SOME POROUS AND MEAGER SETS OF CONTINUOUS MAPPINGS

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Abstract. De Blasi and Myjak showed that in Hilbert space the set of all Banach contractions is σ-lower porous in the space of all nonexpansive mappings. In this paper we generalize this result by considering more general spaces.

1. Introduction

Assume that K is a closed convex and bounded subset of a Hilbert space. De Blasi and Myjak proved [DM2] that the set of all Banach contractions

$$kB = \{f : K \to K : \exists \alpha \in (0,1)\ \forall x, y \in K\ ||f(x) - f(y)|| \leq \alpha ||x - y||\}$$

is σ-porous in the space of all nonexpansive mappings

$$\Omega = \{f : K \to K : \forall x, y \in K\ ||f(x) - f(y)|| \leq ||x - y||\},$$

endowed with the supremum metric:

$$\sup\{||f(x) - g(x)|| : x \in K\}.$$

Since Ω is a complete space and every σ-porous set is meager, the above result shows that kB is a small subset of Ω (note that in [DM1] it was shown that kB is meager in Ω). The most important tool in the proof is the Kirszbraun–Valentine Theorem, used there in a very particular case – for nonexpansive mappings. In our paper we show that using the Kirszbraun–Valentine Theorem in all of its power, we can prove more general results – in particular, instead of the space of nonexpansive mappings, we can consider the space of all uniformly continuous mappings with the modulus of continuity bounded by some fixed function.

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2. Notations and Terminology

We start with giving some notions of porosity. Let (X, d) be a metric space. $B(x, R)$ stands for the open ball with a radius R centered at a point x. We say that $M \subset X$ is *lower porous*, if

$$\forall x \in M \ \exists \alpha > 0 \ \exists R_0 > 0 \ \forall R \in (0, R_0) \ \exists z \in X \ B(z, \alpha R) \subset B(x, R) \backslash M,$$

and M is *α-lower porous*, if

$$\forall x \in M \ \forall \beta \in \left(0, \frac{\alpha}{2}\right) \ \exists R_0 > 0 \ \forall R \in (0, R_0) \ \exists z \in X \ B(z, \beta R) \subset B(x, R) \backslash M.$$

Note that these definitions are equivalent to those given in [Z2, p. 511]. We say that M is *σ-lower porous* if M is a countable union of lower porous sets, and M is *σ-α-lower porous* if M is a countable union of α-lower porous sets. Clearly, every σ-α-lower porous set is σ-lower porous, but the converse is not true – in any "reasonable" metric space there is a σ-lower porous set which is σ-α-lower porous for no $\alpha > 0$ (cf. [Z2, p. 516]). It is worth to mention that this is not the case when we deal with another well known notion of porosity – the upper porosity (cf. [Z2, Proposition 2.9] and [BL, p. 92]).
In [DM2], there was defined another notion of porosity. Namely, we say that $M \subset X$ is *porous*, if

$$\exists \alpha > 0 \ \exists R_0 > 0 \ \forall x \in X \ \forall R \in (0, R_0) \ \exists z \in X \ B(z, \alpha R) \subset B(x, R) \backslash M.$$

Additionally, we define *σ-porosity* in an obvious way.
This notion seems to be stronger than the lower porosity. However, by [Z2, Proposition 2.2], the following conditions are equivalent:

(i) M is σ-lower porous;
(ii) M is σ-porous;
(iii) $M = \bigcup M_n$ and each set M_n is α_n-lower porous for some α_n.

The equivalence of (i) and (ii) shows that mentioned result of De Blasi and Myjak states that kB is σ-lower porous, and the equivalence of (i) and (iii) shows that a σ-lower porous set can be always written as a countable union of α_n-lower porous sets (but the constants α_n may be different).
It can easily be seen that if X is a metric space, $M \subset Y \subset X$ and M is nowhere dense [of

the first category] in Y, then it is also nowhere dense [of the first category] in X.

It turns out that the same holds for lower porosity:

Proposition 1. *Assume that (X,d) is a metric space and $M \subset Y \subset X$. If M is lower-porous [σ-lower porous] in Y, then it is lower-porous [σ-lower porous] in X;*

