K–theory from the point of view of C^*–algebras and Fredholm representations

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1. Topological K–theory as a cohomology theory. Bott periodicity. Relation between real, complex and quaternionian K–theory.
2. Homology K-theory. Algebraic categorical setting.
3. Index of elliptic operators as an invariant of homology K–theory
4. C^*–algebras, Hilbert C^*-modules and Fredholm operators. Homotopical point of view, multipliers.
5. KK–theory and conjectures of Baum–Connes and Novikov. Higher signature, C^*–signature of non simply connected manifolds.

1 Topological K–theory

1.1 Locally trivial bundles, the structure groups, principal bundles

Definition 1 *Let E and B be two topological spaces with a continuous map*

$$p : E \longrightarrow B.$$

The map p is said to define a locally trivial bundle if there is a topological space F such that for any point $x \in B$ there is a neighborhood $U \ni x$ for which the inverse image $p^{-1}(U)$ is homeomorphic to the Cartesian product $U \times F$. Moreover, it is required that the homeomorphism φ fiberwise. This means in algebraic terms that the diagram

$$\begin{array}{ccccc} U \times F & \xrightarrow{\varphi} & p^{-1}(U) & \subset & E \\ \big\downarrow \pi & & \big\downarrow p & & \big\downarrow p \\ U & = & U & \subset & B \end{array}$$

is commutative where

$$\pi : U \times F \longrightarrow U, \quad \pi(x, f) = x$$

is the projection onto the first factor. The space E is called total space of the bundle or the fiberspace, the space B is called the base of the bundle, the space F is called the fiber of the bundle and the mapping p is called the projection.

One can give an equivalent definition using so called *transition functions:*

Definition 2 *Let B and F be two topological spaces and $\{U_\alpha\}$ be a covering of the space B by a family of open sets. The system of homeomorphisms which form the commutative diagram*

$$\begin{array}{ccc} (U_\alpha \cap U_\beta) \times F & \stackrel{\varphi_{\alpha\beta}}{\longrightarrow} & (U_\alpha \cap U_\beta) \times F \\ \downarrow & & \downarrow \\ (U_\alpha \cap U_\beta) & = & (U_\alpha \cap U_\beta \end{array} \tag{1}$$

and satisfy the relations

$$\begin{array}{rl} \varphi_{\alpha\gamma}\varphi_{\gamma\beta}\varphi_{\beta\alpha} = \mathbf{Id}, & \text{for any three indices } \alpha, \beta, \gamma \\ & \text{on the intersection } (U_\alpha \cap U_\beta \cap U_\gamma) \times F \\ \varphi_{\alpha\alpha} = \mathbf{Id} & \text{for each } \alpha. \end{array} \tag{2}$$

By analogy with the terminology for smooth manifolds, the open sets U_α are called *charts*, the family $\{U_\alpha\}$ is called *the atlas of charts*, the homeomorphisms

$$\begin{array}{ccccc} U_\alpha \times F & \stackrel{\varphi_\alpha}{\longrightarrow} & p^{-1}(U_\alpha) & \subset & E \\ \downarrow & & \downarrow & & \downarrow \\ U_\alpha & = & U_\alpha & \subset & B \end{array} \tag{3}$$

are called *the coordinate homeomorphisms* and the $\varphi_{\alpha\beta}$ are called *the transition functions* or *the sewing functions.*

Two systems of the transition functions $\varphi_{\beta\alpha}$, and $\varphi'_{\beta\alpha}$ define isomorphic locally trivial bundles iff there exist fiber preserving homeomorphisms

$$\begin{array}{ccc} U_\alpha \times F & \stackrel{h_\alpha}{\longrightarrow} & U_\alpha \times F \\ \downarrow & & \downarrow \\ U_\alpha & = & U_\alpha \end{array}$$

such that

$$\varphi_{\beta\alpha} = h_\beta^{-1}\varphi'_{\beta\alpha}h_\alpha. \tag{4}$$

Let **Homeo** (F) be the group of all homeomorphisms of the fiber F. Each fiberwise homeomorphism

$$\varphi : U \times F \longrightarrow U \times F, \tag{5}$$

defines a map

$$\bar{\varphi} : U \longrightarrow \mathbf{Homeo}\,(F), \tag{6}$$

So instead of $\varphi_{\alpha\beta}$ a family of functions

$$\bar{\varphi}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbf{Homeo}\,(F),$$

can be defined on the intersection $U_\alpha \cap U_\beta$ and having values in the group **Homeo** (F).

The condition (**??**) means that

$$\begin{gathered}\bar{\varphi}_{\alpha\alpha}(x) = \mathbf{Id}, \\ \bar{\varphi}_{\alpha\gamma}(x)\bar{\varphi}_{\gamma\beta}(x)\bar{\varphi}_{\beta\alpha}(x) = \mathbf{Id}. \\ x \in U_\alpha \cap U_\beta \cap U_\gamma.\end{gathered}$$

and we say that the cochain $\{\bar{\varphi}_{\alpha\beta}\}$ is a *cocycle*.

The condition (4) means that there is a zero dimensional cochain $h_\alpha : U_\alpha \longrightarrow \mathbf{Homeo}\,(F)$ such that

$$\bar{\varphi}_{\beta\alpha}(x) = h_\beta^{-1}(x)\bar{\varphi}'_{\beta\alpha}(x)h_\alpha(x),\ x \in U_\alpha \cap U_\beta.$$

Using the language of homological algebra the condition (4) means that cocycles $\{\bar{\varphi}_{\beta\alpha}\}$ and $\{\bar{\varphi}'_{\beta\alpha}\}$ are cohomologous. Thus the family of locally trivial bundles with fiber F and base B is in one to one correspondence with the one dimensional cohomology of the space B with coefficients in the sheaf of the germs of continuous **Homeo** (F)–valued functions for given open covering $\{U_\alpha\}$. Despite obtaining a simple description of the family of locally trivial bundles in terms of homological algebra, it is ineffective since there is no simple method of calculating cohomologies of this kind.

Nevertheless, this representation of the transition functions as a cocycle turns out very useful because of the situation described below.

