LIE BRACKETS ON AFFINE BUNDLES

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Abstract. Natural analogs of Lie brackets on affine bundles are studied. In particular, a close relation to Lie algebroids and a duality with certain affine analog of Poisson structure is established as well as affine versions of the complete lift and the Cartan exterior calculus.

1. Introduction.

It is known that the framework based on vector bundles is not satisfactory for classical mechanics. For example, the phase space for the charged particle is not the cotangent bundle of the configuration manifold, but an affine bundle over it ([20], [13]). Also in the frame-independent formulation of Newtonian analytical mechanics the phase space is an affine bundle ([1], [17], [7], [14]). The third example is the frame -independent formulation of time -dependent mechanics ([19]). In this case the space of infinitesimal configurations (the domain of a Lagrangian) is no longer a vector bundle (the tangent bundle), but the affine bundle of first jets.

On the other hand, the fundamental structure for the Lagrangian first-order mechanics is the Lie algebroid structure of the tangent bundle. In [21], [9], [12] one can find an attempt to apply more general Lie algebroids to Lagrangian mechanics. It is natural question to combine affine and Lie-algebroidal aspects of mechanics. An attempt to do this has been done in [15,16] in the context of Euler-Lagrange equations.

In this paper we discuss purely geometrical and algebraic aspects motivated by natural examples of objects which carry a structure of an affine bundle with a canonical bracket.

These examples are presented in Section 2, together with basic concepts of differential geometry on affine bundles.

In Section 3 we define an object which is an abstract generalization of properties of algebraic structures studied in the examples of Section 2. We decided to call this object a *Lie affgebroid*. It basicaly coincides with the concept developed in [15]. In Section 3 we also introduce affine versions of Poisson and Jacobi structures.

Section 4 is devoted to studying some cohomology of Lie algebroids which helps to clasify Lie affgeboids up to an isomorphism.

In Section 5 we pass to affine-linear duality, i.e. we study the vector bundle of affine functions on affine bundle. The duality is then used, in Section 6, to show that Lie affgebroids are particular substructures of Lie algebroids. This point of view makes it possible to define complete lifts of sections

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of affgebroid and Cartan calculus as restrictions of appropriate objects from the Lie algebroid. This is done in Section 7.

Finally, in Section 8, we discuss an affine analog of the well-known correspondence between Lie algebroid structure on vector bundle E and linear Poisson structure on the dual E^*. In this context we study natural affine bundle structures associated with epimorphisms of vector bundles. The dual structure for Lie affgebroid bracket appears to be a Poisson-like bracket on sections of a one-dimensional affine bundle. We decided to call it *aff-Poisson structure* to point out the relation to the usual Poisson bracket on functions.

2. Affine bundles: notation and preliminaries.

Let $\tau_i: A_i \to M$ be an affine bundle modeled on a vector bundle $\mathsf{v}(\tau_i): \mathsf{V}(A_i) \to M$, $i = 1, 2, 3$. Note that the space $\mathcal{A}_i$ of sections of τ_i is an affine space modeled on the space $\mathsf{V}(\mathcal{A}_i)$ of sections of $\mathsf{v}(\tau_i)$. Moreover, $\mathsf{V}(\mathcal{A}_i)$ is a $C^\infty(M)$ module. For an affine bundle morphism $\phi: A_1 \to A_2$ we denote by $\phi_{\mathsf{v}}: \mathsf{V}(A_1) \to \mathsf{V}(A_2)$ its linear part, i.e.

$$\phi_{\mathsf{v}}(v) = \phi(a + v) - \phi(a) \quad \text{for} \quad a \in A,\ v \in \mathsf{V}(A), \quad \tau(a) = \mathsf{V}(\tau)(v). \tag{1}$$

Throughout this paper we shall consider only bundle morphisms over the identity on the base. We will denote by $\mathsf{Aff}_M(A_1, A_2)$ (resp. $\mathsf{Hom}_M(V_1, V_2)$) the set of such morphisms in the affine (resp. vector) case. We shall also write $\mathsf{Aff}(A, \mathbb{R})$ instead of $\mathsf{Aff}_M(A, M \times \mathbb{R})$ and $\mathsf{Lin}(V, \mathbb{R})$ instead of $\mathsf{Hom}_M(V, M \times \mathbb{R})$. For a bi-affine mapping

$$F: A_1 \times_M A_2 \to A_3$$

we denote by F^{v} and ${}^{\mathsf{v}}F$, respectively, the mappings

$$\begin{aligned} F^{\mathsf{v}}&: A_1 \times_M \mathsf{V}(A_2) \to \mathsf{V}(A_3) \\ &: (a_1, v_2) \mapsto (F(a_1, \cdot))_{\mathsf{v}}(v_2) \end{aligned} \tag{2}$$

and

$$\begin{aligned} {}^{\mathsf{v}}F&: \mathsf{V}(A_1) \times_M A_2 \to \mathsf{V}(A_3) \\ &: (v_1, a_2) \mapsto (F(\cdot, a_2))_{\mathsf{v}}(v_1). \end{aligned} \tag{3}$$

These mappings are, respectively, affine-linear and linear-affine in the obvious sense. By F_{v} we denote the bilinear part of F, i.e.

$$\begin{aligned} F_{\mathsf{v}}&: \mathsf{V}(A_1) \times_M \mathsf{V}(A_2) \to \mathsf{V}(A_3) \\ &: (v_1, v_2) \mapsto (F^{\mathsf{v}}(\cdot, v_2))_{\mathsf{v}}(v_1) = ({}^{\mathsf{v}}F(v_1, \cdot))_{\mathsf{v}}(v_2) \\ &= F(a_1 + v_1, a_2 + v_2) - F(a_1 + v_2, a_2) + F(a_1, a_2) - F(a_1, a_2 + v_2). \end{aligned} \tag{4}$$

There are obvious generalizations of F^{v} and F_{v} for the case of n-affine mappings.

By a *special vector bundle* we mean a vector bundle with a distinguished non-vanishing section, and by a *special affine bundle* we mean an affine bundle modeled on a special vector bundle. Let $\boldsymbol{V} = (V, \varphi)$ and $\boldsymbol{V}' = (V', \varphi')$ be special vector bundles. A morphism $F: V \to V'$ of vector bundles is called a *morphism of special vector bundles* if φ and φ' are F-related. A *morphism of special affine bundles* is a morphism of affine bundles such that its linear part is a morphism of special vector bundles. We call an n-dimensional affine bundle $\tau: A \to M$ *trivial* if $A = M \times \mathbb{R}^n$ as an affine bundle, and *vectorially trivial* if its model vector bundle is trivial: $\mathsf{V}(A) = M \times \mathbb{R}^n$.

Remark. Any section $\sigma \in \mathsf{Sec}(\tau)$ of A induces an obvious isomorphism $I_\sigma \in \mathsf{Aff}_M(A, \mathsf{V}(A))$ of affine bundles:

$$A_m \ni a \mapsto a - \sigma(m) \in \mathsf{V}(A_m) \tag{5}$$

If A is vectorially trivial, this gives a trivialization

$$I_\sigma: A \to M \times \mathbb{R}^n. \tag{6}$$

Thus, vectorially trivial affine bundles are trivializable but not trivial if no canonical section is given.

An affine morphism

$$D: \mathsf{Sec}(A_1) \to \mathsf{Sec}(A_2)$$

we call an *affine differential operator of order* n if

$$D_{\mathsf{v}}: \mathsf{Sec}(\mathsf{V}(A_1)) \to \mathsf{Sec}(\mathsf{V}(A_2))$$

is a differential operator of order n. Similarly, D is an *affine quasi-derivation* (or, following Mackenzie [10], a *covariant differential operator*) if D_{v} is a quasi-derivation, i.e. if there is a derivation in $C^\infty(M)$ (represented by a vector field on M) $\widehat{D}_{\mathsf{v}}$, called the *anchor* of D, such that

$$D_{\mathsf{v}}(fX) = fD_{\mathsf{v}}(X) + \widehat{D}_{\mathsf{v}}(f)X.$$

If $\mathsf{V}(A_1) = \mathsf{V}(A_2) = M \times \mathbb{R}$, i.e. A_1, A_2 are one-dimensional and vectorially trivial, then D will be called an *affine derivation* if $D_{\mathsf{v}}: C^\infty(M) \to C^\infty(M)$ is a derivation in the algebra $C^\infty(M)$ of smooth functions (D_{v} is represented by a vector field $\widehat{D}_{\mathsf{v}}$ on M).

We denote by $\mathsf{AffDiff}^n(A_1, A_2)$ the space of affine differential operators of order n, by $\mathsf{AffQder}(A_1, A_2)$ the space of affine quasi derivations, and by $\mathsf{AffDer}(A_1, A_2)$ the space of affine derivations (we will write simply $\mathsf{AffDer}(A, \mathbb{R})$ instead of $\mathsf{AffDer}(A, M \times \mathbb{R})$, etc.). These spaces are affine spaces modeled on the corresponding vector spaces $\mathsf{AffDiff}^n(A_1, \mathsf{V}(A_2))$, $\mathsf{AffQder}(A_1, \mathsf{V}(A_2))$, and $\mathsf{AffDer}(A_1, \mathbb{R})$, respectively.

EXAMPLE 1. For a one-dimensional vectorially trivial affine bundle Z, the space $\mathsf{AffDer}(Z, \mathbb{R})$ is not only a vector space, but also a $C^\infty(M)$-module. Let $D: \mathsf{Sec}(Z) \to C^\infty(M)$ be an affine derivation with the linear part $\widehat{D}$. It is easy to see that the formula

$$[D_1, D_2] = \widehat{D}_1 \circ D_2 - \widehat{D}_2 \circ D_1 \tag{7}$$

defines a Lie bracket on $\mathsf{AffDer}(A_1, \mathbb{R})$. Moreover,

$$[D_1, fD_2] = f[D_1, D_2] + \widehat{D}_1(f)D_2, \tag{8}$$

i.e. we have a Lie pseudoalgebra structure on the $C^\infty(M)$-module $\mathsf{AffDer}(A_1, \mathbb{R})$ (cf. [10]).

Let σ be a section of Z. With the trivialization $I_\sigma: Z \to M \times \mathbb{R}$ (see (6))we can identify sections of Z with functions on M. An affine derivation $D \in \mathsf{AffDer}(Z, \mathbb{R})$ induces an affine derivation on $C^\infty(M)$:

$$f \mapsto \widehat{D}(f) + D(\sigma), \tag{9}$$

i.e, $\mathsf{AffDer}(Z, \mathbb{R})$ can be identified with $\mathcal{X}(M) \times C^\infty(M)$. Here $\mathcal{X}(M)$ denotes the Lie algebra of vector fields on M. With this identification, the bracket (7) takes the form

$$[(\widehat{X}, g), (\widehat{Y}, h)] = ([\widehat{X}, \widehat{Y}], \widehat{X}(h) - \widehat{Y}(g)). \tag{10}$$

A basic fact in differential geometry is that derivations on $C^\infty(M)$ can be identified with vector fields, i.e. sections of the Lie algebroid $\mathsf{T}M$. Now, we show that we have a similar interpretation for affine derivations.