Proof. Assume that M is lower porous and fix $x \in M$. Let $\alpha' > 0$ and $R_0' > 0$ be as in the definition of lower porosity, chosen for x. Clearly, we can assume that $\alpha' < \frac{1}{2}$. Now set $R_0 = (1+\alpha')R_0'$ and $\alpha = \frac{\alpha'}{1+\alpha'}$, take $R \in (0, R_0)$ and put $R' = \frac{1}{1+\alpha'}R$. Then $R' \in (0, R_0')$, so there is $y \in Y$ such that ($B_Y(\cdot,\cdot)$ denotes an open ball in Y):

$$B_Y(y, \alpha' R') \subset B_Y(x, R') \setminus M. \tag{1}$$

Since $d(y,x) < R' = (1-\alpha)R$, we have that

$$B(y, \alpha R) \subset B(x, R).$$

On the other hand, $\alpha R = \frac{\alpha'}{1+\alpha'}R = \alpha' R'$, so $B(x, \alpha R) \cap Y = B_Y(x, \alpha' R')$. Hence, by (1), $B(x, \alpha R) \cap M = \emptyset$. This ends the proof. □

The above observations show that it is interesting to find the smallest subspace $Y \subset X$ in which M is σ-lower porous [of the first category]; the most restrictive case is when M is σ-lower porous [of the first category] in itself. Clearly, σ-lower porous subsets of X are small if the Baire Category Theorem holds for X, however, the fact that some set is σ-lower porous in itself give us some interesting information about the structure of it.

Now, let $(X, ||\cdot||)$ be a normed linear space. We say that M is *c-porous*, if its convex hull *convM* is nowhere dense. We say that M is *σ-c-porous*, if M is a countable union of c-porous sets. The notion of c-porosity is closely related to the notions of R-ball porosity and 0-angle porosity (cf. [Z2]) and was discussed further in [S] (cf. [S, Proposition 2.5]). In particular, every σ-c-porous set is σ-lower porous, but the converse need not be true. In fact, c-porosity is one of the most restrictive notions of porosity.

For more information about porosity, we refer the reader to the survey papers [Z1] and [Z2] on porosity on the real line, metric spaces and normed linear spaces.

Now let (X,d) and (Y,ρ) be two metric spaces. If $f : X \to Y$, then the modulus of continuity of A, denoted by ω_f, is defined in the following way:

$$\forall t > 0 \ \omega_f(t) = \sup\{\rho(f(x), f(y)) : x, y \in X,\ d(x,y) \le t\}.$$

We recall the Kirszbraun–Valentine Theorem in its most general form [BL, p. 18]:

Theorem 2. *Let H_1, H_2 be two Hilbert spaces, K be any subset of H_1 and $f : K \to H_2$. Let $\omega : (0,\infty) \to [0,\infty)$ be a nondecreasing, concave function with $\lim_{t\to 0}\omega(t) = 0$. If for every $t > 0$, $\omega_f(t) \leq \omega(t)$, then there exists $\tilde{f} : H_1 \to H_2$ such that $\tilde{f}_{|K} = f$ and for every $t > 0$, $\omega_{\tilde{f}}(t) \leq \omega(t)$.*

By composing $\tilde{f}$ (if needed) with the appropriate projection, and restricting to an arbitrary set, we get the following strengthening of the above result ($\overline{conv}M$ denotes the closed convex hull of the set M):

Corollary 3. *Let H_1, H_2 be two Hilbert spaces, K, L be two subsets of H_1 with $K \subset L$, and $f : K \to H_2$. Let $\omega : (0,\infty) \to [0,\infty)$ be a nondecreasing, concave function with $\lim_{t\to 0}\omega(t) = 0$. If for every $t > 0$, $\omega_f(t) \leq \omega(t)$, then there exists $\tilde{f} : L \to \overline{conv} f(K)$ such that $\tilde{f}_{|K} = f$ and for every $t > 0$, $\omega_{\tilde{f}}(t) \leq \omega(t)$.*

Now we define some topological and metric spaces. Assume that X_1 and X_2 are Banach spaces and $K \subset X_1$. For simplicity, the norms on X_1 and X_2 will be denoted by the same symbol $||\cdot||$ (it will not lead to any confusion). By $C^b(K)$ we denote the space of all continuous bounded functions from K into X_2. We consider $C^b(K)$ as a Banach space with the standard supremum norm (which also will be denoted by $||\cdot||$):

$$||f|| = \sup\{||f(x)|| : x \in K\}.$$

Now let $D \subset X_2$ and $\omega : (0,\infty) \to [0,\infty)$ be any nondecreasing function. Then by $C^b_\omega(K, D)$ we denote the space of all bounded mappings from K into D with the modulus of continuity bounded by ω:

$$C^b_\omega(K, D) = \{f \in C^b(K) : f(K) \subset D \text{ and } \forall t > 0\ \omega_f(t) \leq \omega(t)\}.$$

We consider $C^b_\omega(K, D)$ as a metric subspace of $C^b(K)$. Clearly, if D is a closed subset of X_2, then $C^b_\omega(K, D)$ is a complete space. Moreover, for every $f : K \to X_2$, the following conditions are equivalent:

(i) $\forall t > 0\ \omega_f(t) \leq \omega(t)$;

(ii) $\forall x, y \in K\ ||f(x) - f(y)|| \leq \omega(||x - y||)$.