First of all notice that using the new interpretation a locally trivial bundle is determined by the base B, the atlas $\{U_\alpha\}$ and the functions $\{\varphi_{\alpha\beta}\}$ taking the value in the group $G = \mathbf{Homeo}\,(F)$. The fiber F itself does not directly take part in the description of the bundle. Hence, one can at first describe a locally trivial bundle as a family of functions $\{\varphi_{\alpha\beta}\}$ with values in some topological group G, and after that construct the total space of the bundle with fiber F by additionally defining an action of the group G on the space F, that is, defining a continuous homomorphism of the group G into the group **Homeo** (F).

Secondly, the notion of locally trivial bundle can be generalized and the structure of bundle made richer by requiring that both the transition functions $\bar{\varphi}_{\alpha\beta}$ and the functions h_α are not arbitrary but take values in some subgroup of the homeomorphism group **Homeo** (F).

Thirdly, sometimes information about locally trivial bundle may be obtained by substituting some other fiber F' for the fiber F but using the 'same' transition functions. Thus we come to a new definition of a locally trivial bundle with additional structure — the group where the transition functions take their values, so called *the structure group*.

Definition 3 *A locally trivial bundle with the structure group G is called principal G–bundle if $F = G$ and action of the group G on F is defined by the left translations.*

Theorem 1 *Let $p : E \longrightarrow B$ be a principal G–bundle. Then there is a right action of the group G on the total space E such that:*
1) the right action of the group G is fiberwise,
2) the homeomorphism φ_α^{-1} transforms the right action of the group G on the total space into right translations on the second factor.

Using the transition functions it is very easy to define so called *inverse image* of the bundle. Namely, let $p : E \longrightarrow B$ be a locally trivial bundle with the structure group G and the collection of transition functions $\varphi_{\alpha\beta} : U_{\alpha\beta} \longrightarrow G$ and let $f : B' \longrightarrow B$ be a continuous mapping. Then the inverse image $f^*(p : E \longrightarrow B)$ is defined as a collection of charts $U'_\alpha = f^{-1}(U_\alpha)$ and a collection of transition functions $\varphi'_{\alpha\beta}(x) = \varphi_{\alpha\beta}(f(x))$.

Another geometric definition of the inverse bundle arises from the diagram

$$\begin{array}{ccccc} E' = f^*(E) & \subset & E \times B' & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ B' & = & B' & \xrightarrow{f} & B \end{array} \tag{7}$$

where E' consists of points $(e, b') \in E \times B'$ such that $f(b') = p(e)$. The map $\widehat{f} : f^*(E) \longrightarrow E$ is canonically defined by the map f.

Theorem 2 *Let*

$$\psi : E' \longrightarrow E \tag{8}$$

be a continuous map of total spaces for principal G–bundles over bases B' and B. The map (8) is generated by a continuous map $f : B' \longrightarrow B$ if and only if the map ψ is equvivariant (with respect to right actions of the structure group G on the total spaces).

1.2 Homotopy properties, classifying spaces

Theorem 3 *The inverse images with respect to homotopic mappings are isomorphic bundles.*

Therefore the category of all bundles with structure group G, $Bndls_G(B)$ forms a homotopy functor from the category of CW-spaces to the category of sets.

Definition 4 *A principal bundle $p : E_G \longrightarrow B_G$ is called classifying bundle iff for any CW–space B one has one to one correspondence*

$$Bndls_G(B) \approx [B, B_G] \tag{9}$$

generated by the map

$$\begin{array}{c} \varphi : [B, B_G] \longrightarrow Bngls_G(B), \\ \varphi(f) = f^*(p : E_G \longrightarrow B_G). \end{array} \tag{10}$$

Theorem 4 *The principal G–bundle,*

$$p_G : E_G \longrightarrow B_G \tag{11}$$

is classifying bundle if all homotopy groups of the total space E_G are trivial:

$$\pi_i(E_G) = 0,\ 0 \leq i < \infty. \tag{12}$$

1.3 Characteristic classes

Definition 5 *A mapping $\alpha : Bndls_G(B) \longrightarrow H^*(B)$ is called the characteristic class if the following diagram is commutative*

$$\begin{array}{ccc} Bndls_G(B) & \xrightarrow{\alpha} & H^*(B) \\ \Big\downarrow f^* & & \Big\downarrow f^* \\ Bndls_G(B') & \xrightarrow{\alpha} & H^*(B') \end{array} \tag{13}$$

for any continuous mapping $f : B' \longrightarrow B$, that is α is a natural transformation of functors.

Theorem 5 *The family of all characteristic classes is in one-to-one correspondence with the cohomology $H^*(B_G)$ by the assignment*

$$\begin{array}{c} \alpha(x)(\xi) = f^*(x) \quad \mathit{for} \\ x \in H^*(B_G), \qquad f : B \longrightarrow B_G, \qquad \xi = f^*(p : E_G \longrightarrow B_G). \end{array} \tag{14}$$

1.4 Vector bundles, K–theory, Bott periodicity

The special (and very important) class of locally trivial bundles are so called (real) vector bundles with the structure groups $GL(n, \mathbf{R})$ and the fiber $\mathbf{R}^n$. The structure group can be reduced to the subgroup $O(n)$. If the structure group $O(1)$ can be reduced to the subgroup $G = SO(1)$ then the vector bundle is trivial and denoted by $\overline{\mathbf{1}}$.

Similar version arise for other structure groups:

1) Complex vector bundle with the structure group $GL(n, \mathbf{C})$ and the fiber $\mathbf{C}^n$.

2) Quaternionic vector bundle with the structure group $GL(n, \mathbf{K})$ and the fiber $\mathbf{K}^n$, where $\mathbf{K}$ is the (non commutative) field of quaternions.

All of them admit useful algebraic operation:

1. Direct sum, $\xi = xi_1 \oplus \xi_2$,
2. tensor product,$\xi = xi_1 \otimes \xi_2$,
3. other tensor operations, **HOM** (ξ_1, ξ_2),$\Lambda_k(\xi)$. etc.

Let ξ be a vector bundle over the field $F = \mathbf{R}, \mathbf{C}, \mathbf{K}$. Let $\Gamma(\xi)$ be the space of all sections of the bundle ξ. Then $\Gamma(\overline{\mathbf{1}}) = C(B)$ — the ring of all continuous functions with values in F.