The canonical $\mathbb{R}$-action $\psi: (t, z_p) \mapsto (z_p + t)$ on Z induces an $\mathbb{R}$-action ψ_* on $\mathsf{T}Z$. The space of orbits of this action we will denote $\widetilde{\mathsf{T}}Z$ and we will call it the *reduced tangent bundle* [18]. This is a vector bundle over M and sections of $\widetilde{\mathsf{T}}Z$ can be identified with ψ_*-invariant vector fields on Z. The projection $\tau\colon Z\to M$ induces a projection

$$\widetilde{\tau}=\widetilde{\mathsf{T}}(\tau)\colon \widetilde{\mathsf{T}}Z\to \mathsf{T}M. \tag{11}$$

With this projection, $\widetilde{\mathsf{T}}Z$ becomes a one-dimensional affine bundle modeled on $\mathsf{T}M\times\mathbb{R}$ (for details see Section 8.).

A ψ_*-invariant vector field on Z (a section of $\widetilde{\mathsf{T}}Z$) is a derivation in the algebra of functions on Z which sends affine functions to functions which are constant on fibers. In local affine coordinates (x^a,s) on Z, providing an isomorphism of Z with $\mathsf{V}(Z)=M\times\mathbb{R}$, an invariant vector field reads

$$X=g^a(x)\frac{\partial}{\partial x^a}-\alpha(x)\frac{\partial}{\partial s}=X_M-\alpha(x)\frac{\partial}{\partial s}. \tag{12}$$

The action $\psi(t,x^a,s)=(x^a,s+t)$ gives rise to the corresponding fundamental vector field $X_0=-\frac{\partial}{\partial s}$. We can identify a section σ of Z with an affine function f_σ on Z with the directional coefficient -1 (i.e. $X_0(f)=1$) and such that $f_\sigma\circ\sigma=0$. Consequently, we can interpret a section X of $\widetilde{\mathsf{T}}Z$ as a mapping

$$D_X\colon \mathsf{Sec}(Z)\to C^\infty(M). \tag{13}$$

It is easy to verify that D_X is an affine derivation, i.e. its linear part is a derivation in $C^\infty(M)$. The Lie bracket of invariant vector fields is an invariant vector field. Thus, we have also the Lie bracket of function-valued affine derivations on $\mathsf{Sec}(Z)$. It can be trivially verified that this bracket coincides with the bracket (7).

The Lie bracket on sections of $\widetilde{\mathsf{T}}Z$ defines a Lie algebroid structure with the anchor $\widetilde{\tau}$. It is the principal bundle algebroid of Z interpreted as a principal fibre bundle with the group $(\mathbb{R},+)$. In local coordinates a section $x=(x^a)\mapsto(x,\sigma(x))$ can be represented by the affine function $f_\sigma(x,s)=\sigma(x)-s$. For an invariant vector field X as in (12),

$$Xf_\sigma=g^a\frac{\partial\sigma}{\partial x^a}+\alpha. \tag{14}$$

Thus $D_X(\sigma)=g^a\dfrac{\partial\sigma}{\partial x^a}+\alpha$ is an affine derivation on Z with the linear part represented by the vector field $X_M=g^a(x)\dfrac{\partial}{\partial x^a}=\widetilde{\tau}\circ X$ on M. In fact, $\widetilde{\mathsf{T}}Z$ is a Lie algebroid with the Lie bracket on sections induced from $\mathsf{T}Z$. In local coordinates

$$\begin{aligned} X&=X_M-\alpha\frac{\partial}{\partial s},\\ Y&=Y_M-\beta\frac{\partial}{\partial s},\\ [X,Y]&=[X_M,Y_M]-(X_M(\beta)-Y_M(\alpha))\frac{\partial}{\partial s}. \end{aligned} \tag{15}$$

Moreover, $\widetilde{\mathsf{T}}Z$ has the distinguished section X_0 which corresponds to the fundamental vector field $X_0=-\frac{\partial}{\partial s}$ of the $\mathbb{R}$-action on Z. It follows that $\widetilde{\mathsf{T}}Z$ is a special vector bundle. We summarize these observations in the following.

THEOREM 1. *There is a canonical isomorphism between* $\mathsf{AffDer}(Z,\mathbb{R})$ *with the bracket* (7) *and sections of the reduced tangent bundle* $\widetilde{\mathsf{T}}Z$ *with the canonical Lie algebroid bracket. The fundamental vector field* X_0 *corresponds to the affine derivation*

$$\mathsf{Sec}(Z)\ni\sigma\mapsto 1_M$$

with vanishing linear part.

It is clear now that the space of sections of $\bigwedge^n \widetilde{\mathsf{T}} Z$ corresponds to the space $\bigwedge^n \mathsf{AffDer}(Z,\mathbb{R})$ of affine multiderivations on Z with values in $C^\infty(M)$, i.e. multi-affine skew-symmetric mappings

$$D\colon \mathsf{Sec}(Z) \times \cdots \times \mathsf{Sec}(Z) \to C^\infty(M) \tag{16}$$

such that

$$D(a_1, \dots, a_{n-1}, \cdot)\colon \mathsf{Sec}(Z) \to C^\infty(M) \tag{17}$$

is an affine derivation for any $a_1, \dots, a_{n-1} \in \mathsf{Sec}(Z)$. The graded space

$$\bigwedge \mathsf{AffDer}(Z,\mathbb{R}) = \oplus_{n\in\mathbb{Z}} \bigwedge^n \mathsf{AffDer}(Z,\mathbb{R}), \tag{18}$$

being isomorphic with $\mathsf{Sec}(\bigwedge \widetilde{\mathsf{T}} Z) = \oplus_{n\in\mathbb{Z}} \mathsf{Sec}(\bigwedge^n \widetilde{\mathsf{T}} Z)$ is therefore a Gerstenhaber algebra (see [8]) with the wedge product and the Lie algebroid Schouten-Nijenhuis bracket induced by the Lie algebroid bracket on $\widetilde{\mathsf{T}} Z$. With the identification $\mathsf{Sec}(\widetilde{\mathsf{T}} Z) = \mathcal{X}(M) \times C^\infty(M)$, the Lie algebroid bracket on $\mathsf{Sec}(\widetilde{\mathsf{T}} Z)$ is identified with the Lie algebroid bracket of first-order differential operators on M, so that the Gerstenhaber algebra $\bigwedge \mathsf{AffDer}(Z,\mathbb{R})$ is identified with the Gerstenhaber algebra of skew-symmetric multilinear first-order differential operators. Thus we get the following.

Theorem 2. *There is a canonical Gerstenhaber algebra structure on the graded space $\bigwedge \mathsf{AffDer}(Z,\mathbb{R})$ of multi-affine skew-symmetric derivations on Z with values in $C^\infty(M)$. Using a section σ of Z to identify Z with $M \times \mathbb{R}$ we can identify $\bigwedge^n \mathsf{AffDer}(Z,\mathbb{R})$ with $\mathsf{Sec}(\bigwedge^n \mathsf{T}M) \times \mathsf{Sec}(\bigwedge^{n-1} \mathsf{T}M)$ by*

$$D_{(X_n, X_{n-1})}(\sigma + f_1, \dots, \sigma + f_n) = \langle X_n, \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \rangle + \sum_i (-1)^i \langle X_{n-1}, \mathrm{d}f_1 \wedge .\overset{i}{\vee}. \wedge \mathrm{d}f_n \rangle \tag{19}$$

for $(X_n, X_{n-1}) \in \mathsf{Sec}(\bigwedge^n \mathsf{T}M) \times \mathsf{Sec}(\bigwedge^{n-1} \mathsf{T}M)$. With this identification, the wedge product is

$$(X_n, X_{n-1}) \wedge (Y_k, Y_{k-1}) = (X_n \wedge Y_k, X_{n-1} \wedge Y_k + (-1)^n X_n \wedge Y_{k-1}) \tag{20}$$

and the Schouten-Nijenhuis bracket reads

$$[(X_n, X_{n-1}), (Y_k, Y_{k-1})]^{SN} = ([X_n, Y_k]^{SN}, [X_{n-1}, Y_k]^{SN} + (-1)^{n-1}[X_n, Y_{k-1}]^{SN}), \tag{21}$$

where the Schouten-Nijenhuis bracket on the right-hand side is the classical Schouten-Nijenhuis bracket of multivector fields.

Example 2. Let Z be as above. Consider now the affine space $\mathcal{A} = \mathsf{AffDer}(Z,Z)$ modeled on $\mathsf{V}(\mathcal{A}) = \mathsf{AffDer}(Z,\mathbb{R})$. It is easy to see that the commutator

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1 \tag{22}$$

defines a bracket on $\mathcal{A}$ with values in $\mathsf{V}(\mathcal{A})$. This bracket is an affine analog of the Lie bracket of vector fields. It is bi-affine and it has the following properties:

(1) it is skew-symmetric: $[D_1, D_2] = -[D_2, D_1]$,
(2) it satisfies the Jacobi identity:

$$[D_1, [D_2, D_3]]^{\mathsf{v}} + [D_2, [D_3, D_1]]^{\mathsf{v}} + [D_3, [D_1, D_2 .]]^{\mathsf{v}} = 0,$$

(3) for any $D \in \mathcal{A}$ the map $\mathsf{ad}_D = [D, \cdot]$ is an affine quasi-derivation, i.e.

$$[D, fX]^{\mathsf{v}} = f[D,X]^{\mathsf{v}} + \widehat{D}(f)X,$$

for $f \in C^\infty(M)$, $X \in \mathsf{V}(\mathcal{A})$, and some $\widehat{D} \in \mathcal{X}(M)$, where $\mathcal{X}(M)$ denotes the Lie algebra of vector fields on M, We write here $\widehat{D}$ instead of D_{v} to denote the vector field representing the linear part of D. The affine-linear part $[\,,\,]^{\mathsf{v}}$ of $[\,,\,]$ is given by

$$[D,X]^{\mathsf{v}} = \widehat{D}\circ X - X\circ D,$$

$D\in\mathsf{AffDer}(Z,Z)$, $X\in\mathsf{AffDer}(Z,\mathbb{R})$. The linear part $[\,,\,]_{\mathsf{v}}$ of $[\,,\,]$ coincides with the Lie algebra bracket on $\mathsf{AffDer}(Z,\mathbb{R})$ from the previous example. Given a global section σ of Z, we get (6) an isomorphism $I_\sigma\colon Z\to M\times\mathbb{R}$ and, consequently, we can identify $\mathsf{AffDer}(Z,Z)$ with $\mathcal{X}(M)\times C^\infty(M)$ and sections of Z with functions on M. With these identifications,

$$\begin{aligned}(X,g)(f) &= X(f)+g\,,\\ [(X,g),(Y,h)] &= ([X,Y],X(h)-Y(g)+g-h)\,.\end{aligned}\tag{23}$$

Let us recall that I_σ provides also the identification of $\mathsf{AffDer}(Z,\mathbb{R})$ with $\mathcal{X}(M)\times C^\infty(M)$, but the induced bracket is $[(X,g),(Y,h)] = ([X,Y],X(h)-Y(g))$. In the previous example we identified an element of $\mathsf{AffDer}(Z,\mathbb{R})$ with a section of the reduced tangent bundle $\widetilde{\mathsf{T}}Z$. Now, we give a similar construction for $\mathsf{AffDer}(Z,Z)$.