Hence if $X_1 = X_2$, K is bounded, $D = K$ and $\omega(t) = t$, $t > 0$, then $C^b_\omega(K, D)$ coincides with the space of all nonexpansive mappings considered in [DM2].

Now we will deal with the case where the mappings may be unbounded. It turns out that in this case we can define a natural topology as well. Let us define the following sets (K, D and ω have the same meaning as above):

$$C(K) = \{f : K \to X_2 : f \text{ is continuous}\};$$

$$C_\omega(K, D) = \{f \in C(K) : f(K) \subset D \text{ and } \forall t > 0\ \omega_f(t) \le \omega(t)\}.$$

Define the topology τ on $C(K)$ in the following way:

$$\tau = \{U \subset C(K) : \forall f \in U\ \exists n \in \mathbb{N}\ \exists \epsilon > 0\ B(f, (n, \epsilon)) \subset U\},$$

where

$$B(f, (n, \epsilon)) = \{g \in C(K) : \forall x \in B(0, n) \cap K,\ ||f(x) - g(x)|| \le \epsilon\}$$

and 0 is the origin of the space X_1. Note that τ can be considered as a topology generated by the uniformity with the base $\{E(n, \epsilon) : n \in \mathbb{N},\ \epsilon > 0\}$, where

$$E(n, \epsilon) = \{(f, g) \in C(K) \times C(K) : \forall x \in B(0, n) \cap K,\ ||f(x) - g(x)|| \le \epsilon\}.$$

By [E, Theorem 8.1.21], $C(K)$ is metrizable. It can easily be seen that $C(K)$ endowed with the metric defined in [E, Theorem 8.1.21] is complete. It is also easy to prove that if D is a closed subset of H_2, then $C_\omega(K, D)$ is a closed subset of $C(K)$, hence it is also completely metrizable.

Now let $Y \subset C(K)$. Then the relative topology induced from $C(K)$ can be described in the following way:

$$\tau_{|Y} = \{U \subset Y : \forall f \in U\ \exists n \in \mathbb{N}\ \exists \epsilon > 0\ B^Y(f, (n, \epsilon)) \subset U\},$$

where $B^Y(f, (n, \epsilon)) = B(f, (n, \epsilon)) \cap Y$. This shows that if $C_\omega(K, D)$ is the space of all nonexpansive mappings ($K = D$ and $\omega(t) = t$), then the topology on it is the same as that considered by Reich and Zaslavski [RZ2].

It is also easy to see that if $M \subset Y \subset C(K)$, then M is nowhere dense in Y if and only if

$$\forall f \in M\ \forall n \in \mathbb{N}\ \forall \epsilon > 0\ \exists g \in Y\ \exists m \in \mathbb{N}\ \exists \epsilon_1 > 0\ B^Y(g, (m, \epsilon_1)) \subset B^Y(f, (n, \epsilon)) \setminus M.$$

In the sequel, we will write $B(f, (n, \epsilon))$ instead of $B^Y(f, (n, \epsilon))$ – this will not lead to any confusion.

3. A generalization of the De Blasi and Myjak Result

In this section we assume that H_1, H_2 are Hilbert spaces, $K \subset H_1$ contains a nontrivial segment (i.e., there are $x, y \in K$, such that $x \neq y$ and $\{tx + (1-t)y : t \in [0,1]\} \subset K$), $D \subset H_2$ is closed, convex and contains at least two elements, and ω is a nondecreasing, concave function with $\omega(t) > 0$ for $t > 0$, and $\lim_{t\to 0} \omega(t) = 0$. Note that these assumptions imply that ω is continuous, and hence:

$$(0, \sup_{t>0} \omega(t)) = \omega((0,\infty)). \tag{2}$$

If $\lambda > 0$, then $\lambda\omega$ denotes the function defined by $\lambda\omega(t) = \lambda \cdot \omega(t)$ for $t > 0$. The main result of this section is the following:

Theorem 4. *Assume that $\lambda \in (0,1)$. Then the following statements hold:*

(i) *The set $C^b_{\lambda\omega}(K,D)$ is $\frac{(1-\lambda)^2}{16}$-lower porous in $C^b_\omega(K,D)$;*
(ii) *The set $C_{\lambda\omega}(K,D)$ is nowhere dense in $C_\omega(K,D)$.*