Theorem 6 *The space has a natural structure of (left) $C(B)$–module by fiberwise multiplication. If B is compact space then $\Gamma(\xi)$ is projective finite generated $C(B)$–module. Inverse, each projective finite generated $C(B)$–module can be presented as a space $\Gamma(\xi)$ for a vector bundle ξ.*

1.4.1 Different tools for description of vector bundles

1. System of transition functions,
2. Projective modules,
3. Projective values function,
4. Difference construction,
5. Exact sequence of vector bundles.

Definition 6 *Let $K(X)$ denotes the Abelian group where the generators are (isomorphism classes of) vector bundles over the base X subject to the following relations:*

$$[\xi]+[\eta]-[\xi\oplus\eta]=0 \tag{15}$$

for vector bundles ξ and η, and where $[\xi]$ denotes the element of the group $K(X)$ defined by the vector bundle ξ.

The group defined in Definition 6 is called *the Grothendieck group* of the category of all vector bundles over the base X.

To avoid confusion, the group generated by all real vector bundles will be denoted by $K_O(X)$, the group generated by all complex vector bundles will be denoted by $K_U(X)$ and the group generated by all quaternionic vector bundles will be denoted by $K_{Sp}(X)$.

Let $K^0(X,x_0)$ denote the kernel of the homomorphism $K(X)\longrightarrow K(x_0)$:

$$K^0(X,x_0)=\mathbf{Ker}\ \left(K(X)\longrightarrow K(x_0)\right).$$

Elements of the subring $K^0(X,x_0)$ are represented by differences $[\xi]-[\eta]$ for which $\dim\xi=\dim h$. The elements of the ring $K(X)$ are called *virtual* bundles and elements of the ring $K^0(X,x_0)$ are *virtual* bundles of trivial dimension over the point x_0.

Now consider a pair (X,Y) of the cellular spaces, $Y\subset X$. Denote by $K^0(X,Y)$ the ring

$$K^0(X,Y)=K^0(X/Y,[Y]).$$

For any negative integer $-n$, let

$$K^{-n}(X,Y)=K^0(S^nX,S^nY)$$

where $S^n(X)$ denotes the n–times suspension of the space X:

$$S^nX=(S^n\times X)/(S^n\vee X).$$

Theorem 7 *The pair (X,Y) induces an exact sequence*

$$\begin{array}{ll} K^0(Y,x_0)\longleftarrow K^0(X,x_0)\longleftarrow K^0(X,Y)\longleftarrow & \\ \quad\longleftarrow K^{-1}(Y,x_0)\longleftarrow K^{-1}(X,x_0)\longleftarrow K^{-1}(X,Y)\longleftarrow & \\ \quad\longleftarrow K^{-2}(Y,x_0)\longleftarrow K^{-2}(X,x_0)\longleftarrow K^{-2}(X,Y)\longleftarrow\ldots & \\ \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \ldots\ \longleftarrow K^{-n}(Y,x_0)\longleftarrow K^{-n}(X,x_0)\longleftarrow K^{-n}(X,Y)\longleftarrow\ldots & \\ \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \end{array} \tag{16}$$

Consider a complex n–dimensional vector bundle ξ over the base X and let $p : E \longrightarrow X$ be the projection of the total space E onto the base X. Consider the space E as a new base space and a complex of vector bundles

$$0 \longrightarrow \Lambda_0\eta \xrightarrow{d_0} \Lambda_1\eta \xrightarrow{d_1} \Lambda_2\eta \xrightarrow{d_2} \ldots \xrightarrow{d_{n-1}} \Lambda_n\eta \longrightarrow 0, \tag{17}$$

where $\eta = p^*\xi$ is inverse image of the bundle ξ and the homomorphism

$$d_k : \Lambda_k\eta \longrightarrow \Lambda_{k+1}\eta$$

is defined as the exterior multiplication by the vector $y \in E,\ y \in \xi_x,\ x = p(y)$. It is known that if the vector $y \in \xi_x$ is nonzero, $y \neq 0$, then the complex (17) is exact. Consider the subspace $D(\xi) \subset E$ which consists of all vectors $y \in E$ such that $|y| \leq 1$. Then the subspace $S(\xi) \subset D(\xi)$ of all unit vectors gives the pair $(D\xi), S(\xi))$ for which the complex (17) is exact on $S(\xi)$. Denote the element defined by (17) by $\beta(\xi) \in K^0(D(\xi), S(\xi))$. Then one has homomorphism given by multiplication by the element $\beta(\xi)$

$$\beta : K(X) \longrightarrow K^0\left(D(\xi), S(\xi)\right). \tag{18}$$

The homomorphism (18) is called *the Bott homomorphism.*

In particular the Bott element $\beta \in K^0(\mathbf{S}^2, s_0) = K^{-2}(\mathbf{S}^0, s_0) = \mathbf{Z}$ generates a homomorphism $\tilde{h}$:

$$K^{-n}(X, Y) \xrightarrow{\otimes\beta} K^{-(n+2)}(X, Y) \tag{19}$$

which is called *the Bott periodicity isomorphism* and hence forms a periodic cohomology K–theory.

1.5 Relations between complex, symplectic and real bundles

Let G be a compact Lie group. A *G–space* X is a topological space X with continuous action of the group G on it. The map $f : X \longrightarrow Y$ is said to be *equivariant* if

$$f(gx) = gf(x), g \in G.$$

Similarly, if f is a locally trivial bundle and also equivariant then f is called an *equivariant* locally trivial bundle. An equivariant vector bundle is defined similarly. The theory of equivariant vector bundles is very similar to the classical theory. In particular, equivariant vector bundles admit the operations of direct sum and tensor product. In certain simple cases the description of equivariant vector bundles is reduced to the description of the usual vector bundles.

The category of G– equivariant vector bundles is good place to give consistent descriptions of three different structures on vector bundles — complex, real and symplectic.

Consider the group $G = \mathbf{Z}_2$ and a complex vector bundle ξ over the G–space X. This means that the group G acts on the space X. Let E be the total space of the bundle ξ and let

$$p : E \longrightarrow X$$

be the projection in the definition of the vector bundle ξ. Let G act on the total space E as a fiberwise operator which is linear over the reals and anticomplex over complex numbers, that is, if $\tau \in G = \mathbf{Z}_2$ is the generator then

$$\tau(\lambda x) = \bar{\lambda}\tau(x), \ \lambda \in \mathbf{C}, \ x \in E. \tag{20}$$

A vector bundle ξ with the action of the group G satisfying the condition (20) is called a *KR–bundle* . The operator τ is called the anticomplex involution. The corresponding Grothendieck group of KR–bundles is denoted by $KR(X)$.