There is a canonical diffeomorphism $\mathsf{T}Z=\widetilde{\mathsf{T}}Z\times_M Z$ and a $\mathbb{R}^2$-group action on $\mathsf{T}Z$:

$$((t,r),(v,z))\mapsto(v+rX_0,z+t),\tag{24}$$

where X_0 is the distinguished section X_0 of $\widetilde{\mathsf{T}}Z$ which corresponds to the fundamental vector field of the $\mathbb{R}$-action on Z (Example 1). In local coordinates this action reads

$$(x^a,s,\dot{x}^b,\dot{s})\mapsto(x^a,t+s,\dot{x}^b,\dot{s}-r).\tag{25}$$

We reduce $\mathsf{T}Z$ with respect to the group homomorphism

$$\mathbb{R}^2\to\mathbb{R}\colon(r,t)\mapsto r+t\tag{26}$$

i.e., we consider a manifold $\overline{\mathsf{T}}Z$ of orbits of the kernel group $(r,-r)$ with the induced group action

$$(t,[(v,z)])\mapsto[(v,z+t)]=[(v+tX_0,z)].\tag{27}$$

$\overline{\mathsf{T}}Z$ is an affine bundle over M and its model bundle is canonically isomorphic to $\widetilde{\mathsf{T}}Z$. The pair $(\overline{\mathsf{T}}Z,X_0)$ is a special affine bundle over M denoted by $\overline{\mathsf{T}}\boldsymbol{Z}$. Sections of $\overline{\mathsf{T}}Z$ correspond to vector fields on $\mathsf{T}Z$, invariant with respect to the $\mathbb{R}$-action

$$(r,(v,z))\mapsto(v+rX_0,z-r).\tag{28}$$

Such a vector field has the local form

$$X=g^a(x)\frac{\partial}{\partial x^a}+(s-g(x))\frac{\partial}{\partial s}=X_M+(s-g(x))\frac{\partial}{\partial s}.\tag{29}$$

The vector field X preserves, as the derivation in $C^\infty(Z)$, the affine subspace of functions with the directional coefficient -1, i.e. such that $X_0(f)=1$. Since such affine functions represent sections of Z, we get a mapping

$$D_X\colon\mathsf{Sec}(Z)\to\mathsf{Sec}(Z).\tag{30}$$

Let σ be a section of Z and f_σ the corresponding function on Z. In local coordinates

$$f_\sigma(x,s)=\sigma(x)-s\tag{31}$$

and

$$X(f_\sigma) = X_M(\sigma) + g(x) - s. \tag{32}$$

Consequently,

$$D_X(\sigma) = X_M(\sigma) + g. \tag{33}$$

The linear part of D_X is a derivation in the algebra $C^\infty(M)$ given by the vector field X_M. We conclude that sections of $\overline{\mathsf{T}}Z$ can be viewed as the affine derivations $D\colon \mathsf{Sec}(Z) \to \mathsf{Sec}(Z)$, i.e. we have the canonical isomorphism of the space of the affine derivations $\mathsf{AffDer}(Z,Z)$ and the sections of $\overline{\mathsf{T}}Z$. Moreover,

$$[X_M + (s - g(x))\frac{\partial}{\partial s}, Y_M + (s - h(x))\frac{\partial}{\partial s}] = [X_M, Y_M] - (X_M(h) - Y_M(g) + g - h)\frac{\partial}{\partial s}, \tag{34}$$

so that $[D_X, D_Y]$ corresponds exactly to the element from $\mathsf{AffDer}(Z,\mathbb{R})$ associated with the invariant vector field $[X,Y]$. Thus we get a theorem analogous to Theorem 1.

THEOREM 3. *There is a canonical isomorphism between $\mathsf{AffDer}(Z,Z)$ with the bracket (22) and sections of the affine bundle $\overline{\mathsf{T}}Z$ with the canonical bracket.*

♠

3. Lie brackets on affine bundles.

The Example 2 justifies the following definition.

DEFINITION 1. An *affine Lie bracket* on an affine space $\mathcal{A}$ is a bi-affine map

$$[\cdot,\cdot]\colon \mathcal{A} \times \mathcal{A} \to \mathsf{V}(\mathcal{A})$$

which

(1) is skew-symmetric: $[\alpha_1, \alpha_2] = -[\alpha_2, \alpha_1]$ and
(2) satisfies the Jacobi identity:

$$[\alpha_1, [\alpha_2, \alpha_3]]^{\mathsf{v}} + [\alpha_2, [\alpha_3, \alpha_1]]^{\mathsf{v}} + [\alpha_3, [\alpha_1, \alpha_2]]^{\mathsf{v}} = 0.$$

An affine space equipped with an affine Lie bracket we shall call a *Lie affgebra*. Note that the term *affine Lie algebra* has been already used for certain types of Kac-Moody algebras.

DEFINITION 2. If A is an affine bundle over M modeled on $\mathsf{V}(A)$ than a *Lie affgebroid structure* on A is an affine Lie bracket on sections of A and a morphism $\gamma\colon A \to \mathsf{T}M$ of affine bundles (over id on M) such that $[\alpha, \cdot]^{\mathsf{v}}$ is a quasi-derivation with the anchor $\gamma(\alpha)$, i.e.

$$[\alpha, fX]^{\mathsf{v}} = f\,[\alpha, X]^{\mathsf{v}} + \gamma(\alpha)(f)X \tag{35}$$

for all $\alpha \in \mathcal{A}$, $X \in \mathsf{V}(\mathcal{A})$, $f \in C^\infty(M)$.

Remark. The above definition goes back to the one used in [15] but without the additional assumption that the base manifold M is fibered over $\mathbb{R}$ and that $\gamma(\alpha)$ are vector fields projectable onto $\frac{\partial}{\partial t}$. This requirement was motivated by the particular aim of [15] to get time-dependent Euler-Lagrange equations, but it makes impossible to get all Lie affgebras. We will see later (Theorem 13) that the particular case of [15] means just vanishing of certain cohomology.

The following fact has been proved, in principle, in [15], Proposition 1 .

THEOREM 4. *For every Lie affgebroid structure* $[\cdot,\cdot]\colon \mathcal{A}\times\mathcal{A}\to \mathsf{V}(\mathcal{A})$ *its vector part*

$$[\cdot,\cdot]_{\mathsf{v}}\colon \mathsf{V}(\mathcal{A})\times\mathsf{V}(\mathcal{A})\to\mathsf{V}(\mathcal{A})$$

is a Lie algebroid structure with the anchor map γ_{v}. *Moreover,* $[\alpha,\cdot]^{\mathsf{v}}$ *is a derivation of* $[\cdot,\cdot]_{\mathsf{v}}$ *for every* $\alpha\in\mathcal{A}$. *Conversely, if we have an affine-linear map* $[\cdot,\cdot]^0\colon \mathcal{A}\times\mathsf{V}(\mathcal{A})\to\mathsf{V}(\mathcal{A})$ *which satisfies (35) for certain affine morphism* $\gamma\colon A\to \mathsf{T}M$ *and such that* $[\cdot,\cdot]_0=[\cdot,\cdot]^0_{\mathsf{v}}$ *is skew symmetric and* $[\alpha,\cdot]^0$ *is a derivation of* $[\cdot,\cdot]_0$, *for any* $\alpha\in\mathcal{A}$, *then there is a unique Lie affgebroid bracket* $[\cdot,\cdot]$ *on* $\mathcal{A}$ *such that* $[\cdot,\cdot]^0=[\cdot,\cdot]^{\mathsf{v}}$.

DEFINITION 3. Let $\tau\colon Z\to M$ be a one-dimensional affine bundle with $\mathsf{V}(Z)=M\times\mathbb{R}$. An affine Lie bracket on $\mathsf{Sec}(Z)$

$$\{\,,\,\}\colon \mathsf{Sec}(Z)\times\mathsf{Sec}(Z)\to C^\infty(M)$$

is called an *aff-Poisson* (resp.*aff-Jacobi*) *bracket* if

$$\{\alpha,\cdot\}\colon \mathsf{Sec}(Z)\to C^\infty(M) \tag{36}$$

is an affine derivation (resp. an affine first order differential operator) for every $\alpha\in\mathsf{Sec}(Z)$.

We use the term *aff-Poisson*, since *affine Poisson structure* has already a different meaning in the literature.

THEOREM 5. *For every aff-Poisson (resp. aff-Jacobi) bracket*

$$\{\,,\,\}\colon \mathsf{Sec}(Z)\times\mathsf{Sec}(Z)\to C^\infty(M)$$

its vector part

$$\{\,,\,\}_{\mathsf{v}}\colon C^\infty(M)\times C^\infty(M)\to C^\infty(M)$$

is a Poisson (resp. aff-Jacobi) bracket. Moreover,

$$\{\alpha,\cdot\}^{\mathsf{v}}\colon C^\infty(M)\to C^\infty(M)$$

is a derivation (resp. first-order differential operator) for every section $\alpha\in\mathsf{Sec}(Z)$, *which is simultaneously a derivation of the bracket* $\{\,,\,\}_{\mathsf{v}}$. *Conversely, if we have a Poisson (resp. Jacobi) bracket* $\{\,,\,\}_0$ *on* $C^\infty(M)$ *and a derivation (resp. a first-order differential operator)*

$$D\colon C^\infty(M)\to C^\infty(M)$$

which is simultaneously a derivation of the bracket $\{\,,\,\}_0$, *then there is a unique aff-Poisson (resp. aff-Jacobi) bracket* $\{\,,\,\}$ *on* $\mathsf{Sec}(Z)$ *such that* $\{\,,\,\}_0=\{\,,\,\}_{\mathsf{v}}$ *and* $D=\{\alpha,\cdot\}^{\mathsf{v}}$ *for a chosen section* $\alpha\in\mathsf{Sec}(Z)$.

PROOF: The above theorem is a direct consequence of Theorem 4. ■

Remark. Using a section σ to identify $\mathsf{Sec}(Z)$ with $C^\infty(M)$, we get the aff-Poisson (resp. aff-Jacobi) bracket on $\mathsf{Sec}(Z)$ in the form

$$\{f,g\}=D(g-f)+\{f,g\}_{\mathsf{v}}, \tag{37}$$

where D is a vector field (resp first-order differential operator) which is a derivation of the Poisson (resp. Jacobi) bracket $\{\,,\,\}_{\mathsf{v}}$. We have clearly $\{f,\cdot\}^{\mathsf{v}}=D+\{f,\cdot\}_{\mathsf{v}}$. In particular, for the Poisson case, the aff-Poisson bracket is a bi-affine skew-symmetric derivation, so it is represented by the pair (Λ,D), where Λ is the Poisson tensor of $\{\,,\,\}_{\mathsf{v}}$, in accordance with Theorem 2. The Jacobi identity means that $[(\Lambda,D),(\Lambda,D)]^{SN}=0$, i.e. $[\Lambda,\Lambda]^{SN}=0$ and $[D,\Lambda]^{SN}=0$.

