{-\lambda s} g_2(x_1(s,a_1), x_2(s,a_1))ds. \end{aligned} \tag{5.0.4}$$

Let us first observe that, as in Example 0.2, any bounded continuous solution of (5.0.3), (5.0.4) is a solution to the original system (5.0.1) satisfying the initial condition $x_1(0) = a_1$ and $x_2(0) = -\int_0^\infty e^{-\lambda s} g_2(x_1(s, a_1), x_2(s, a_1))ds$. If we recall that $\mathbf{x}(t)$ depends on a_1 in a continuous way we see that in fact $x_2(0) = \phi(a_1)$ so that there is a curve $x_2(0) = \phi(x_1(0))$ such that the solutions starting from points of this curve remain bounded. The problem is to show that the integral equation has solutions that are bounded. We shall show it by successive approximations. Let us define, for a moment formally,

$$\begin{aligned}
\mathbf{x}^0(t, \mathbf{a}) &= \mathbf{0}, \\
\mathbf{x}^n(t, \mathbf{a}) &= \mathcal{U}(t)\mathbf{a} + \int_0^t \mathcal{U}(t-s)\mathbf{g}(\mathbf{x}^{n-1}(s, \mathbf{a}))ds - \int_t^\infty \mathcal{V}(t-s)\mathbf{g}(\mathbf{x}^{n-1}(s, \mathbf{a}))ds.
\end{aligned}$$

Since by the assumption $\mathbf{g}(\mathbf{0}) = \mathbf{0}$, we have

$$\|\mathbf{x}^1(t, \mathbf{a}) - \mathbf{x}^0(t, \mathbf{a})\| = \|\mathcal{U}(t)\mathbf{a}\| \leq e^{-(\alpha+\sigma)t}\|\mathbf{a}\| \leq e^{-\alpha t}\|\mathbf{a}\|.$$

Clearly, if $\|\mathbf{a}\| < k$, then $\|\mathbf{x}^1(t, \mathbf{a})\| < k$ for all times and we can consider the second iterate

$$\mathbf{x}^2(t, \mathbf{a}) = \mathcal{U}(t)\mathbf{a} + \int_0^t \mathcal{U}(t-s)\mathbf{g}(\mathbf{x}^1(s, \mathbf{a}))ds - \int_t^\infty \mathcal{V}(t-s)\mathbf{g}(\mathbf{x}^1(s, \mathbf{a}))ds.$$

and estimate, provided $\|\mathbf{x}^1(t, \mathbf{a}))\| < \delta$,

$$\begin{aligned}
\|\mathbf{x}^2(t, \mathbf{a}) - \mathbf{x}^1(t, \mathbf{a})\| &\leq \int_0^t \|\mathcal{U}(t-s)\|\|\mathbf{g}(\mathbf{x}^1(t, \mathbf{a})) - \mathbf{g}(\mathbf{x}^0(t, \mathbf{a}))\|ds \\
&\quad + \int_t^\infty \|\mathcal{V}(t-s)\|\|\mathbf{g}(\mathbf{x}^1(t, \mathbf{a})) - \mathbf{g}(\mathbf{x}^0(t, \mathbf{a}))\|ds \\
\leq \epsilon\|\mathbf{a}\| \int_0^t e^{-(\sigma+\alpha)(t-s)}e^{-\alpha s}ds &+ \epsilon\|\mathbf{a}\| \int_t^\infty e^{\sigma(t-s)}e^{-\alpha s}ds \\
= \epsilon\|\mathbf{a}\| \left(\frac{e^{-(\sigma+\alpha)t}}{\sigma}(e^{\sigma t} - 1) \right. &+ \left. e^{\sigma t}\frac{e^{-(\sigma+\alpha)t}}{\alpha+\sigma} \right) \\
&\leq \frac{2\epsilon\|\mathbf{a}\|}{\sigma}e^{-\alpha t}
\end{aligned}$$

where we used $e^{\sigma t} - 1 \leq e^{\sigma t}$ and $1/(\alpha + \sigma) \leq 1/\sigma$. Let us assume that $\epsilon/\sigma < 1/4$, then

$$\|\mathbf{x}^2(t, \mathbf{a}) - \mathbf{x}^1(t, \mathbf{a})\| \leq \frac{\|\mathbf{a}\|}{2}e^{-\alpha t}.$$

Let us conjecture the induction assumption

$$\|\mathbf{x}^j(t, \mathbf{a}) - \mathbf{x}^{j-1}(t, \mathbf{a})\| \leq \frac{\|\mathbf{a}\|}{2^{j-1}}e^{-\alpha t}, \tag{5.0.5}$$

for $j \leq n$. Note, that if this assumption is satisfied for all $j \leq n$, then

$$\|\mathbf{x}^n(t, \mathbf{a})\| \leq \|\mathbf{x}^n(t, \mathbf{a}) - \mathbf{x}^{n-1}(t, \mathbf{a})\| + \ldots + \|\mathbf{x}^1(t, \mathbf{a})\| = \|a\|e^{-\alpha t}(2^{-(n-1)} + \ldots + 1) \leq 2\|a\|e^{-\alpha t}$$

so that if $\|\mathbf{a}\| < \delta/2$, then $\|\mathbf{x}^n(t, \mathbf{a})\| < \delta$ and the estimates for iterates can be carried for the next step. Thus, we have

$$\|\mathbf{x}^{n+1}(t, \mathbf{a}) - \mathbf{x}^n(t, \mathbf{a})\| \leq \int_0^t \|\mathcal{U}(t-s)\|\|\mathbf{g}(\mathbf{x}^n(t, \mathbf{a})) - \mathbf{g}(\mathbf{x}^{n-1}(t, \mathbf{a}))\|ds$$