Below we describe some of the relations with classical real and complex vector bundles.

Proposition 1 *Suppose that the G–space X has the form $X = Y \times \mathbf{Z}_2$ and the involution τ transposes the second factor. Then the group $KR(X)$ is naturally isomorphic to the group $K_U(Y)$ and this isomorphism coincides with restriction of a vector bundle to the summand $Y \times \{1\}$, $1 \in G = \mathbf{Z}_2$, ignoring the involution τ.*

Proposition 2 *Suppose the involution τ on X is trivial. Then*

$$KR(X) \approx K_O(X). \tag{21}$$

The isomorphism (21) associates to any KR–bundle the fixed points of the involution τ.

Proposition 3 *The operation of forgetting the involution induces a homomorphism*

$$KR(X) \longrightarrow K_U(X)$$

and when the involution is trivial on the base X this homomorphism coincides with complexification

$$c : K_O(X) \longrightarrow K_U(X).$$

Moreover, the proof of Bott periodicity can be extended word by word to KR–theory:

Theorem 8 *There is an element $\beta \in KR(D^{1,1}, S^{1,1}) = KR^{-1,-1}(\mathbf{pt})$ such that the homomorphism given by multiplication by β*

$$\beta : KR^{p,q}(X,Y) \longrightarrow KR^{p-1,q-1}(X.Y) \tag{22}$$

is an isomorphism.

It turns out that this scheme can be modified so that it includes another type of K–theory – that of quaternionic vector bundles.

Let K be the(noncommutative) field of quaternions. As for real or complex vector bundles we can consider locally trivial vector bundles with fiber K^n and structure group $\mathbf{GL}(n, K)$, the so called quaternionic vector bundles. Each quaternionic vector bundle can be considered as a complex vector bundle

$p : E \longrightarrow X$ with additional structure defined by a fiberwise anticomplex linear operator J such that

$$J^2 = -1, IJ + JI = 0,$$

where I is fiberwise multiplication by the imaginary unit.

More generally, let J be a fiberwise anticomplex linear operator which acts on a complex vector bundles ξ and satisfies

$$J^4 = 1, IJ + JI = 0. \tag{23}$$

Then the vector bundle ξ can be split into two summands $\xi = \xi_1 \oplus \xi_2$ both invariant under action of J, that is, $J = J_1 \oplus J_2$ such that

$$J_1^2 = 1,\ J_2^2 = -1. \tag{24}$$

Hence the vector bundle ξ_1 is the complexification of a real vector bundle and ξ_2 is quaternionic vector bundle.

Consider a similar situation over a base X with an involution τ such that the operator (23) commutes with τ. Such vector bundle will be called a KRS–*bundle* .

Lemma 1 *A KRS–bundle ξ is split into an equivariant direct sum $\xi = \xi_1 \oplus \xi_2$ such that $J^2 = 1$ on ξ_1 and $J^2 = -1$ on ξ_2.*

Lemma 1 shows that the Grothendieck group $KRS(X)$ generated by KRS–bundles has a $\mathbf{Z}_2$–grading, that is,

$$KRS(X) = KRS_0(X) \oplus KRS_1(X).$$

It is clear that $KRS_0(X) = KR(X)$. In the case when the involution τ acts trivially, $KRS_1(X) = K_Q(X)$, that is,

$$KRS(X) = K_O(X) \oplus K_Q(X)$$

where $K_Q(X)$ is the group generated by quaternionic bundles.

q	-8	-7	-6	-5	-4	-3	-2	-1	0
K_O	$\mathbf{Z}$ $\alpha=\gamma^2$	0	0	0	$\mathbf{Z}$ $u\gamma$	0	$\mathbf{Z}_2$ h^2	$\mathbf{Z}_2$ h	$\mathbf{Z}$ $u^2=4$
K_Q	$\mathbf{Z}$ $u\gamma^2$	0	$\mathbf{Z}_2$ $h^2\gamma$	$\mathbf{Z}_2$ $h\gamma$	$\mathbf{Z}$ γ	0	0	0	$\mathbf{Z}$ u

Figure 1: A list of the groups K_O and K_Q.

2 Index of elliptic operators as an invariant of homology K–theory, Atiyah homology K–theory as an ancestor of KK–theory

2.1 Homology K–theory. Algebraic categorical setting

A naive point of view on homology theory is that the homology groups dual to the cohomology groups $h^*(X)$ should be considered as

$$h_*(X) \stackrel{def}{=} \mathbf{Hom}\,(h^*(X), \mathbf{Z}). \tag{25}$$

This naive definition is not good since gives non exact functor. More appropriate definition is the following.

Consider a natural transformation of functors

$$\alpha_Y : h^*(X \times Y) \longrightarrow h^*(Y) \tag{26}$$

which is the homomorphism of $h^*(Y)$–modules and for continuous mapping $f : Y' \longrightarrow Y$ gives the commutative diagram

$$\begin{array}{ccc} h^*(X \times Y) & \xrightarrow{\alpha} & h^*(Y) \\ \Big\downarrow (\mathbf{Id} \times f)^* & & \Big\downarrow f^* \\ h^*(X \times Y') & \xrightarrow{\alpha} & h^*(Y') \end{array} \tag{27}$$

Let be the family of all natural transformations of the type (26, 27). The functor $h^*(X)$ defines a homology theory.

2.2 PDO

Consider a linear differential operator A which acts on the space of smooth functions of n real variables:

$$A : C^\infty(\mathbf{R}^n) \longrightarrow C^\infty(\mathbf{R}^n).$$

and is presented as a finite linear combination of partial derivatives

$$A = \sum_{|\alpha| \le m} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}. \tag{28}$$

Put

$$a(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha i^{|\alpha|}.$$

The function $a(x, \xi)$ is called the *symbol of a differential operator* A. The operator A can be reconstructed from its symbol as

$$A = a\left(x, \frac{1}{i}\frac{\partial}{\partial x}\right).$$

Since the symbol is polynomial with respect to variables ξ, it can be split into homogeneous summands

$$a(x, \xi) = a_m(x, \xi) + a_{m-1}(x, \xi) + \ldots + a_0(x, \xi).$$

The highest term $a_m(x, x)$ is called the *principal symbol* of the operator A while whole symbol sometimes is called the *full symbol*.