Before we prove the theorem, we will give the most important corollaries of it. At first, let us define some additional sets:

$$kB^b_\omega(K,D) = \{f \in C^b_\omega(K,D) : \exists \lambda \in (0,1)\ \forall t > 0\ \omega_f(t) \leq \lambda\omega(t)\} = \bigcup_{\lambda\in(0,1)} C^b_{\lambda\omega}(K,D);$$

$$L^b_\omega(K,D) = \{f \in C^b(K) : f(K) \subset D \text{ and } \exists M > 0\ \forall t > 0\ \omega_f(t) \leq M\omega(t)\} = \bigcup_{M>0} C^b_{M\omega}(K,D).$$

Similarly we define $kB_\omega(K,D)$ and $L_\omega(K,D)$. It is easy to see that if $K = D$ and $\omega(t) = t$ for $t > 0$, then $kB^b_\omega(K,K)$ is the set of all Banach contractions with a bounded image, $kB_\omega(K,K)$ is the set of all Banach contractions, $L^b_\omega(K,K)$ is the set of all Lipschitzian self-mappings with a bounded image, and $L_\omega(K,K)$ is the set of all Lipschitzian self-mappings.

Corollary 5. *The following assertions hold:*

(i) *The metric space $kB^b_\omega(K,D)$ is σ-lower porous in itself. In particular, the set $kB^b_\omega(K,D)$ is σ-lower porous in $C^b_\omega(K,D)$ and σ-c-porous in $C^b(K)$;*
(ii) *The metric space $L^b_\omega(K,D)$ is σ-lower porous in itself. In particular, the set $L^b_\omega(K,D)$ is σ-c-porous in $C^b(K)$;*
(iii) *The topological space $kB_\omega(K,D)$ is of the first category in itself. In particular, the set $kB_\omega(K,D)$ is of the first category in $C_\omega(K,D)$;*

(iv) *The topological space $L_\omega(K,D)$ is of the first category in itself. In particular, the set $L_\omega(K,D)$ is of the first category in $C(K)$.*

Proof. We will prove only part (i), since the proofs of the other parts are very similar. We have

$$kB^b_\omega(K,D) = \bigcup_{n\in\mathbb{N}} C^b_{(1-\frac{1}{n})\omega}(K,D).$$

By Theorem 4, for any $m > n$, $C^b_{(1-\frac{1}{n})\omega}(K,D)$ is a lower porous subset of $C^b_{(1-\frac{1}{m})\omega}(K,D)$, hence by Proposition 1, is lower porous in $kB^b_\omega(K,D)$. Thus $kB^b_\omega(K,D)$ is σ-lower porous in itself and, again by Proposition 1, in $C^b_\omega(K,D)$. Moreover, $C^b_{(1-\frac{1}{n})\omega}(K,D)$ is a convex (and, of course, nowhere dense) subset of $C^b(K)$, so we get (i). □

Now we give the proof of Theorem 4. We will write $B(f,(n,\epsilon))$ instead of $B^{C_\omega(K,D)}(f,(n,\epsilon))$.

Proof. Put $\alpha = \frac{(1-\lambda)^2}{32}$. It is enough to show that there is $R_0 > 0$ such that for any $R \in (0, R_0)$ and $f \in C_{\lambda\omega}(K,D)$, there is $g \in C_\omega(K,D)$ such that

(a) $\sup_{x\in K} ||g(x) - f(x)|| \le \frac{1}{2}R$;
(b) If $f(K)$ is bounded, then $g(K)$ is bounded;
(c) $\exists n \in \mathbb{N}\ \forall h \in C(K)\ \Big(\sup_{x\in B(0,n)\cap K} ||h(x) - g(x)|| < \alpha R \Rightarrow h \notin C_{\lambda\omega}(K,D)\Big)$.

Indeed, let $f \in C^b_{\lambda\omega}(K,D)$ and $R \in (0,R_0)$. Take g fulfilling (a)–(c). By (b) and (a), $g \in C^b_\omega(K,D)$ and $||f-g|| < (1-\alpha)R$. Finally, (c) implies that if $||g-h|| < \alpha R$, then $h \notin C^b_{\lambda\omega}(K,D)$. Hence we get (i).

Now let $f \in C_{\lambda\omega}(K,D)$, $m \in \mathbb{N}$ and $\epsilon > 0$. Set $R = \min\{\epsilon, \frac{1}{2}R_0\}$ and take g and $n \in \mathbb{N}$ as above. Now if $n' = \max\{n,m\}$, then (a) and (c) easily imply that $B(g,(n',\frac{\alpha}{2}R)) \subset B(f,(m,\epsilon)) \setminus C_{\lambda\omega}(K,D)$. Hence we get (ii).