$$+\int_t^\infty \|\mathcal{V}(t-s)\|\|\mathbf{g}(\mathbf{x}^n(t,\mathbf{a}))-\mathbf{g}(\mathbf{x}^{n-1}(t,\mathbf{a}))\|ds$$

$$\begin{aligned}
&\leq \epsilon\|\mathbf{a}\|\int_0^t e^{-(\sigma+\alpha)(t-s)}2^{-(n-1)}e^{-\alpha s}ds \quad + \quad \epsilon\|\mathbf{a}\|\int_t^\infty e^{\sigma(t-s)}2^{-(n-1)}e^{-\alpha s}ds\\
&= \frac{\epsilon\|\mathbf{a}\|}{2^{n-1}}\left(\frac{e^{-(\sigma+\alpha)t}}{\sigma}(e^{\sigma t}-1) \quad + \quad e^{\sigma t}\frac{e^{-(\sigma+\alpha)t}}{\alpha+\sigma}\right)\\
&\leq \quad \frac{\epsilon\|\mathbf{a}\|}{2^{n-2}\sigma}e^{-\alpha t} \leq \frac{\|\mathbf{a}\|}{2^n}e^{-\alpha t}
\end{aligned}$$

where we used $\epsilon/\sigma < 1/4$ in the last step.

Therefore, as in the Picard theorem, the sequence $\mathbf{x}^n(t,\mathbf{a})$ converges uniformly on $[0,\infty)$ to some continuous function $\mathbf{x}(t,\mathbf{a})$ satisfying

$$\|\mathbf{x}(t,\mathbf{a})\| \leq 2\|\mathbf{a}\|e^{-\alpha t}$$

for $t \geq 0$ and $\|\mathbf{a}\| < \delta/2$. As in Picard's theorem we find that this $\mathbf{x}(t,\mathbf{a})$ is the solution of the integral equation (5.0.4) and therefore it is also a unique solution of the differential equation (5.0.1) satisfying the initial condition $x_1(0) = a_1, x_2(0) = -\int_0^\infty e^{-\lambda s}g_2(x_1(s,a_1),x_2(s,a_1))ds$ so that we found a curve C, defined in some neighbourhood of the origin, with the property that any solution emanating from C tends to zero as $t \to \infty$.

Let us consider a solution $\mathbf{y}(t)$ of (5.0.1) with $\mathbf{y}(0) = \mathbf{b}$, where $\mathbf{b}$ is small but not on C. Assume that $\|\mathbf{y}(t)\| \leq \delta$, where δ is defined as above. The solution satisfies

$$\mathbf{y}(t) = e^{t\mathcal{A}}\mathbf{b} + \int_0^t e^{(t-s)\mathbf{A}}\mathbf{g}(\mathbf{y}(s))ds,$$

and, as before, we write this solution as

$$\mathbf{y}(t) = \mathcal{U}(t)\mathbf{b} + \mathcal{V}(t)\mathbf{b} + \int_0^t \mathcal{U}(t-s)\mathbf{g}(\mathbf{y}(s))ds + \int_0^\infty \mathcal{V}(t-s)\mathbf{g}(\mathbf{y}(s))ds - \int_t^\infty \mathcal{V}(t-s)\mathbf{g}(\mathbf{y}(s))ds \qquad (5.0.6)$$

where all the integrals exist due to the bound on $\mathcal{V}(t)$ and since $\mathbf{g}(\mathbf{y}(s))$ is bounded whenever $\|\mathbf{y}(s)\| \leq \delta$. Since $\mathcal{V}(t)$ is just a multiplication of the second coordinate by $e^{\lambda t}$, we can rewrite the above as

$$\mathbf{y}(t) = \mathcal{U}(t)\mathbf{b} + \mathcal{V}(t)\mathbf{c} + \int_0^t \mathcal{U}(t-s)\mathbf{g}(\mathbf{y}(s))ds - \int_t^\infty \mathcal{V}(t-s)\mathbf{g}(\mathbf{y}(s))ds$$

where

$$\mathbf{c} = \mathbf{b} + \int_0^\infty \mathcal{V}(-s)\mathbf{g}(\mathbf{y}(s))ds.$$

Since $\|\mathbf{y}(t)\| \leq \delta$ and all the terms on the right-hand side, except possibly $\mathcal{V}(t)\mathbf{c}$, are clearly bounded. Thus, $\mathcal{V}(t)\mathbf{c}$ must be also bounded but $\mathcal{V}(t)\mathbf{c} = (0, e^{\lambda t c_2})$, where c_2 is the second component of $\mathbf{c}$. Hence $c_2 = 0$ but this implies $\mathcal{V}(t)\mathbf{c} = 0$ for all $t \geq 0$ so that $\mathbf{y}(0)$ is on the curve C, contrary to the assumption.

This result allows to establish that C consists of orbits. In fact, let $\mathbf{x}(t)$ be a solution emanating from C, converging to $\mathbf{0}$. If it does not stay on C, then for some time t_*, $\mathbf{x}(t_*) \notin C$. Considering $\hat{\mathbf{x}}(t) = \mathbf{x}(t+t_*)$, we see that $\hat{\mathbf{x}}(t)$ is a solution with initial condition not on C that converges to $\mathbf{0}$ which is a contradiction with the previous part of the proof.

The statement concerning the curve C' is obtained by changing the direction of time in (5.0.1) and noting that the properties of the system relevant to the proof do not change hance C for the system with reversed time becomes C' for the original system. ■