Proposition 4 *Let $y = y(x)$ be a smooth change of variables. Then in the new coordinate system the operator B defined by the formula*

$$(Bu)(y) = (Au\,(y(x)))_{x=x(y)}$$

is again a differential operator of order m for which the principal symbol is

$$b_m(y, \eta) = a_m\left(x(y), \eta \frac{\partial y(x(y))}{\partial x}\right). \tag{29}$$

The formula (29) shows that variables ξ change as a tensor of valency $(0,1)$, that is, as components of a cotangent vector.

The concept of a differential operator exists on an arbitrary smooth manifold M. The concept of a whole symbol is not well defined but the principal symbol can be defined as a function on the total space of the cotangent bundle T^*M. It is clear that the differential operator A does not depend on the principal symbol alone but only up to the addition of an operator of smaller order.

The notion of a differential operator can be generalized in various directions. First of all notice that

$$(Au)(x) = \mathbf{F}_{\xi \longrightarrow x}\left(a(x, \xi)\left(\mathbf{F}_{x \longrightarrow \xi}(u)(\xi)\right)\right), \tag{30}$$

where $\mathbf{F}$ is the Fourier transformation.

Hence we can enlarge the family of symbols to include some functions which are not polynomials.

Then the operator A defined by formula (30) with non polynomial symbol is called a *pseudodifferential operator of order* m (more exactly, not greater than m). The pseudodifferential operator A acts in the Schwartz space $\mathbf{S}$.

This definition of a pseudodifferential operator can be extended to the Schwartz space of functions on an arbitrary compact manifold M. Let $\{U_\alpha\}$ be an atlas of charts with a local coordinate system x_α. Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to the atlas of charts, that is,

$$0 \le \varphi_\alpha(x) \le 1, \ \sum_\alpha \varphi_\alpha(x) \equiv 1, \ \mathbf{Supp}\ \varphi_\alpha \subset U_\alpha.$$

Let $\psi_\alpha(x)$ be functions such that

$$\mathbf{Supp}\ \psi_\alpha \subset U_\alpha, \ \varphi_\alpha(x)\psi_\alpha(x) \equiv \varphi_\alpha(x).$$

Define an operator A by the formula

$$A(u)(x) = \sum_\alpha \psi_\alpha(x) A_\alpha \left(\varphi_\alpha(x)u(x)\right), \tag{31}$$

where A_α is a pseudodifferential operator on the chart U_α (which is diffeomorphic to $\mathbf{R}^n$) with principal symbol $a_\alpha(x_\alpha, \xi_\alpha) = a(x, \xi)$.

In general, the operator A depends on the choice of functions ψ_α, φ_α and the local coordinate system x_α, uniquely up to the addition of a pseudodifferential operator of order strictly less than m.

The next useful generalization consists of a change from functions on the manifold M to smooth sections of vector bundles.

The crucial property of the definition (31) is the following

Proposition 5 *Let*

$$a : \pi^*(\xi_1) \longrightarrow \pi^*(\xi_2), ; \ b : \pi^*(\xi_a) \longrightarrow \pi^*(\xi_3), ;$$

be two symbols of orders m_1, m_2. Let $c = ba$ be the composition of the symbols. Then the operator

$$b(D)a(D) - c(D) : \Gamma^\infty(\xi_1) \longrightarrow \Gamma^\infty(\xi_3)$$

is a pseudodifferential operator of order $m_1 + m_2 - 1$.

Proposition 5 leads to a way of solving equations of the form

$$Au = f \tag{32}$$

for certain pseudodifferential operators A. To find a solution of (32), it sufficient to construct a left inverse operator B, that is, $BA = 1$. Usually it is not possible, but a weaker condition can be realized.

Condition 1 *$a(x, \xi)$ is invertible for sufficiently large $|\xi| \ge C$.*

The pseudodifferential operator $A = a(D)$ is called an *elliptic* if Condition 1 holds. If A is elliptic operator than there is an (elliptic) operator $B = b(D)$ such that $AD - \mathbf{Id}$ is the operator of order -1.

The final generalization for elliptic operators is the substitution of a sequence of pseudodifferential operators for a single elliptic operator. Let $\xi_1, \xi_2, \ldots, \xi_k$ be a sequence of vector bundles over the manifold M and let

$$0 \longrightarrow \pi^*(\xi_1) \xrightarrow{a_1} \pi^*(\xi_2) \xrightarrow{a_2} \ldots \xrightarrow{a_{k-1}} \pi^*(\xi_k) \longrightarrow 0 \tag{33}$$

be a sequence of symbols of order $(m_1, \ldots, m_{k-1})$. Suppose the sequence (33) forms a complex, that is, $a_s a_{s-1} = 0$. Then the sequence of operators

$$0 \longrightarrow \Gamma^\infty(\xi_1) \xrightarrow{a_1(D)} \Gamma^\infty(\xi_2) \longrightarrow \ldots \longrightarrow \Gamma^\infty(\xi_k) \longrightarrow 0 \tag{34}$$

in general, does not form a complex because we can only know that the composition $a_k(D)a_{k-1}(D)$ is a pseudodifferential operator of the order less then $m_s + m_{s-1}$.

If the sequence of pseudodifferential operators forms a complex and the sequence of symbols (33) is exact away from a neighborhood of zero section in T^*M then the sequence (34) is called an *elliptic complex* of pseudodifferential operators.

2.3 Fredholm operators

The bounded operator $K : H \longrightarrow H$ is said to be a *compact* if any bounded subset $X \subset H$ is mapped to a precompact set, that is, the set $\overline{F(X)}$ is compact. If $\dim \mathbf{Im}\,(K) < \infty$ then K is called finite dimensional operator. Each finite dimensional operator is compact. If $\lim_{n\to\infty} \|K_n - K\| = 0$ and K_n are compact operators, then K is again compact operator. More of that each compact operator K can be presented as $K = \lim_{n\to\infty} K_n$, where K_n are finite dimensional operators.

The operator F is said to be Fredholm if there is an operator G such that both $K = FG - 1$ and $K' = GF - 1$ are compact.