Let $x_0, y' \in K$ be such that $x_0 \neq y'$ and the segment $[x_0,y'] \subset K$. Since $\lim_{t\to 0}\omega(t) = 0$, there exists $r' > 0$ such that

$$r' < ||x_0 - y'|| \quad \text{and} \quad \omega(r') < \frac{1}{4} diam D.$$

Note that if $diam D = \infty$, then the second inequality means that $\omega(t') < \infty$ (and is satisfied for every positive real). Set $R_0 = \omega(r')$ and let $R \in (0,R_0)$ and $f \in C_{\lambda\omega}(K,D)$. Define $z_0 = f(x_0)$. Since $\omega(r') < \frac{1}{4} diam D$, there exists $s' \in D \setminus \{z_0\}$ such that

$$||s' - z_0|| > \omega(r').$$

By (2), there exists $r_1 > 0$ such that $\omega(r_1) = \frac{R}{4}$. Since ω is concave, nondecreasing, continuous and $\lim_{t\to 0}\omega(t) = 0$, we have that

$$\omega\left(\frac{r_1}{2}\right) \geq \frac{R}{8} \tag{3}$$

and $r_1 < r'$. Now let $r_2 > 0$ be such that

$$\omega(r_2) = \frac{1-\lambda}{2}\omega(\frac{r_1}{2}). \tag{4}$$

Clearly, $r_2 < \frac{r_1}{2} < r'$ and $\omega(r_2) < ||s' - z_0||$. Now let $y_0 \in [x_0, y']$ and $s_0 \in [z_0, s']$ be such that

$$||y_0 - x_0|| = r_2 \quad \text{and} \quad ||s_0 - z_0|| = \omega(r_2). \tag{5}$$

We are ready to define g:

- $g(x_0) = f(x_0) = z_0$;
- $g(y_0) = s_0$;
- If $x \in K_1 = \{y \in K : ||y - x_0|| \geq r_1\}$, we set $g(x) = f(x)$.

Before we define g on the rest of the set K, let us note that the function $g : \{x_0, y_0\} \cup K_1 \to D$, that we have already defined, has the modulus of continuity bounded by ω. Indeed, we have

$$||g(y_0) - g(x_0)|| = ||s_0 - z_0|| = \omega(r_2) = \omega(||y_0 - x_0||), \tag{6}$$

and if $y \in K_1$, then

$$||g(x_0) - g(y)|| = ||f(x_0) - f(y)|| \leq \lambda\omega(||x_0 - y||).$$

Moreover, $||y_0 - y|| > \frac{r_1}{2}$ since $||y_0 - x_0|| < \frac{r_1}{2}$, so

$$||g(y_0)-g(y)|| \leq ||g(y_0)-g(x_0)||+||g(x_0)-g(y)|| \leq \omega(r_2)+||f(x_0)-f(y_0)||+||f(y_0)-f(y)|| \leq$$

$$2\omega(r_2) + \lambda\omega(||y_0 - y||) = (1-\lambda)\omega\left(\frac{r_1}{2}\right) + \lambda\omega(||y_0 - y||) \leq$$

$$(1-\lambda)\omega(||y_0 - y||) + \lambda\omega(||y_0 - y||) = \omega(||y_0 - y||),$$

and, finally, if $z \in K_1$, then

$$||g(z) - g(y)|| = ||f(z) - f(y)|| \leq \omega(||z - y||).$$

Since $g(\{x_0, y_0\} \cup K_1) \subset D$ and D is convex and closed, by Corollary 3, we can extend g to the mapping $\tilde{g}$ so that $\tilde{g} \in C_\omega(K, D)$. For simplicity of notation, we will write g instead of

$\tilde{g}$. To complete the proof, we need to show that g satisfies (a), (b) and (c).

Clearly, $\sup_{x\in K}||f(x)-g(x)|| = \sup_{x\in K\setminus K_1}||f(x)-g(x)||$, and for any $x \in K\setminus K_1$,

$$||f(x)-g(x)|| \leq ||f(x)-f(x_0)|| + ||g(x_0)-g(x)|| \leq$$
$$2\omega(||x_0-x||) \leq 2\omega(r_1) = \frac{R}{2},$$

which shows (a). (b) follows immediately from (a). Now let $n \in \mathbb{N}$ be such that $x_0, y_0 \in B(0,n)$, and $h \in C(K)$ be such that $\sup_{x\in B(0,n)\cap K}||h(x)-g(x)|| < \alpha R$. Then

$$||g(x_0)-g(y_0)|| \leq ||g(x_0)-h(x_0)|| + ||h(x_0)-h(y_0)|| + ||h(y_0)-g(y_0)||.$$

Hence and by (3), (5), (6), (4) and the fact that $\alpha = \frac{(1-\lambda)^2}{32}$, we get

$$\omega_h(r_2) \geq ||h(x_0)-h(y_0)|| > \omega(r_2) - 2\alpha R = \omega(r_2) - \frac{(1-\lambda)^2}{16}R \geq$$
$$\omega(r_2) - \frac{(1-\lambda)^2}{2}\omega\left(\frac{r_1}{2}\right) = \omega(r_2) - (1-\lambda)\omega(r_2) = \lambda\omega(r_2).$$

Thus $h \notin C_{\lambda\omega}(K,D)$ and the result follows. □

A natural question arises, whether the set $kB^b_\omega(K,D)$ is σ-α-lower porous in $C^b_\omega(K,D)$ for some $\alpha > 0$. We will give a partially negative answer.