4. Lie affgebroid brackets and $Qder$-cohomology.

Every Lie affgebroid structure on A determines a Lie algebroid structure on $\mathsf{V}(A)$. Moreover, by Theorem 4, every section α of A determines a derivation $[\alpha,\cdot]^{\mathsf{v}}$ of the Lie algebroid bracket on $\mathsf{Sec}(\mathsf{V}(A))$ which at the same time is a quasi-derivation on sections of $\mathsf{V}(A)$. Passing to another section we change this derivation by an inner derivation. As in the case of a Lie algebra, this determines certain cohomology class of the Lie algebroid $\mathsf{V}(A)$. To be more precise, consider a Lie algebroid structure on a vector bundle E over M. Let $C^n_{Qder}(E,E)$ be the space of n-cochains with coefficients in the adjoint representation which are quasi-derivations with respect to each argument. This means that elements of $C^n_{Qder}(E,E)$ are skew-n-linear maps

$$\mu\colon \mathsf{Sec}(E)\times\cdots\times\mathsf{Sec}(E)\to\mathsf{Sec}(E),$$

such that for $X_1,\dots,X_n\in\mathsf{Sec}(E)$, $f\in C^\infty(M)$,

$$\mu(fX_1,X_2,\dots,X_n)=f\mu(X_1,\dots,X_n)+\widehat{\mu}_{(X_2,\dots,X_n)}(f)X_1 \tag{38}$$

for certain vector field $\widehat{\mu}_{(X_2,\dots,X_n)}$ on M.

THEOREM 6. *The standard Chevalley coboundary operator*

$$\partial\mu(X_0,X_1,\dots,X_n)=\sum_i(-1)^i[X_i,\mu(X_0,\overset{i}{\check{\dots}},X_n)]+\sum_{i<j}(-1)^{i+j}\mu([X_i,X_j],X_1,\overset{i}{\check{\dots}}\overset{j}{\check{\dots}},X_n) \tag{39}$$

can be restricted to the subcomplex of multi-quasi-derivations, i.e.

$$\partial C^n_{Qder}(E,E)\subset C^{n+1}_{Qder}(E,E). \tag{40}$$

PROOF: The proof consists of standard calculations in order to show that, for $\mu\in C^n_{Qder}(E,E)$,

$$\partial\mu(fX_0,X_1,\dots,X_n)=f\partial\mu(X_0,X_1,\dots,X_n)+\widehat{D}(f)X_0 \tag{41}$$

for certain vector field $\widehat{D}$ on M depending on $X_1,\dots,X_n$ and μ. ■

The corresponding cohomology spaces will be denoted by $H^n_{Qder}(E,E)$. Let us notice that a Qder-1-cocycle is a quasi-derivation $D\colon\mathsf{Sec}(E)\to\mathsf{Sec}(E)$ such that

$$\partial D(X_0,X_1)=[X_0,D(X_1)]+[D(X_0),X_1]-D([X_0,X_1])=0, \tag{42}$$

i.e. D is a derivation of the Lie bracket. Any Qder-1-coboundary is of the form

$$(\partial X_0)(X)=[X,X_0], \tag{43}$$

so that 1-coboundaries are just inner derivations. Hence,

$$H^1_{Qder}(E,E)=\mathsf{Der}_{Qder}(E)/\mathsf{InnDer}(E). \tag{44}$$

THEOREM 7. *For an affine bundle A over M, every Lie affgebroid bracket on A has the form*

$$[\alpha_0+X,\alpha_0+Y]=D(Y)-D(X)+[X,Y]_{\mathsf{v}}, \tag{45}$$

where $\alpha_0\in\mathsf{Sec}(A)$, $X,Y\in\mathsf{Sec}(\mathsf{V}(A))$, $D=[\alpha_0,\cdot]^{\mathsf{v}}$. The isomorphism class of a Lie affgebroid structure on an affine bundle $\tau\colon A\to M$ with the prescribed vector part $[\,,\,]_{\mathsf{v}}$ on $\mathsf{V}(A)$ is determined by the class $[D]\in H^1_{Qder}(\mathsf{V}(A),\mathsf{V}(A))$ for the Lie algebroid $(\mathsf{V}(A),[\,,\,]_{\mathsf{v}})$.

PROOF: We have already seen that a Lie affgebroid structure on A determines a Lie algebroid structure on $\mathsf{V}(A)$ and a class from $H^1_{Qder}(\mathsf{V}(A),\mathsf{V}(A))$ represented by $D=[\alpha_0,\cdot]^{\mathsf{v}}$ for a section $\alpha_0\in\mathsf{Sec}(A)$. Let now be given a Lie algebroid bracket $[\,,]_{\mathsf{v}}$ on $\mathsf{Sec}(\mathsf{V}(A))$ and $[D]\in H^1_{Qder}(\mathsf{V}(A),\mathsf{V}(A))$. We choose a section $\alpha_0\in\mathsf{Sec}(A)$ and define a bracket $[\,,]\colon\mathsf{Sec}(A)\times\mathsf{Sec}(A)\to\mathsf{Sec}(\mathsf{V}(A))$ by

$$[\alpha_0+X,\alpha_0+Y]=D(Y)-D(X)+[X,Y]_{\mathsf{v}}. \tag{46}$$

It is easy to see that this formula defines a Lie affgebroid bracket on A with the linear part $[\,,]_{\mathsf{v}}$. The Jacobi identity follows from the fact that D is a derivation of $[\,,]_{\mathsf{v}}$, and the existence of the anchor follows from the fact that D is a quasi-derivation. If we choose another representant D' of $[D]$, then $D'=D+\mathsf{ad}_{X_0}$ for certain $X_0\in\mathsf{Sec}(\mathsf{V}(A))$, and we get another bracket

$$[\alpha_0+X,\alpha_0+Y]'=(D+\mathsf{ad}_{X_0})(Y-X)+[X,Y]_{\mathsf{v}}. \tag{47}$$

We have clearly an isomorphism between $[\,,]$ and $[\,,]'$ induced by the isomorphism $A\ni\alpha\mapsto\alpha+X_0(\tau(\alpha))$. ∎

The next theorem shows that every Lie affgebroid structure can be extended to a standard Lie algebroid.

THEOREM 8. *Let $[\,,]_0$ be a Lie algebroid bracket on sections of a vector bundle E over M and let $[D]\in H^1_{Qder}(E,E)$, i.e. D is a* Qder*-1-cocycle. Then, the bracket defined on sections of $E\oplus\mathbb{R}$ by*

$$[(X,f),(Y,g)]=\Big([X,Y]_0+fD(Y)-gD(X),(\widehat{X}+f\widehat{D})(g)-(\widehat{Y}+g\widehat{D})(f)\Big) \tag{48}$$

is a Lie algebroid bracket on $E\oplus\mathbb{R}$ extending $[\,,]_0$ on sections of $E=E\oplus\{0\}\subset E\oplus\mathbb{R}$ and such that $[\mathsf{Sec}(A),\mathsf{Sec}(A)]\subset\mathsf{Sec}(E)$, where A is the affine subbundle $A=E\oplus\{1\}$ of $E\oplus\mathbb{R}$, modeled on $\mathsf{V}(A)=E=E\oplus\{0\}$.

Conversely, every Lie algebroid extension of $[\,,]_0$ with the above properties has the form (48) and the isomorphism class of such a Lie algebroid bracket on $E\oplus\mathbb{R}$ is determined by the class $[D]$. The Lie algebroid bracket (48) reduced to the sections of A gives a Lie affgebroid bracket (45) on A with $\alpha_0=(0,1)\in\mathsf{Sec}(E\oplus\mathbb{R})$.

PROOF: The $C^\infty(M)$-module $\mathsf{Sec}(E\oplus\mathbb{R})$ is generated by the sections of E and the section $\alpha_0=(0,1)\in\mathsf{Sec}(E\oplus\mathbb{R})$. There is a unique extension of the bracket $[\,,]_0$ on sections of E to a Lie algebroid bracket on $\mathsf{Sec}(E\oplus\mathbb{R})$ for which $\mathsf{ad}_{\alpha_0}=D$, namely the bracket (48). The Jacobi identity follows from the fact that D is a derivation of $[\,,]_0$, and the existence of the anchor follows from the fact that D is a quasi-derivation. If we choose another representant $D'\in[D]$, $D'=D+\mathsf{ad}_Y$, the corresponding Lie algebroid bracket $[\,,]'$ on $\mathsf{Sec}(E\oplus\mathbb{R})$ goes to (48) with respect to the isomorphisms $(X_p,t)\to(X_p+tY(p),t)$ of $\mathsf{Sec}(E\oplus\mathbb{R})$. Indeed, with respect to this isomorphism (X,f) goes to $(X+fY,f)$, i.e. sections X of E are preserved and α_0 goes to α_0+Y. Since $\mathsf{ad}_{\alpha_0+Y}=D+\mathsf{ad}_Y=D'$, the bracket $[\,,]$ goes to $[\,,]'$.

Conversely, if $[\,,]$ is a Lie algebroid bracket on $\mathsf{Sec}(E\oplus\mathbb{R})$ extending $[\,,]_0$ on the sections of E and such that $[\mathsf{Sec}(A),\mathsf{Sec}(A)]\subset\mathsf{Sec}(E)$, then the bracket has clearly the form (48) for $D=\mathsf{ad}_{\alpha_0}$. Since

$$[(X,1),(Y,1)]=[\alpha_0+X,\alpha_0+Y]=([X,Y]_0+D(Y-X),0),$$

the bracket (48), reduced to A, gives (45). ∎

Remark. The mentioned isomorphisms are, in general, non-canonical and depend on the choice of a section or a representant of the class in $H^1_{Qder}(\mathsf{V}(A),\mathsf{V}(A))$. A canonical way to interpret a Lie affgebroid as an affine subbundle of a Lie algebroid will be described in the next section.

5. Affine - special linear duality.

For an affine bundle $\tau: A \to M$, let $\tau^\dagger: A^\dagger \to M$ be the vector bundle of affine functions on fibers of A, i.e. the fiber $A^\dagger_p$ at $p \in M$ is the space $\mathsf{Aff}(A_p, \mathbb{R})$ of affine functions on A_p. It is easy to see that $A^\dagger$ carries a canonical smooth vector bundle structure. Locally, for coordinates (x^a) on M and for a local basis $e_0, e_1, \ldots, e_n$ of sections of A (i.e. $e_1 - e_0, e_2 - e_0, \ldots, e_n - e_0$, form a local basis of sections of $\mathsf{V}(A)$), we have the corresponding coordinates (x^a, s^i) on A such that for $\alpha \in A_p$

$$\alpha = e_0(p) + \sum_i s^i(\alpha)(e_i(p) - e_0(p)).$$

The adapted local coordinates on $A^\dagger$ are (x^a, a_i, b) and for $f \in A^\dagger_p$, we have

$$f(\alpha) = a_i(f)s^i(\alpha) + b(f).$$

Note that $A^\dagger$ is a special vector bundle with the distinguished section 1_A corresponding to the constant function with the value 1. The special vector bundle $(A^\dagger, 1_A)$ will be called the *vector dual* to the affine bundle A. Conversely, with a special vector bundle $\boldsymbol{V} = (V, \varphi_0)$ we associate an affine bundle $\boldsymbol{V}^\ddagger$ which is identified with the affine subbundle of the dual vector bundle V^*, defined by the equation

$$(V, \varphi_0)^\ddagger = \{f \in V^*: \langle f, \varphi_0\rangle = 1\}. \tag{49}$$

For the detailed discussion on affine structures involved see Section 8. Since for any vector bundle V we have the obvious identification

$$\mathsf{Sec}(V^*) \ni X \mapsto \imath_X \in \mathsf{Lin}(V, \mathbb{R})$$

of sections of V^* with linear functions on the dual bundle V, sections of $(V, \varphi_0)^\ddagger$ can be viewed as those linear functions on V which take the value 1 on $\varphi_0(M) \subset V$.
The evaluation mappings

$$\begin{aligned} \langle\cdot,\cdot\rangle_A &: A \times_M A^\dagger \to M \times \mathbb{R} \\ &: (a, \varphi) \mapsto \varphi(a) \end{aligned} \tag{50}$$

and

$$\begin{aligned} \langle\cdot,\cdot\rangle_{\varphi_0} &: (V, \varphi_0)^\ddagger \times_M V \to M \times \mathbb{R} \\ &: (f, v) \mapsto f(v) \end{aligned} \tag{51}$$

are affine-linear and such that $\langle f, \varphi_0\rangle_{\varphi_0} = 1$, $\langle a, 1_A\rangle_A = 1$, i.e. for each $a \in A_p$, $f \in (V, \varphi_0)^\ddagger_p$, the mappings

$$\langle f, \cdot\rangle_{\varphi_0}: V_p \to \mathbb{R}, \qquad \langle a, \cdot\rangle_A \to \mathbb{R} \tag{52}$$

are morphisms of special vector spaces.