Theorem 9 *Let F be a Fredholm operator. Then*

1. $\dim \mathbf{Ker}\, F < \infty$, $\dim \mathbf{Coker}\, F < \infty$ *and the image,* $\mathbf{Im}\, F$, *is closed. The number* $\mathbf{index}\, F = \dim \mathbf{Ker}\, F - \dim \mathbf{Coker}\, F$ *is called the index of the Fredholm operator F.*

2. $\mathbf{index}\, F = \dim \mathbf{Ker}\, F - \dim \mathbf{Ker}\, F^*$, *where F^* is the adjoint operator.*

3. *there exists $\varepsilon > 0$ such that if $\|F - G\| < \varepsilon$ then G is a Fredholm operator and*
$$\mathbf{index}\, F = \mathbf{index}\, G,$$

4. *if K is compact then $F + K$ is also Fredholm and*
$$\mathbf{index}\,(F + K) = \mathbf{index}\, F. \tag{35}$$

5. *If F and G are Fredholm operators then the composition FG is Fredholm and*
$$\mathbf{index}\,(FG) = \mathbf{index}\ F + \mathbf{index}\ G.$$

The notion of a Fredholm operator has an interpretation in terms of the finite dimensional homology groups of a complex of Hilbert spaces. In general, consider a sequence of Hilbert spaces and bounded operators

$$0 \longrightarrow C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} C_n \longrightarrow 0. \tag{36}$$

We say that the sequence (36) is *Fredholm complex* if $d_k d_{k-1} 0$, $\mathbf{Im}\ d_k$ is a closed subspace and

$$\dim\left(\mathbf{Ker}\ d_k / \mathbf{Coker}\ d_{k-1}\right) = \dim H\left(C_k, d_k\right) < \infty.$$

Then the *index of Fredholm complex* (36) is defined by the following formula:

$$\mathbf{index}\,(C,d) = \sum_k (-1)^k \dim H(C_k, d_k).$$

Theorem 10 *Let*
$$0 \longrightarrow C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} C_n \longrightarrow 0 \tag{37}$$
be a sequence satisfying the condition that each $d_k d_{k-1}$ is compact. Then the following conditions are equivalent:

1. *There exist operators $f_k : C_k \longrightarrow C_{k-1}$ such that $f_{k+1} d_k + d_{k-1} f_k = 1 + r_k$ where each r_k is compact.*

2. *There exist compact operators s_k such that the sequence of operators $d'_k = d_k + s_k$ forms a Fredholm complex. The index of this Fredholm complex is independent of the operators s_k.*

2.4 Sobolev spaces

Consider an arbitrary compact manifold M and vector bundle ξ. One can define a Sobolev norm on the space of the sections $\Gamma^\infty(M,\xi)$, using the formula

$$\|u\|_s^2 = \int_{\mathbf{R}^n} \bar{u}(x)(1+\Delta)^s u(x)dx,$$

where Δ is the Laplace-Beltrami operator on the manifold M with respect to a Riemannian metric. The Sobolev norm depends on the choice of Riemannian metric, inclusion of the bundle ξ into trivial bundle uniquely equivalent norms. Hence the completion of the space of sections $\Gamma^\infty(M,\xi)$ is defined correctly. We shall denote this completion by $H_s(M,\xi)$.

Theorem 11 *Let M be a compact manifold, ξ be a vector bundle over M and $s_1 < s_2$. Then the natural inclusion*

$$H_{s_2}(M,\xi) \longrightarrow H_{s_1}(M,\xi) \tag{38}$$

is a compact operator.

Theorem 12 *Let*

$$a(D) : \Gamma^{\infty}(M,\xi_1) \longrightarrow \Gamma^{\infty}(M,\xi_2) \tag{39}$$

be a pseudodifferential operator of order m. Then there is a constant C such that

$$\|a(D)u\|_{s-m} \leq C\|u\|_s, \tag{40}$$

that is, the operator $a(D)$ can be extended to a bounded operator on Sobolev spaces:

$$a(D) : H_s(M,\xi_1) \longrightarrow H_{s-m}(M,\xi_2). \tag{41}$$

Using theorems 11 and 12 it can be shown that an elliptic operator is Fredholm for appropriate choices of Sobolev spaces.

Theorem 13 *Let $a(D)$ be an elliptic pseudodifferential operator of order m as in (39). Then its extension (41) is Fredholm. The index of the operator (41) is independent of the choice of the number s.*

2.5 Index of elliptic operators

An elliptic operator $\sigma(D)$ is defined by a symbol

$$\sigma : \pi^*(\xi_1) \longrightarrow \pi^*(\xi_2) \tag{42}$$

which is an isomorphism away from a neighborhood of the zero section of the cotangent bundle T^*M. Since M is a compact manifold, the symbol (42) defines a triple $(\pi^*(\xi_1), \sigma, \pi^*(\xi_2))$ which in turn defines an element

$$[\sigma] \in K_c(T^*M).$$

Theorem 14 *The index* **index** $\sigma(D)$ *of the Fredholm operator $\sigma(D)$ depends only on the element $[\sigma] \in K_c(T^*M)$. The mapping*

$$\mathbf{index} \; : K_c(T^*M) \longrightarrow \mathbf{Z}$$

is an additive homomorphism.

More of that

$$\mathbf{index}\, \sigma(D) = p_*[\sigma], \tag{43}$$

where

$$p_* : K_c(T^*M) \longrightarrow K_c(pt) = \mathbf{Z}$$

is the direct image homomorphism induced by the trivial mapping $p : M \longrightarrow pt$.