Proposition 6. *Let $\lambda \in (0,1)$ and $\alpha > 2\frac{1-\lambda}{2-\lambda}$. If (additionally) D has a nonempty interior, then the set $C^b_{\lambda\omega}(K,D)$ is not α-lower porous in $C^b_\omega(K,D)$.*

Proof. Fix any $\lambda \in (0,1)$ and let $\delta \in (0,1)$ be such that $\frac{1-\lambda}{2-\lambda} < \delta < \frac{\alpha}{2}$. In particular,

$$(1-\lambda)(1-\delta) < \delta. \tag{7}$$

Now take any $y \in \operatorname{Int} D$, and let $r > 0$ be such that $B(y, 2r) \subset D$. Define $f(x) = y$ for every $x \in K$. Then, clearly, $f \in C^b_{\lambda\omega}(K,D)$. Now let $g \in C^b_\omega(K,D)$ be such that

$$B(g,\delta r) \subset B(f,r). \tag{8}$$

It is enough to show that $B(g,\delta r) \cap C^b_{\lambda\omega}(K,D) \neq \emptyset$. At first observe that

$$||f-g|| \leq (1-\delta)r. \tag{9}$$

Indeed, assume on the contrary that it is not the case. Then for some $x_0 \in K$, $||g(x_0)-y|| > (1-\delta)r$, so we can take $z \in D$ with $||z-y|| = r$ and $||z-g(x_0)|| < \delta r$. Now for every $x \in K$, define $\tilde{g}(x) = g(x) + z - g(x_0)$. Since for every $x \in K$,

$$||\tilde{g}(x)-y|| \leq ||\tilde{g}(x)-g(x)|| + ||g(x)-y|| \ \leq \ ||z-g(x_0)|| + ||f-g|| < \delta r + r < 2r,$$

we get that $\tilde{g}(x) \in D$, and therefore $\tilde{g} \in C^b_\omega(K,D)$. On the other hand, $||f-\tilde{g}|| \geq ||y-\tilde{g}(x_0)|| = ||y-z|| = r$ and $||g-\tilde{g}|| = ||z-g(x_0)|| < \delta r$, so we get the contradiction with (8). Hence (9) holds. Now define the mapping a in the following way:

$$\forall_{x\in K}\ a(x) = \lambda g(x) + (1-\lambda)y.$$

It can easily be checked that $a \in C^b_{\lambda\omega}(K,D)$. Moreover, by (9) and (7), for every $x \in K$,

$$||a(x) - g(x)|| = (1-\lambda)||g(x) - y|| \leq (1-\lambda)(1-\delta)r < \delta r,$$

so $||a-g|| < \delta r$. This ends the proof. □

Since $\lim_{\lambda\to 1} 2\frac{1-\lambda}{2-\lambda} = 0$, there is no $\alpha > 0$ such that for every $\lambda \in (0,1)$, $C^b_{\lambda\omega}(K,D)$ is α-lower porous. However, the question whether the set $kB^b_\omega(K,D)$ is σ-α-lower porous (for some $\alpha > 0$) in $C^b_\omega(K,D)$, is open.

4. Mappings with unbounded domains

Assume additionally that K is convex and unbounded and also D is unbounded. Then we can strengthen part (ii) of Theorem 4 and part (iii) of Corollary 5. If $t_0 > 0$ and $\lambda \in (0,1)$, then by $C^{\lambda,t_0}_\omega(K,D)$ we denote the following set:

$$C^{\lambda,t_0}_\omega(K,D) = \{f \in C_\omega(K,D) : \omega_f(t_0) \leq \lambda\omega(t_0)\}.$$

Theorem 7. *For every $t_0 > 0$ and $\lambda \in (0,1)$, the set $C^{\lambda,t_0}_\omega(K,D)$ is nowhere dense in $C_\omega(K,D)$.*

Before we prove the above result, we will present its corollary:

Corollary 8. *The set*

$$C_{<\omega}(K,D) = \{f \in C_\omega(K,D) : \exists t > 0\ \omega_f(t) < \omega(t)\}$$

is of the first category in $C_\omega(K,D)$.