By a *pairing* between an affine bundle A and a special vector bundle $\boldsymbol{V} = (V, \varphi_0)$ over M we mean an affine-linear mapping

$$\langle\cdot,\cdot\rangle: A \times_M V \longrightarrow M \times \mathbb{R} \tag{53}$$

with the following properties:

(i) $\langle a, \varphi_0\rangle = 1$ for each $a \in A$,
(ii) it is nondegenerate, i.e. the induced morphisms $V \to A^\dagger$ and $A \to V^*$ are injective.

Note, that the image of the induced mapping $A \to V^*$ is contained in $\boldsymbol{V}^\ddagger$.

THEOREM 9. *Let*

$$\langle\cdot,\cdot\rangle: A\times_M V\to M\times\mathbb{R} \tag{54}$$

be a pairing between an affine bundle A and a special vector bundle $\boldsymbol{V}=(V,\varphi_0)$. Then the induced morphisms $V\to A^\dagger$ and $A\to \boldsymbol{V}^\ddagger$ are isomorphisms.

COROLLARY 1. *We have natural isomorphisms*

(a) $A\simeq (A^\dagger, 1_A)^\ddagger$,
(b) $(V,\varphi_0)\simeq (((V,\varphi_0)^\ddagger)^\dagger, 1_{(V,\varphi_0)^\ddagger})$.

PROOF OF THE THEOREM: Let $\phi_r: V\to A^\dagger$ and $\phi_l: A\to \boldsymbol{V}^\ddagger$ be the mappings induced by the pairing $\langle\cdot,\cdot\rangle$. Since they are injective, it is enough to prove that they are surjective. Since $1\in\phi_r(V_p)$ and the intersection of the kernels $\bigcap_{v\in V_p}\mathsf{Ker}\langle\cdot,v\rangle_{\mathsf{v}}$ is trivial, ϕ_r is an epimorphism. Moreover, $\phi_r(\varphi_0)=1_A$.

It remains to show that $A\simeq (V,\varphi_0)^\ddagger$. The map $A_p\ni a\mapsto \phi_l(a)=\langle a_p,\cdot\rangle\in V_p^*$ induces an affine injective map $\phi_l: A\to \boldsymbol{V}^\ddagger$. Moreover $\phi_l(A)=(V,\varphi_0)^\ddagger$, since if $\operatorname{codim}\phi_l(A_p)>1$ then there are infinitely many $\varphi\in V_p$ for which $\langle\cdot,\varphi\rangle=1$, in spite of the fact that ϕ_r is injective. ■

In this way we have defined duality between affine bundles on one hand and special vector bundles on the other hand. The vector bundle $(A^\dagger)^*$ we denote by $\widehat{A}$ and call the *vector hull* of the affine bundle A. The affine bundle A is canonically isomorphic to the affine subbundle of $\widehat{A}$ described as a 1-level set of the linear function $\imath_{\varphi_0}$ on $\widehat{A}$ defined by a non-vanishing section φ_0 of $A^\dagger$.

EXAMPLE 3. Let A be an affine bundle modeled on a one-dimensional vector bundle $\mathsf{V}(A)$ over M. A fiber of $A^\dagger$ is a vector space of dimension 2 and for each $a\in A_p$ the fiber $A^\dagger_p$ can be split into a direct sum of two 1-dimensional subspaces: one contains constant functions and the other one consists of functions vanishing at a. The first subspace can be parameterized by reals and the second by the linear parts, i.e. by elements of $(\mathsf{V}(A))^*_p$. We have then a surjective mapping

$$\begin{aligned}\eta&: A\times_M \mathsf{V}(A)^*\times\mathbb{R}\to A^\dagger\\ &: (a,\varphi,t)\mapsto (b\mapsto \varphi(b-a)+t)\end{aligned} \tag{55}$$

which is a linear isomorphism for each $a\in A$. The equality $\eta(a,\varphi,t)=\eta(a',\varphi',t')$, i.e.

$$\varphi(b-a)+t=\varphi'(b-a')+t'=\varphi'(b-a)+\varphi'(a-a')+t',$$

is equivalent to $\varphi=\varphi'$ and $t=t'+\varphi'(a-a')$. It follows that the level sets of η are orbits of the fiber-wise action of $\mathsf{V}(A)$ on $A\times_M\mathsf{V}(A)^*\times\mathbb{R}$:

$$(u,(a,\varphi,t))\mapsto (a+u,\varphi,t+\varphi(u)). \tag{56}$$

Since $\eta_a=\eta(a,\cdot,\cdot)$ is an isomorphism for each a, we have also the dual isomorphism for $a\in A_p$

$$(\eta_a)^*: (A^\dagger_p)^*\to (\mathsf{V}(A)^*_p\times\mathbb{R})^*=\mathsf{V}(A)_p\times\mathbb{R}. \tag{57}$$

The level sets of the dual action

$$\eta^*: A\times_M\mathsf{V}(A)\times\mathbb{R}\to (A^\dagger)^*=\widehat{A}, \tag{58}$$

$(\eta^*)_a=((\eta_a)^*)^{-1}$, are the orbits of the dual action of $\mathsf{V}(A)$ on $A\times_M\mathsf{V}(A)\times\mathbb{R}$, given by the formula

$$(u,(a,v,t))\mapsto (a-u,v+su,t), \tag{59}$$

and $\widehat{A}$ can be viewed as the vector bundle of the orbits $[(a,v,t)]$. In the representations given by η and η^*, the canonical pairing between $A^\dagger$ and $\widehat{A}$ takes the form

$$\langle[(a,\varphi,t)],[(a',v,s)]\rangle=\varphi(v)+s\varphi(a'-a)+st \tag{60}$$

and the embedding $A\hookrightarrow\widehat{A}$ is given by $a\mapsto[(a,0,1)]$. ♠

Example 4. Let Z be a one-dimensional affine bundle over M modeled on the trivial bundle $M\times\mathbb{R}$. In this case

$$Z\times_M \mathsf{V}(Z)^*\times\mathbb{R} = Z\times\mathbb{R}^2 = Z\times_M \mathsf{V}_(Z)\times\mathbb{R}. \tag{61}$$

With these identifications the level sets of η and η^* are, respectively, the orbits of $\mathbb{R}$-actions:

$$(\alpha,(z,s,t))\mapsto(z+\alpha,s,t+s\alpha)$$

and

$$(\alpha,(z,s,t))\mapsto(z-\alpha,s+\alpha t,t).$$

We can identify $z\in Z_p$ with an affine function $Z_p\ni y\mapsto z-y$ represented by the triple $(z,-1,0)$. Thus we have inclusions $Z\hookrightarrow\widehat{Z}$ and $Z\hookrightarrow Z^\dagger$ which define an isomorphism of $Z^\dagger$ and $\widehat{Z}$, and the pairing

$$\begin{aligned}\langle\cdot,\cdot\rangle_Z\colon Z^\dagger\times_M\widehat{Z} = Z^\dagger\times_M Z^\dagger &\to\mathbb{R}\\ :([(z,s,t)],[(z',s',t')]) &\mapsto st'-s's(z-z')-s't = s(t'+s'(y-z'))-s'(t+s(y-z)).\end{aligned} \tag{62}$$

Let f,g be affine functions represented by (z,s,t) and (z',s',t') respectively, i.e. $f(y)=t+s(y-z)$ and $g(y)=t'+s'(y-z')$. Then the formula (62) reads

$$\langle f,g\rangle = X_0(g)f-X_0(f)g, \tag{63}$$

where $X_0=-\frac{\partial}{\partial y}$ is the fundamental vector field associated with the $\mathbb{R}$-action on Z. For $s\neq 0$ the equivalence classes can be parameterized by $Z\times\mathbb{R}_*$:

$$(z,s)\mapsto[(z,s,0)] \tag{64}$$

and for $s=0$ by $\mathbb{R}$:

$$t\to[(z,0,t)]. \tag{65}$$

It follows that $Z^\dagger$ can be parameterized by the set $Z^0=(Z\times\mathbb{R}_*)\cup(M\times\mathbb{R}\times\{0\})$ with the obvious projection on M. In this representation the vector bundle structure of $Z^\dagger$ is given by the formulae

$$\begin{aligned}(s,a)+(s',a') &= \begin{cases}(s+\frac{a'}{a+a'}(s'-s),a+a') & \text{if } a+a'\neq 0\\ (a(s'-s),0) & \text{if } \ a+a'=0\end{cases}\\ (t,0)+(t',0) &= (t+t',0)\\ (t,0)+(s,a) &= (s-\frac{t}{a},a)\\ \lambda(s,a) &= (s,\lambda a)\\ \lambda(t,0) &= (\lambda t,0).\end{aligned} \tag{66}$$

The vector (s,a) represents the affine function on Z_p vanishing at s and with directional coefficient a, $a\neq 0$ and $(t,0)$ represents the constant affine function equal to t. ♠

Example 5. Let Z be a one-dimensional affine bundle which is vectorially trivial. The vector bundle $\widetilde{\mathsf{T}}Z$ introduced in Example 1 has a non-vanishing distinguished section X_0, so that it is a special vector bundle. An element of the dual affine bundle $\widetilde{\mathsf{T}}^\ddagger Z=(\widetilde{\mathsf{T}}Z)^\ddagger$ is a hyperspace in $\widetilde{\mathsf{T}}_pZ$, transversal to $X_0(p)$. It follows that a section of $\widetilde{\mathsf{T}}^\ddagger Z$ can be identified with a ψ-invariant horizontal distribution, i.e. a connection on Z. Standard representation of such connection is a ψ-invariant 1-form α on Z such that $\langle\alpha,X_0\rangle=1$. In [18] the affine bundle $\widetilde{\mathsf{T}}^\ddagger Z$ has been denoted by $\mathsf{P}Z$. This is an affine version of the cotangent bundle T^*M. ♠

6. The Lie algebroid hull of a Lie affgebroid.

We begin this section with the following lemma.