2.6 The Atiyah homology theory

The naive idea is that cohomology K–group (with compact supports) of the total space of cotangent bundle of the manifold M, $K_c(T^*M)$, should be isomorphic to a homology K–group due to a Poincare duality, $K_*(T^*M) \approx K_*(M)$. This identification can be arrange as a natural transformation of functors

$$\begin{array}{rcl} D : K_c(T^*M) & \approx & K_*(M), \\ D(\sigma) : K^*(M \times N) & \longrightarrow & K^*(N) \\ D(\sigma)(\xi) & = & \mathbf{index}\,(\sigma \otimes \xi) \in K^*(N). \end{array} \tag{44}$$

Therefore the homology K–groups, $K_*(M)$ can be identified with the collection of triples $\sigma = (H, F, \rho)$, where H is a Hilbert space, F ia a Fredholm operator, $\rho : C(M) \longrightarrow B(H)$ is a representation of the algebra $C(M)$ of all continuous functions to the algebra $B(H)$ of bounded operators, such that for any function $f \in C(M)$ the operator $Ff - fF : H \longrightarrow H$ is compact. If ξ is a vector bundle on M then the space $\Gamma(\xi)$ is the projective finite generated module over $C(M)$. Therefore the operator $F \otimes \mathbf{Id}_\xi : H \otimes_{C(M)} \Gamma(\xi) \longrightarrow H \otimes_{C(M)} \Gamma(\xi)$ is Fredholm. Hence one obtains a natural transformation

$$\begin{array}{rcl} \sigma : K^*(M \times N) & \longrightarrow & K^*(N), \\ \sigma(\xi) & = & \mathbf{index}\,(F \otimes \mathbf{Id}_\xi) \in K^*(N). \end{array} \tag{45}$$

This definition was an ancestor of KK–theory.

3 C^*–algebras, Hilbert C^*–modules and C^*–Fredholm operators

3.1 C^*–algebras of continuous functions

The simplest case of C^*–algebras is the case of commutative C^*–algebras. The Gelfand–Naimark theorem says that any commutative C^*–algebra with unit ia isomorphic to an algebra $C(X)$ of continuous functions on a compact space X. Therefore $K_*(C(X)) = K^*(X)$.

3.2 Hilbert C^*–modules

Due to Paschke the Hilbert A–module M is a Banach A–modules with additional structure of inner product $\langle x, y\rangle \in A,\ x, y \in M$ which possesses natural properties of inner product.

If ξ is a finite dimensional vector bundle over a compact space X, then $\Gamma(\xi)$ is the projective finite generated Hilbert $C(X)$–module.

3.3 Fredholm operators, Calkin algebra, multiplier

If ξ is a finite dimensional vector bundle over a compact space X, then it can be described as a continuous family of projectors, that is a continuous matrix valued function $P = P(x),\ x \in X,\ P(x) \in \mathbf{Mat}(N,N),\ P(x)P(x) = P(x)$, $P(x) : \mathbf{C}^N \longrightarrow \mathbf{C}^N$. This means that $\xi = \mathbf{Im}\, P$. Hence if $\eta = \ker P$ then

$$\xi \oplus \eta = \overline{N}. \tag{46}$$

Then $\xi \oplus \sum_{k=1}^{\infty} \overline{N}_k \approx \sum_{k=1}^{\infty} \overline{N}_k \approx H \otimes X$ where H is the Hilbert space, or $\xi \oplus H \otimes X = H \otimes X$. Hence there is a continuous Fredholm family

$$\begin{array}{l} F(x) : H \longrightarrow H, \\ \mathbf{Ker}\ (F) = \xi, \\ \mathbf{Coker}\ (F) = 0. \end{array} \tag{47}$$

3.4 K–theory for C^*–algebras, Chern character

Generalization of K–theory for C^*–algebra A. $K_A(X)$ is the Grothendieck group generated by vector bundles whose fibers M are projective finite generated A–modules, and the structure groups $\mathbf{Aut}\ _A(M)$. $K^*_A(X)$ is corresponding periodic cohomology theory.

The Chern character

$$\mathbf{ch} : K^*_A(X) \longrightarrow H^*(X; K^*_A(\mathbf{pt}) \otimes Q).$$

Theorem 15 *If X is a finite CW–space then the Chern character induces the isomorphism*

$$\mathbf{ch} : K^*_A(X) \otimes Q \longrightarrow H^*(X; K^*_A(\mathbf{pt}) \otimes Q).$$

3.5 Kuiper, Cuntz-Higson, Troitsky theorem

Let H be the Hilbert space, $GL(H) \subset B(H)$ be the group of invertible operators. The Kuiper theorem says that

$$\pi_i(GL(H)) = 0,\ 0 \le i < \infty. \tag{48}$$

A generalization of the Kuiper theorem for the group $GL^*_A(l_2(A))$.

Let $\mathcal{F}$ be the space of all Fredholm operators. Then

$$K^*(X) \approx [X, \mathcal{F}]. \tag{49}$$

Let $\mathcal{Q} = B(H)/\mathcal{K}$ be the Calkin algebra, where $\mathcal{K}$ is the subalgebra of all compact operators. Let $G(\mathcal{Q})$ be the group of invertible elements in the algebra $\mathcal{Q}$. Then one has a homomorphism

$$[X, \mathcal{F}] \longrightarrow [X, \mathcal{Q}],$$

hence a homomorphism

$$K^0(X) \longrightarrow K^1_{\mathcal{Q}}(X)$$

which is an isomorphism.

3.6 Non simply connected manifolds and canonical line vector bundle

Let π be a finite presented group which can serve as a fundamental group of a compact connected manifold M, $\pi = \pi_1(M, x_0)$. Let $B\pi$ be the classifying space for the group π. Then there is continuous mapping

$$f_M : M \longrightarrow B\pi \tag{50}$$

such that induced homomorphism

$$(f_M)_* : \pi_1(M, x_0) \longrightarrow \pi_1(B\pi, b_0) = \pi \tag{51}$$

is the isomorphism. Then one can construct the line vector bundle ξ_A over M with the fiber A, one dimensional free module over the group C^*–algebra $A = C^*[\pi]$ using the representation $\pi \subset \mathbf{C}^*[\pi]$. This canonical line vector bundle can be used to construct so called assembly map

$$\mu : K_*(B\pi) \longrightarrow K_*(\mathbf{C}^*[\pi]) \tag{52}$$

4 KK–theory and conjectures by Baum–Connes and Novikov

4.1 Representations of group and K–theory of classifying spaces. Atiyah–Hirzebruch theorem, Jackowski–Oliver theorem, Lusztig theorem

4.2 Fredholm representations

4.3 KK–theory by Kasparov

4.4 Symmetric C^*–signature

Let M be a closed oriented non simply connected manifold with fundamental group π. Let $B\pi$ be the classifying space for the group π and let

$$f_M : M \longrightarrow B\pi,$$

be a map inducing the isomorphism of fundamental groups.

Let $\Omega_*(B\pi)$ denote the bordism group of pairs (M, f_M). Recall that $\Omega_*(B\pi)$ is a module over the ring $\Omega_* = \Omega_*(\text{ pt })$.