Proof. Let $\mathbb{Q}$ stand for the set of all positive rationals. Then we have

$$C_{<\omega}(K,D) = \bigcup_{q\in\mathbb{Q}} \bigcup_{n\in\mathbb{N}} C^{\frac{n-1}{n},q}_\omega(K,D).$$

Indeed, if $f \in C_\omega(K,D)$ is such that for any $q \in \mathbb{Q}$ and any $n \in \mathbb{N}$, $f \notin C_\omega^{\frac{n-1}{n},q}(K,D)$, then for any $q \in \mathbb{Q}$, $\omega_f(q) = \omega(q)$. Since ω is continuous and ω_f is nondecreasing, $\omega_f(t) = \omega(t)$ for every $t > 0$. The result follows. □

In [RZ3] (see also [RZ1] and [R]) it was shown that if K is a closed convex and bounded subset of any Banach space, then the set of all Rakotch contractions is large in the space of all nonexpansive mappings in the sense that its complement is σ-porous ($f : K \to K$ is said to be a Rakotch contraction, if there exists a nonincreasing function $\phi : [0,\infty) \to [0,1]$ such that $\phi(t) < 1$ for every $t > 0$, and for any $x, y \in K$, $||f(x) - f(y)|| \leq \phi(||x-y||)||x-y||$). Corollary 8 shows that

Corollary 9. *If K is an unbounded subset of a Hilbert space, then the set of all Rakotch contractions is meager in the space of all nonexpansive self-mappings.*

Now we will give the proof of Theorem 7:

Proof. It is obvious that if $f \in C_\omega^{\lambda,t_0}(K,D)$, then

$$\text{either } \left(\sup_{t>0} \omega_f(t) \leq \lambda\omega(t_0)\right) \text{ or } \left(\omega_f(t_0) \leq \lambda\omega(t_0) \text{ and } \sup_{t>0} \omega_f(t) > \lambda\omega(t_0)\right).$$

Hence it suffices to show that for every $\delta \in (0, \omega(t_0))$, the following sets

$$C_1 = \left\{ f \in C_\omega(K,D) : \sup_{t>0} \omega_f(t) \leq \delta \right\}$$

and

$$C_2 = \left\{ f \in C_\omega(K,D) : \omega_f(t_0) \leq \delta \text{ and } \sup_{t>0} \omega_f(t) > \delta \right\}$$

are nowhere dense in $C_\omega(K,D)$.

We first show that C_1 is nowhere dense. Let $f \in C_1$, $n \in \mathbb{N}$ and $\epsilon > 0$. W have to show that there exist $g \in C_\omega(K,D)$, $m \in \mathbb{N}$ and $\epsilon_1 > 0$ such that $B(g,(m,\epsilon_1)) \subset B(f,(n,\epsilon)) \setminus C_1$ (again, instead of $B^{C_\omega(K,D)}(f,(n,\epsilon))$, we simply write $B(f,(n,\epsilon))$). Since $\omega_f(t) \leq \delta$ for any $t > 0$, we have that $f(B(0,n) \cap K)$ is bounded. Put

$$p(z) = \sup\{||z-y|| : y \in f(B(0,n) \cap K)\} \text{ for any } z \in D.$$

It is well known that $p : D \to [0,\infty)$ is continuous (even nonexpansive). Since $diam f(B(0,n) \cap K) \leq \sup_{t>0} \omega_f(t)$, we have that $p(z) \leq \delta$ for any $z \in f(B(0,n) \cap K)$. By the fact that D is unbounded and connected, we obtain $[\delta,\infty) \subset p(D)$. Therefore there exist $t' > 0$ and $z_0 \in D$

such that $\delta < p(z_0) < \omega(t')$. Now let $x_0 \in K$ be such that $||x_0|| > n + t'$. We are ready to define g:

- $g(x_0) = z_0$;
- $g(x) = f(x)$ for $x \in B(0,n) \cap K$.