LEMMA 1. *Let A be an affine subspace of a finite dimensional vector space V, $A=\{v\in V\colon \varphi(v)=1\}$ for certain $\varphi\in V^*$, $\varphi\neq 0$. Then,*

(1) $A^{\otimes n}=\{a_1\otimes\ldots\otimes a_n\colon a_i\in A\}$ *spans* $V\otimes\ldots\otimes V$ *– the n-fold tensor product of* V*;*
(2) $A^{\wedge n}=\{a_1\wedge\ldots\wedge a_n\colon a_i\in A\}$ *spans* $\bigwedge^n V$*;*
(3) $A\wedge \mathsf{V}(A)^{\wedge n}=\{a\wedge v_1\wedge\ldots\wedge v_n : a\in A, v_i\in\mathsf{V}(A)\}$ *spans* $\bigwedge^{n+1}V$*.*

PROOF: (1) Inductively, $A=a_0+\mathsf{V}(A)$ for any $a_0\in\mathsf{Sec}(A)$ and $a_0\notin\mathsf{V}(A)$, so

$$\mathsf{span}A=\mathsf{span}\{a_0,\mathsf{V}(A)\}=V.$$

Assume, that $A^{\otimes n}$ spans the n-fold tensor product of V and let $v_1,\ldots,v_{n+1}\in V$. Since $v_1\otimes\ldots\otimes v_n$ is a linear combination $\sum\lambda_j u_j$ for a basis u_j of $V^{\otimes n}$ from $A^{\otimes n}$ and $v_{n+1}=\lambda a_0+v$, $v\in\mathsf{V}(A)$, we can write $v_{n+1}=a_0+v+(\lambda-1)a_0$, so that

$$\begin{aligned}v_1\otimes\ldots\otimes v_{n+1}=v_1\otimes\ldots\otimes v_n\otimes((a_0+v)+(\lambda-1)a_0)&=\sum\lambda_i u_i\otimes((a_0+v)+(\lambda-1)a_0)\\&=\sum\lambda_i u_i\otimes(a_0+v)+\sum(\lambda-1)\lambda_i u_i\otimes a_0,\end{aligned}\tag{67}$$

i.e. $v_1\otimes\ldots\otimes v_{n+1}$ is a linear combination of elements of $A^{\otimes(n+1)}$.
(2) follows easily from (1).
(3) It follows from (2) that $A^{\wedge(n+1)}$ spans $\bigwedge^{(n+1)}V$. Since for $a_0\in A$, $v_i\in\mathsf{V}(A)$,

$$(a_0+v_1)\wedge\ldots\wedge(a_0+v_{n+1})=\sum_{i\geq 2}^{n+1}(-1)^{i+1}a_0\wedge v_1\wedge.\overset{i}{\vee}.\wedge v_{n+1}\;+\;(a_0+v_1)\wedge v_1\wedge\ldots\wedge v_{n+1},$$

$A\wedge(V_A)^{\wedge n}$ spans $A^{\wedge(n+1)}$, so $\bigwedge^{n+1}V$. ■

THEOREM 10. *For every Lie affgebroid bracket $[\,,]$ on an affine bundle A over M there is a unique Lie algebroid bracket $[\,,]^\wedge$ on $\widehat{A}$ such that $[\,,]$ is the restriction of $[\,,]^\wedge$ to sections of A.*

PROOF: For a given section α_0 of A there is a unique isomorphism of A with the affine subbundle $A'=\mathsf{V}(A)\oplus\{1\}$ of $\mathsf{V}(A)\oplus\mathbb{R}$ such that α_0+X corresponds to $(X,1)$ for any section X of $\mathsf{V}(A)$. This isomorphism extends uniquely to an isomorphism of vector bundles $\widehat{A}$ and $\mathsf{V}(A)\oplus\mathbb{R}$ which maps sections $f\alpha_0+X$ to (X,f). According to Theorem 8, there is a Lie algebroid bracket $[\,,]$ on $\mathsf{V}(A)\oplus\mathbb{R}$ which, restricted to $\mathsf{Sec}(A)\simeq\mathsf{Sec}(A')$, gives a Lie affgebroid bracket, so the existence follows. This Lie affgebroid bracket defines the bracket and anchors for a generating set of sections of $\widehat{A}$ (A spans $\widehat{A}$), so the Lie algebroid bracket on $\widehat{A}$ extending the Lie affgebroid bracket on A is unique. ■

Note that the above result has been already mentioned in [15].

Now, let $\tau\colon A\to M$ be an affine bundle and let $\varphi=1_A$ be the section of $A^\dagger$ such that A is the level-1-set of φ in $\widehat{A}=(A^\dagger)^*$. Let $[\,,]$ be a Lie algebroid bracket on sections of $\widehat{A}$, let Λ be the corresponding linear Poisson tensor on $A^\dagger$ and let d be the corresponding exterior derivative. Let $\mathsf{Sec}(\widehat{A})\ni X\mapsto X^c\in\mathfrak{X}(\widehat{A})$ be the complete lift of sections of the Lie algebroid $\widehat{A}$ to vector fields on $\widehat{A}$ (cf. [11,3,4]).

THEOREM 11. *The following are equivalent*

(1) *the restriction of $[\,,]$ to the sections of A is a Lie affgebroid bracket;*
(2) $[X,Y]\in\mathsf{Sec}(\mathsf{V}(A))$ *for each* $X,Y\in\mathsf{Sec}(A)$*;*
(3) $\mathrm{d}\varphi=0$*;*
(4) $\mathfrak{L}_{V_{\mathsf{T}}(\varphi)}\Lambda=0$*, where $\mathfrak{L}$ is the Lie derivative and $V_{\mathsf{T}}(\varphi)$ is the vertical lift of φ;*
(5) *the complete Lie algebroid lift X^c of any section X of A is a vector field on $\widehat{A}$ which is tangent to $A\subset\widehat{A}$.*

PROOF: (1)$\Rightarrow$(2) is trivial.

(2)$\Rightarrow$(3) Let $X, Y \in \mathsf{Sec}(A)$. Then

$$\mathrm{d}\varphi(X,Y) = \widehat{X}(\varphi(Y)) - \widehat{Y}(\varphi(X)) - \varphi([X,Y]) = 0,$$

since $\varphi(X) = \varphi(Y) = 1$ and φ vanishes on $\mathsf{V}(A)$. It follows from Lemma 1 that $\mathrm{d}\varphi = 0$.

(3)$\Leftrightarrow$(4) follows from the identity $[\Lambda, V_{\mathsf{T}}(\mu)]^{SN} = V_{\mathsf{T}}(\mathrm{d}\mu)$ ([2], Thm 15.d), where μ is a multisection of $A^{\dagger}$ and $[\,,\,]^{SN}$ is the Schouten-Nijenhuis bracket.

(4)$\Rightarrow$(5) For every $\mu \in \mathsf{Sec}(A^{\dagger})$, $X^c(\imath_\mu) = \imath_{\pounds_X \mu}$, where $\pounds$ is the Lie derivative and $\imath_\mu$ is the linear function on $\widehat{A}$ corresponding to μ (see [11,3,4]). Since, for $X \in \mathsf{Sec}(A)$,

$$\pounds_X \varphi = \mathrm{i}_X \mathrm{d}\varphi + \mathrm{d}\mathrm{i}_X \varphi = \mathrm{i}_X 0 + \mathrm{d}(1) = 0,$$

we get $X^c(\imath_\varphi) = 0$. Since A is a level-set of $\imath_\varphi$, it follows that X^c is tangent to A.

(5)$\Rightarrow$(1) Let $X, Y \in \mathsf{Sec}(A)$. Since X^c is tangent to A, the function $X^c(\imath_\varphi)$ vanishes on A. But the function $X^c(\imath_\varphi)$ is linear and A spans $\widehat{A}$, so that $X^c(\imath_\varphi) = 0$. Since $V_{\mathsf{T}}(Y)(\imath_\varphi) = V_{\mathsf{T}}(\varphi(Y)) = 1$, and (cf. [3,4]) $V_{\mathsf{T}}([X,Y])(\imath_\varphi) = [X^c, V_{\mathsf{T}}(Y)](\imath_\varphi)$, we get $V_{\mathsf{T}}([X,Y])(\imath_\varphi) = 0$. But $V_{\mathsf{T}}([X,Y])(\imath_\varphi) = V_{\mathsf{T}}(\varphi([X,Y]))$, so $\varphi([X,Y]) = 0$ and $[X,Y] \in \mathsf{Sec}(\mathsf{V}(A))$. Now, the Jacobi identity and the existence of the anchor map for the bracket restricted to a map on $\mathsf{Sec}(A) \times \mathsf{Sec}(A)$ follows directly from the corresponding properties of the Lie algebroid bracket $[\,,\,]$. ■

According to Theorem 11 (5), the complete lift $X^c$of any section X of A, so also for any section of $\mathsf{V}(A)$, is tangent to the submanifold A of $\widehat{A}$ and we can define the *Lie affgebroid complete lift* X^A as the restriction of the vector field X^c to A. Since $V_{\mathsf{T}}(X)(\imath_\varphi) = \langle X, \varphi\rangle \circ \tau$, also the vertical lift of any section X of $\mathsf{V}(A)$ is tangent to A. Its restrictions to A we will denote $V^A_{\mathsf{T}}(X)$ and call the *Lie affgebroid vertical lift* of X.

THEOREM 12. *For any Lie affgebroid bracket $[\,,\,]$ on A and all sections $X_1, X_2 \in \mathsf{Sec}(A)$, $Y_1, Y_2 \in \mathsf{Sec}(\mathsf{V}(A))$,*

(a) $[X_1^A, X_2^A] = ([X_1, X_2])^A$,
(b) $[Y_1^A, Y_2^A] = ([Y_1, Y_2])^A$,
(c) $[X_1^A, V_{\mathsf{T}}(Y_1)] = V^A_{\mathsf{T}}([X_1, Y_1])$,
(d) $[V^A_{\mathsf{T}}(Y_1), V^A_{\mathsf{T}}(Y_2)] = 0$.

Here the brackets on the left-hand sides are clearly the brackets of vector fields on A.

PROOF: The above identities follow immediately from the analogous identities for the complete and vertical lifts of sections of a Lie algebroid (cf. [3,4]). ■

Theorem 11 implies that a Lie affgebroid structure on A is determined, in fact, by a Lie algebroid structure on a vector bundle E (which is $\widehat{A}$) and a non-vanishing 1-cocycle $\varphi \in \mathsf{Sec}(E^*)$. This is a particular case of a *generalized Lie algebroid* in the sense of Iglesias and Marrero [6] or a *Jacobi algebroid* in the sense of [5]. In the particular case, when the 1-cocycle φ is trivial, we get the affine Lie algebroid in the sense of [15] as shows the following.

THEOREM 13. *In the notation preceding Theorem 11, assume that we have a Lie affgebroid bracket on A with an anchor γ such that $\varphi = \mathrm{d}t$ for certain $t \in C^\infty(M)$. Then $t\colon M \to \mathbb{R}$ is a fibration over the open subset $t(M)$ of $\mathbb{R}$ and γ maps sections of A to vector fields on M which are projectable onto the canonical vector field $\frac{\partial}{\partial t}$ on $\mathbb{R}$.*

PROOF: Since

$$\gamma(X)(t) = \langle X, \mathrm{d}t\rangle = \langle X, \varphi\rangle = 1 \tag{68}$$

for any section X of A, the function t is regular, so that $t : M \to \mathbb{R}$ is a fibration and $\gamma(X)$ is projectable onto $\frac{\partial}{\partial t}$. ■

Remark. In the case when φ is not trivial one can make it trivial for the prize of extending the base manifold by $\mathbb{R}$. This construction for generalized Lie algebroids can be found in [6]. The sections of the corresponding Lie algebroid can be viewed as time-dependent sections of the original one.