One can construct a homomorphism

$$\sigma : \Omega_*(B\pi) \longrightarrow L_*(\mathbf{C}\pi) \tag{53}$$

which for every manifold (M, f_M) assigns the element $\sigma(M) \in L_*(\mathbf{C}\pi)$, so called symmetric $\mathbf{C}\pi$–signature, where $L_*(\mathbf{C}\pi)$ is the Wall group for the group ring $\mathbf{C}\pi$.

The homomorphism σ satisfies the following conditions:
(a) σ is homotopy invariant,
(b) if N is a simply connected manifold and $\tau(N)$ is its signature then

$$\sigma(M \times N) = \sigma(M)\tau(N) \in L_*(\mathbf{C}\pi).$$

We shall be interested only in the groups after tensor multiplication with the field $\mathbf{Q}$, in other words in the homomorphism

$$\sigma : \Omega_*(B\pi) \otimes \mathbf{Q} \longrightarrow L_*(\mathbf{C}\pi) \otimes \mathbf{Q}.$$

However

$$\Omega_*(B\pi) \otimes \mathbf{Q} \approx H_*(B\pi; \mathbf{Q}) \otimes \Omega_*.$$

Hence one has

$$\sigma : H_*(B\pi; \mathbf{Q}) \longrightarrow L_*(\mathbf{C}\pi) \otimes \mathbf{Q}.$$

Therefore the homomorphism σ represents the cohomology class

$$\bar{\sigma} \in H^*(B\pi; L_*(\mathbf{C}\pi) \otimes \mathbf{Q}).$$

Then for any manifold (M, f_M) one has

$$\sigma(M, f_M) = \langle L(M) f_M^*(\bar{\sigma}), [M] \rangle \in L_*(\mathbf{C}\pi) \otimes \mathbf{Q}. \tag{54}$$

Hence if $\alpha : L_*(\mathbf{C}\pi) \otimes \mathbf{Q} \longrightarrow \mathbf{Q}$ is an additive functional and $\alpha(\bar{\sigma}) = x \in H^*(B\pi; \mathbf{Q})$ then

$$\operatorname{sign}_x(M, f_M) = \langle L(M) f_M^*(x), [M] \rangle \in \mathbf{Q}$$

should be the homotopy-invariant higher signature. This gives a description of the family of all homotopy-invariant higher signatures. Therefore one should study the cohomology class

$$\bar{\sigma} \in H^*(B\pi; L_*(\mathbf{C}\pi) \otimes \mathbf{Q}) = H^*(B\pi; \mathbf{Q}) \otimes L_*(\mathbf{C}\pi) \otimes \mathbf{Q}.$$

4.4.1 Combinatorial description of symmetric $\mathbf{C}\pi$–signature

Here we give an economic description of algebraic Poincare complexes as a graded free $\mathbf{C}\pi$–module with the boundary operator and the Poincare duality operator. Consider a chain complex of $\mathbf{C}\pi$–modules C, d:

$$C = \bigoplus_{k=0}^{n} C_k,$$
$$d = \bigoplus_{k=1}^{n} d_k,$$
$$d_k : C_k \longrightarrow C_{k-1}$$

and a Poincare duality homomorphism

$$D : C^* \longrightarrow C, \quad \deg D = n.$$

They form the diagram

$$\begin{array}{ccccccc}
C_0 & \xleftarrow{d_1} & C_1 & \xleftarrow{d_2} & \cdots & \xleftarrow{d_n} & C_n \\
\uparrow D_0 & & \uparrow D_1 & & & & \uparrow D_n \\
C_n^* & \xleftarrow{d_n^*} & C_{n-1}^* & \xleftarrow{d_{n-1}^*} & \cdots & \xleftarrow{d_1^*} & C_0^*
\end{array}$$

with the following properties:

$$\begin{aligned}
d_{k-1} d_k &= 0, \\
d_k D_k + (-1)^{k+1} D_{k-1} d_{n-k+1}^* &= 0, \\
D_k &= (-1)^{k(n-k)} D_{n-k}^*.
\end{aligned} \tag{55}$$

Assume that the Poincare duality homomorphism induces an isomorphism in homology groups. Then the triple (C, d, D) is called *algebraic Poincare complex.* This definition allows to construct algebraic Poincare complex $\sigma(X)$ for each triangulation of combinatorial manifold X: $\sigma(X) = (C, d, D)$, where

$$C = C_*(X; \mathbf{C}\pi)$$

is the graded chain complex of the manifold X with local system of coefficients induced by the natural inclusion of fundamental group $\pi = \pi_1(X)$ into the group ring $\mathbf{C}\pi$, d is boundary homomorphism,

$$D = \oplus D_k, \quad D_k = \frac{1}{2}\left(\cap[X] + (-1)^{k(n-k)}(\cap[X])^*\right),$$

where $\cap[X]$ is the intersection with the open fundamental cycle of the manifold X.

Put

$$F_k = i^{k(k-1)} D_k. \tag{56}$$

Then the diagram

$$\begin{array}{ccccccc}
C_0 & \xleftarrow{d_1} & C_1 & \xleftarrow{d_2} & \cdots & \xleftarrow{d_n} & C_n \\
\uparrow F_0 & & \uparrow F_1 & & & & \uparrow F_n \\
C_n^* & \xleftarrow{d_n^*} & C_{n-1}^* & \xleftarrow{d_{n-1}^*} & \cdots & \xleftarrow{d_1^*} & C_0^*
\end{array} \tag{57}$$

possesses more natural conditions of commutativity and conjugacy

$$\begin{aligned}
d_k F_k + F_{k-1} d_{n-k+1}^* &= 0, \\
F_k &= (-1)^{\frac{n(n-1)}{2}} F_{n-k}^*.
\end{aligned} \tag{58}$$

Let

$$F = \oplus_{k=0}^{n} F_k, \ F, \quad \deg F = n.$$

Over completion to a regular C^*–algebra $C^*[\pi]$ one can define an element of hermitian K–theory using the non degenerated selfadjoint operator

$$G = d + d^* + F : C \longrightarrow C.$$

Then

$$\text{sign}\,[C, G] = \text{sign}\,(C, d, D) \in K_0^h(C^*[\pi]). \tag{59}$$

4.5 Higher signature, conjecture by Baum–Connes and Novikov

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