Note that if $x \in B(0,n) \cap K$, then $t' < ||x_0|| - n < ||x_0|| - ||x|| \leq ||x - x_0||$, so

$$||g(x) - g(x_0)|| \leq p(z_0) \leq \omega(t') \leq \omega(||x - x_0||).$$

Hence the modulus of continuity of $g : \{x_0\} \cup (B(0,n) \cap K) \to D$ is bounded by ω. By Corollary 3, we can extend g to a mapping $\tilde{g}$ so that $\tilde{g} \in C_\omega(K, D)$. Denote this extension also by g. Now let $m \in \mathbb{N}$ be such that $||x_0|| < m$ (in particular, $m > n$). Since $p(z_0) > \delta$, there exists $x_1 \in B(0,n) \cap K$ with $||z_0 - f(x_1)|| > \delta$. Set

$$\epsilon_1 = \min\left\{\epsilon, \frac{||z_0 - f(x_1)|| - \delta}{4}\right\}.$$

We will show that $B(g,(m,\epsilon_1)) \subset B(f,(n,\epsilon)) \setminus C_1$. If $h \in B(g,(m,\epsilon_1))$, then

$$\omega_h(||x_0 - x_1||) \geq ||h(x_0) - h(x_1)|| \geq ||g(x_0) - g(x_1)|| - 2\epsilon_1 \geq$$
$$||z_0 - f(x_1)|| - \frac{||z_0 - f(x_1)|| - \delta}{2} = \frac{||z_0 - f(x_1)|| + \delta}{2} > \delta,$$

so $h \notin C_1$. On the other hand, if $x \in B(0,n) \cap K$, then $x \in B(0,m) \cap K$, and therefore

$$||h(x) - f(x)|| = ||h(x) - g(x)|| < \epsilon_1 \leq \epsilon,$$

so $h \in B(f,(n,\epsilon))$. Thus C_1 is nowhere dense.

Now we prove that C_2 is nowhere dense. Let $f \in C_2$, $n \in \mathbb{N}$ and $\epsilon > 0$. Since $\sup_{t>0} \omega_f(t) > \delta$, there exist $x_0, y_0 \in K$ such that $||f(x_0) - f(y_0)|| > \delta$. Let $n' \geq n$ be such that $x_0, y_0 \in B(0,n') \cap K$. Since K is unbounded, we can choose $x' \in K$ such that $||x_0 - x'|| > 3n' + ||x_0|| + t_0$. Then

$$||x'|| \geq ||x_0 - x'|| - ||x_0|| > 3n' + t_0 > 3n'.$$

Fix $y' \in [x_0, x']$ so that $||x' - y'|| = t_0$. Then also

$$||y'|| \geq ||x_0 - y'|| - ||x_0|| > 3n' + ||x_0|| - ||x_0|| = 3n'.$$

Since $\delta < \omega(t_0)$ and $\delta < ||f(x_0) - f(y_0)||$, there exists $\gamma \in (0,1)$ such that

$$\omega(t_0) > (1-\gamma)||f(x_0) - f(y_0)|| > \delta. \tag{10}$$

We are ready to define g:

- $g(x') = f(x_0)$;
- $g(y') = \gamma f(x_0) + (1-\gamma) f(y_0)$;
- $g(x) = f(x)$ for $x \in B(0,n) \cap K$.

Now if $x \in B(0,n) \cap K$, then

$$||x - x_0|| \leq ||x|| + ||x_0|| \leq n + n' \leq 2n'$$

and

$$||x - x'|| \geq ||x'|| - ||x|| > 3n' - n \geq 2n',$$

hence

$$||g(x) - g(x')|| = ||f(x) - f(x_0)|| \leq \omega(||x - x_0||) \leq \omega(2n') \leq \omega(||x - x'||).$$

Similarly, $||x - y_0|| \leq 2n'$ and $||x - y'|| \geq 2n'$, so

$$||g(x) - g(y')|| = ||f(x) - \gamma f(x_0) - (1-\gamma) f(y_0)|| \leq \gamma ||f(x) - f(x_0)|| + (1-\gamma)||f(x) - f(y_0)|| \leq$$

$$\leq \gamma\omega(2n') + (1-\gamma)\omega(2n') \leq \omega(||x - y'||).$$

Moreover, by (10), we have

$$||g(x') - g(y')|| = (1-\gamma)||f(x_0) - f(y_0)|| \leq \omega(t_0) = \omega(||x' - y'||).$$

Therefore we can extend g to the whole set K so that $g \in C_\omega(K, D)$. Now let $m \in \mathbb{N}$ be such that

$$(K \cap B(0,n)) \cup \{x', y'\} \subset B(0,m)$$

and let

$$\epsilon_1 = \min\left\{\epsilon, \frac{||g(x') - g(y')|| - \delta}{4}\right\}.$$

By (10), $\epsilon_1 > 0$. If $h \in B(g, (m, \epsilon_1))$, then

$$\omega_h(t_0) \geq ||h(x') - h(y')|| \geq ||g(x') - g(y')|| - 2\epsilon_1 \geq \frac{||g(x') - g(y')|| + \delta}{2} > \delta,$$

hence $h \notin C_2$. On the other hand, $B(g, (m, \epsilon_1)) \subset B(f, (n, \epsilon))$, so the result follows. □

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