7. Exterior calculus.

DEFINITION 4. An *affine k-form* on an affine bundle A over M is a skew-symmetric k-affine map

$$\overline{\mu}\colon \mathsf{Sec}(A)\times\mathsf{Sec}(A)\times\cdots\times\mathsf{Sec}(A)\to C^\infty(M).$$

which comes from an affine morphism

$$A\times_M\cdots\times_M A\to M\times\mathbb{R},$$

i.e. $\overline{\mu}(a_1,\dots,a_{k-1},\cdot)_{\mathsf{v}}$ is $C^\infty(M)$-linear for any $a_1,\dots,a_{k-1}\in\mathsf{Sec}(A)$.

Remark. This definition coincides with that one in [15]. The space $\mathsf{Aff}\Omega^k(A)$ of affine k-forms is a $C^\infty(M)$-module in the obvious way. We have also the standard wedge product on the graded space $\mathsf{Aff}\Omega(A)=\bigoplus_{k\in\mathbb{Z}}\mathsf{Aff}\Omega^k(A)$ which makes $\mathsf{Aff}\Omega(A)$ into a graded associative commutative algebra. Here, $\mathsf{Aff}\Omega^k(A)=\{0\}$ for $k<0$ and $\mathsf{Aff}\Omega^0(A)=C^\infty(M)$.

THEOREM 14. *There is a canonical isomorphism between the space $\mathsf{Aff}\Omega^k(A)$ of affine k-forms on A and the space $\Omega^k(\widehat{A})=\mathsf{Sec}(\wedge^k A^\dagger)$ of sections of the k-th exterior power of $A^\dagger$, given by restrictions of k-forms on $\widehat{A}$ to A. This isomorphism can be extended to an isomorphism of graded associative algebras $\mathsf{Aff}\Omega(A)$ and $\Omega(\widehat{A})$.*

PROOF: By definition, every affine 1-form $\mu\colon\mathsf{Sec}(A)\to C^\infty(M)$ is represented by a unique section of $A^\dagger$ and vice-versa. Sections of $\wedge^k A^\dagger$ can be identified with k-forms on $\widehat{A}=(A^\dagger)^*$ and the restriction of any k-form

$$\mu\colon\mathsf{Sec}(\widehat{A})\times\cdots\times\mathsf{Sec}(\widehat{A})\to C^\infty(M)$$

to $\mathsf{Sec}(A)\times\cdots\times\mathsf{Sec}(A)$ gives clearly an affine k-form $\overline{\mu}$ on A. Conversely, let $\overline{\mu}$ be an affine k-form on A. Let us choose a section α_0 of A. Let $v^o_1,\dots,v^o_n$ be a local basis of sections of $\mathsf{V}(A)$. Then $\alpha^o_0=\alpha_0$ and $\alpha^o_i=\alpha_0+v^o_i$, $i=1,\dots,n$, is a basis of sections of $\widehat{A}$. There is a unique k-form μ on $\widehat{A}$ such that

$$\mu(\alpha^o_{i_1},\dots,\alpha^o_{i_k})=\overline{\mu}(\alpha^o_{i_1},\dots,\alpha^o_{i_k})$$

for all $i_1,\dots,i_k\in\{0,\dots,n\}$. We will show that $\overline{\mu}$ is the restriction of μ to sections of A. First, we observe that, since $\overline{\mu}$ is skew-symmetric and affine,

$$\overline{\mu}(\alpha_0+v_1,\dots,\alpha_0+v_k)=\sum(-1)^{i+1}\overline{\mu}^{\mathsf{v}}(\alpha_0,v_1,\overset{i}{.\check{}.},v_k)+\overline{\mu}_{\mathsf{v}}(v_1,\dots,v_k)\tag{69}$$

for certain affine-multilinear and linear maps $\overline{\mu}^{\mathsf{v}}$ and $\overline{\mu}_{\mathsf{v}}$, respectively, and for all $v_1,\dots,v_k\in\mathsf{Sec}(\mathsf{V}(A))$. A similar formula for μ reads

$$\mu(\alpha_0+v_1,\dots,\alpha_0+v_k)=\sum(-1)^{i+1}\mu(\alpha_0,v_1,\overset{i}{.\check{}.},v_k)+\mu(v_1,\dots,v_k).\tag{70}$$

Hence,

$$\overline{\mu}^{\mathsf{v}}(\alpha_0,v_2,\dots,v_k)=\overline{\mu}(\alpha_0,\alpha_0+v_2,\cdots\alpha_0+v_k),$$

so that

$$\overline{\mu}^{\mathsf{v}}(\alpha^o_0,v^o_{i_2},\dots,v^o_{i_k})=\overline{\mu}(\alpha^o_0,\alpha^o_{i_2},\dots,\alpha^o_{i_k})=\mu(\alpha^o_0,\alpha^o_{i_2},\dots,\alpha^o_{i_k})=\mu(\alpha_0,v^o_{i_2},\dots,v^o_{i_k}).\tag{71}$$

This implies further, by (69), that

$$\overline{\mu}_{\mathsf{v}}(v^o_{i_1},\dots,v^o_{i_k})=\mu(v^o_{i_1},\dots,v^o_{i_k})\tag{72}$$

for all $i_1,\dots,i_k\in\{0,\dots,n\}$. Since both μ and $\overline{\mu}^{\mathsf{v}}$ are $C^\infty(M)$-linear in the last (k-1)-arguments, (71) implies that

$$\overline{\mu}^{\mathsf{v}}(\alpha_0^o,v_2,\dots,v_k)=\mu(\alpha_0,v_2,\dots,v_k)$$

for any $v_2,\dots,v_k\in\mathsf{Sec}(\mathsf{V}(A))$. Similarly, (72) implies

$$\overline{\mu}_{\mathsf{v}}(v_1,v_2,\dots,v_k)=\mu(v_1,v_2,\dots,v_k)$$

and finally, due to (69), that

$$\overline{\mu}(\alpha_0+v_1,\dots,\alpha_0+v_k)=\mu(\alpha_0+v_1,\dots,\alpha_0+v_k),$$

i.e. that $\overline{\mu}$ is the restriction of μ to sections of A. ■

It is well known that a Lie algebroid structure determines an exterior derivative on forms. So, the Lie algebroid structure on $\widehat{A}$, extending the Lie affgebroid structure on A, gives rise to an exterior derivative

$$\mathrm{d}\colon\Omega^n(\widehat{A})\to\Omega^{n+1}(\widehat{A})$$

which, in view of the previous theorem, can be seen as an exterior derivative on $\mathsf{Aff}\Omega^n(A)$:

$$\mathrm{d}\colon\mathsf{Aff}\Omega^n(A)\to\colon\mathsf{Aff}\Omega^{n+1}(A).$$

The derivation property of d means that

$$\mathrm{d}(\mu\wedge\nu)=\mathrm{d}\mu\wedge\nu+(-1)^{deg\mu}\mu\wedge\mathrm{d}\nu.$$

Since we obtain affine k-forms on A from k-forms on $\widehat{A}$ by restriction, the Cartan-like formula for d is clearly

$$\begin{aligned}\mathrm{d}\overline{\mu}(\alpha_0,\dots,\alpha_n)&=\sum(-1)^{i+1}\gamma(\alpha_i)(\mu(\alpha_0,\overset{i}{\ldots\vee\ldots},\alpha_n))+\sum_{i<j}(-1)^{i+j}\mu([\alpha_i,\alpha_j],\alpha_0,\overset{i}{\ldots\vee}\ldots\overset{j}{\vee\ldots},\alpha_n)=\\&\sum(-1)^{i+1}\gamma(\alpha_i)(\overline{\mu}(\alpha_0,\overset{i}{\ldots\vee\ldots},\alpha_n))-\sum_{i<j}(-1)^{n+i+j}\overline{\mu}^{\mathsf{v}}(\alpha_0,\overline{\alpha}_1,\overset{i}{\ldots\vee}\ldots\overset{j}{\vee\ldots},\overline{\alpha}_n,[\alpha_i,\alpha_j]),\end{aligned}\tag{73}$$

where $\alpha_i\in\mathsf{Sec}(A)$ and $\overline{\alpha}_i=\alpha_i-\alpha_0$ for $i>0$. The corresponding affine de Rham complex on A is equivalent, via the isomorphism described in Theorem 14, to the de Rham complex of the Lie algebroid $\widehat{A}$. Using the well-known fact that graded derivations of degree 1 on $\Omega(\widehat{A})$ defined by the Cartan formula

$$\mathrm{d}\mu(X_0,X_1,\dots,X_n)=\sum_i(-1)^i\widehat{X}_i(\mu(X_0,\overset{i}{\ldots\vee\ldots},X_n))+\sum_{i<j}(-1)^{i+j}\mu([X_i,X_j],X_1,\overset{i}{\ldots\vee}\ldots\overset{j}{\vee\ldots},X_n)\tag{74}$$

satisfy $\mathrm{d}^2=0$ if and only if the bracket $[\,,\,]$ on sections of $\widehat{A}$ is a Lie algebroid bracket with the anchor $X\mapsto\widehat{X}$, we get immediately the following [15]:

THEOREM 15. *For a graded derivation on $\mathsf{Aff}\Omega(A)$ of degree 1*

$$\mathrm{d}\colon\mathsf{Aff}\Omega(A)\to\mathsf{Aff}\Omega(A),$$

defined by (73), $\mathrm{d}^2=0$ if and only if $[\,,\,]$ is a Lie affgebroid bracket with the anchor γ.

Remark. The 1-cocycle φ determining A inside $\widehat{A}$ (cf. Theorem 11) represents the affine 1-form with the constant value 1 on $\mathsf{Sec}(A)$. The ideal $\Omega(\widehat{A})_\varphi$ generated by φ in the exterior algebra $\Omega(\widehat{A})$ is a differential ideal and, since $A^\dagger/\langle\varphi\rangle = \mathsf{V}(A)^*$, the differential algebra

$$\Omega(\mathsf{V}(A)) = \Omega(\widehat{A})/\Omega(\widehat{A})_\varphi$$

represents just the exterior calculus for the Lie algebroid structure on $\mathsf{V}(A)$.

8. Affine bundles of vector bundle epimorphisms.

According to Corollary 1, sections of A may be identified with linear functions on $A^\dagger$ which take the value 1 on the section $\varphi = 1_A$ and, with respect to this identification, the Lie affgebroid bracket on A may be identified with a linear Poisson bracket on $A^\dagger$ restricted to this affine subspace of functions. We shall use also another identification in which the Lie affgebroid bracket comes from an aff-Poisson bracket. To do this we need some preparations.

Let $\tau_i\colon E_i \to M$, i=1,2, be vector bundles and let $\rho \in \mathsf{Hom}_M(E_1, E_2)$ be a vector bundle epimorphism, i.e. $\rho(E_1) = E_2$. The kernel of ρ is a vector subbundle of E_1 which will be denoted by K_ρ.

PROPOSITION 1. *The fibration $\rho\colon E_1 \to E_2$ is an affine fibration modeled on the vector bundle $E_2 \times_M K_\rho$ over E_2.*

PROOF: Let $e_1, e_2 \in (E_1)_m$ with $\rho(e_1) = \rho(e_2) \in (E_2)_m$. Then $\rho(e_1 - e_2) = 0$, i.e. $e_1 - e_2 \in (K_\rho)_m$. It is easy to see that this defines a smooth affine bundle structure on the total space E_1 and base E_2 which is modeled on $E_2 \times_M K_\rho$. ■

The canonical embedding $K_\rho \hookrightarrow E_1$ induces the projection

$$\rho^0\colon E_1^* \to K_\rho^*. \tag{75}$$

The kernel of this projection can be identified with E_2^* embedded by

$$\rho^*\colon E_2^* \to E_1^*. \tag{76}$$

Let us denote:

(a) by $\mathsf{LinSec}(\rho)$ - the affine subspace of those sections $\sigma \in \mathsf{Sec}(\rho)$ which are morphisms of vector bundles, i.e.

$$\mathsf{LinSec}(\rho) = \{\sigma \in \mathsf{Hom}_M(E_2, E_1)\colon \rho\circ\sigma = id_{E_2}\}. \tag{77}$$

Since for $\sigma_1, \sigma_2 \in \mathsf{LinSec}(\rho)$, we have $\rho\circ(\sigma_1 - \sigma_2) = 0$, the affine space $\mathsf{LinSec}(\rho)$ is modeled on $\mathsf{Hom}_M(E_2, K_\rho)$;

(b) by $\mathsf{Proj}(E_1, K_\rho)$ - the affine space of those $\sigma \in \mathsf{Hom}_M(E_1, K_\rho)$ which are projections. For $\sigma_1, \sigma_2 \in \mathsf{Proj}(E_1, K_\rho)$, we have $(\sigma_1 - \sigma_2)|_{K_\rho} = 0$, i.e. $(\sigma_1 - \sigma_2) \in \mathsf{Hom}_M(E_1/K_\rho, K_\rho) \simeq \mathsf{Hom}_M(E_2, K_\rho)$ – the model vector space for $\mathsf{Proj}(E_1, K_\rho)$.

(c) by $\mathsf{ProjSec}(E_1^* \otimes_M K_\rho)$ the set of those sections $\sigma \in \mathsf{Sec}(E_1^* \otimes_M K_\rho)$ which represent projections onto K_ρ, i.e. $\langle X, \sigma\rangle = X$ for any $X \in \mathsf{Sec}(K_\rho)$, where $\langle , \rangle$ is the obvious contraction. The model space is here clearly $\mathsf{Sec}(E_2^* \otimes_M K_\rho)$.

THEOREM 16. *The following affine spaces are canonically isomorphic:*

(a) $\mathsf{LinSec}(\rho)$,
(b) $\mathsf{Proj}(E_1, K_\rho)$,
(c) $\mathsf{ProjSec}(E_1^* \otimes_M K_\rho)$,
(d) $\mathsf{LinSec}(\rho^0)$,
(e) $\mathsf{Proj}(E_1^*, E_2^*)$,
(f) $\mathsf{ProjSec}(E_1^* \otimes_M E_2^*)$.

PROOF: Any linear section $\sigma: E_2 \to E_1$ induces the vector bundle decomposition

$$E_1 = K_\rho \oplus \sigma(E_2), \tag{78}$$

so the projection of E_1 onto K_ρ, and vice versa. Any such projection can be represented by the corresponding section of $E_1 \otimes_M K_\rho$. A projection $p: E_1 \to K_\rho$ corresponds, by duality, to an embedding $p^*: K_\rho^* \to E_1^*$ which is a linear section of ρ^0 and we can repeat all the identification on the dual side for $\rho^0: E_1^* \to K_\rho^*$. ■

Remark. The linear part of the defined above isomorphism between $\mathsf{LinSec}(\rho)$ and $\mathsf{Proj}(E_2, K_\rho)$ is minus the identity on $\mathsf{Hom}_M(E_1, K_\rho)$. To see it better, let E_2 be a subspace of E_1, i.e. $E_1 = E_2 \oplus K_\rho$. Then a section $\sigma \in \mathsf{LinSec}(\rho)$ is identified with a linear mapping $\overline{\sigma}: E_2 \to K_\rho$ by $\sigma(e) = (e, \overline{\sigma}(e))$. Similarly, a projection $\lambda \in \mathsf{Proj}(E_1, K_\rho)$ is identified with $\overline{\lambda}: E_2 \to K_\rho$ by $\overline{\lambda}(e) = \lambda(e, 0)$. In both cases the affine structure is the canonical affine structure of the vector bundle $\mathsf{Hom}_M(E_1, K_\rho)$. But the isomorphism of the theorem is given by $\lambda \circ \sigma = 0$, i.e. $\lambda(e, \overline{\sigma}(e)) = \overline{\lambda}(e) + \overline{\sigma}(e) = 0$.

THEOREM 17. *Assume that K_ρ is generated by a single section φ of $\tau_1: E_1 \to M$. Then we have the identification*

$$\sigma \longleftrightarrow f_\sigma \longleftrightarrow a_\sigma,$$

where

$$\begin{aligned} &\sigma \in \mathsf{Sec}(\rho), \\ f_\sigma = \imath_{a_\sigma} &\in \mathsf{Lin}_\varphi(E_1, \mathbb{R}) = \{\mathsf{Lin}(E_1, \mathbb{R}) \ni f: f \circ \varphi = 1\}, \\ &a_\sigma \in \mathsf{Sec}((E_1, \varphi)^\ddagger), \end{aligned} \tag{79}$$

where $(E_1, \varphi)^\ddagger = \{E_1^* \ni v: \imath_\varphi(v) = 1\}$ and the correspondence between σ and f_σ is determined by the equation

$$f_\sigma(e) \cdot \varphi(\tau_1(e)) = e - \sigma \circ \rho(e) \tag{80}$$

for all $e \in E_1$.

PROOF: The correspondence $a_\sigma \longleftrightarrow f_\sigma$ is obvious. Let us see that (80) defines a unique $f_\sigma \in \mathsf{Lin}_\varphi(E_1, \mathbb{R})$ for any $\sigma \in \mathsf{LinSec}(\rho)$. Since for any $e \in E_m$ the right-hand side of (80) is an element of $(K_\rho)_m$, the coefficient $f_\sigma(e)$ is well defined. Moreover, by linearity of ρ and σ,

$$f_\sigma(e_1 + e_2)\varphi(m) = (e_1 - \sigma \circ \rho(e_1)) + (e_2 - \sigma \circ \rho(e_2)) = (f_\sigma(e_1) + f_\sigma(e_2))\varphi(m) \tag{81}$$

for any $e_1, e_2 \in (E_1)_m$, so f_σ is linear. Further, by definition,

$$f_\sigma(\varphi(m)) \cdot \varphi(m) = \varphi(m) - \sigma \circ \rho(\varphi(m)) = \varphi(m), \tag{82}$$

so $f_\sigma \circ \varphi = 1$.

Conversely, let $f \in \mathsf{Lin}_\varphi(E_1, \mathbb{R})$. Let us notice that the mapping $S: e \mapsto e - f(e) \cdot \varphi(\tau_1(e))$ is constant along fibers of ρ. Indeed, for $e \in (E_1)_m$ and $v \in (K_\rho)_m$, $v = a\varphi(m)$, we have

$$\begin{aligned} S(e + v) &= (e + v) - f(e + v)\varphi(m) \\ &= (e - f(e)\varphi(m)) + (v - f(v)\varphi(m)) \\ &= S(e) + a\varphi(m) - af(\varphi(m))\varphi(m) = S(e). \end{aligned} \tag{83}$$

Here we have used the linearity of f and the fact that $f \circ \varphi = 1$. Since $S(e)$ depends only on $\rho(e)$, we can define $\sigma: E_2 \to E_1$ by

$$\sigma(\rho(e)) = S(e). \tag{84}$$

Since $\rho(S(e)) = \rho(e)$, the mapping σ is a section. Moreover, the linearity of f implies the linearity of σ. ■

Let us see how the above correspondences look like in local coordinates. Let us choose local coordinates (x^a) on M, linear coordinates (x^a, y_i) on E_2, linear coordinates (x^a, y_i, z) on E_1. The dual coordinates on E_1^* will be denoted by (x^a, ξ^i, η). Then $K_\rho = \{e \in E_1: y_i(e) = 0\}$ is generated by the section $\varphi: (x^a) \mapsto (x^a, 0, 1)$ and $(E_1, \varphi)^\ddagger = \{E_1^* \ni e^*: \eta = 1\}$. For the linear section $\sigma(x^a, y_i) = (x^a, y_i, f^i(x)y_i)$ we have $a_\sigma(x^a) = (x^a, -f^i(x), 1)$ and $f_\sigma = \imath_{a_\sigma}(x^a, y_i, z) = -f^i(x)y_i + z$. This implies easily the following.

THEOREM 18. *In the notation of Theorem 17, $f_\sigma = \overline{\rho}^*(\sigma)$ (i.e. f_σ is the pull-back of the section σ) with respect to the morphism of the special affine bundles*

$$\begin{array}{ccc} E_1 \times \overline{\mathbb{R}} & \xrightarrow{\overline{\rho}} & E_1 \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \rho} \\ E_1 & \xrightarrow{\rho} & E_2 = E_1/\langle\varphi\rangle \end{array}, \tag{85}$$

where $\overline{\mathbb{R}}$ is the special vector space $(\mathbb{R}, -1)$ and $\overline{\rho}(e_m, t) = e_m - t\varphi(m)$.

Let us remark that, completely analogously, one can consider the affine bundles obtained from affine projections $\rho: A_1 \to A_2$ of affine bundles instead of vector bundles. We get results similar to the above ones for affine sections and functions instead of linear ones.

THEOREM 19. *There is a one-to-one correspondence between Lie affgebroid brackets $[\,,\,]$ on an affine bundle A and linear aff-Poisson brackets $\{\,,\,\}$ on the 1-dimensional affine bundle*

$$\rho: A^\dagger \to A^\dagger/\langle 1_A\rangle \simeq \mathsf{V}(A)^* \tag{86}$$

determined by

$$\{\sigma_1, \sigma_2\} = \imath_{[a_{\sigma_1}, a_{\sigma_2}]}. \tag{87}$$

Remark. Here we call an aff-Poisson structure linear if the bracket of linear sections is a linear function on $A^\dagger/\langle 1_A\rangle$. Since $A^\dagger/\langle 1_A\rangle$ can be canonically identified with $\mathsf{V}(A)^*$, the linear aff-Poisson structure restricted to $\mathsf{LinSec}(\rho)$ gives a bracket

$$\{\,,\,\}: \mathsf{LinSec}(\rho) \times \mathsf{LinSec}(\rho) \to \mathsf{Lin}(\mathsf{V}(A)^*, \mathbb{R}). \tag{88}$$

Since, due to Theorem 17, we have the canonical identification $\mathsf{LinSec}(\rho) \simeq \mathsf{Sec}(A)$ and $\mathsf{Lin}(V(A)^*, \mathbb{R}) \simeq \mathsf{V}(A)$, any Lie affgebroid bracket on A corresponds to a unique Lie affgebroid bracket on $\mathsf{LinSec}(\rho)$ and vice-versa.

LEMMA 2. *There is a one-to one correspondence between quasi-derivations $D: \mathsf{Sec}(E) \to \mathsf{Sec}(E)$ of a vector bundle E over M and linear vector fields $\overline{D}$ on E^* such that*

$$\imath_{D(X)} = \overline{D}(\imath_X). \tag{89}$$

PROOF: In local linear coordinates (x^a, y_i) on E^* associated with a basis (e_i) of sections of E and local coordinates (x^a) on M, we put

$$\overline{D} = f_i^j(x) y_j \partial_{y_i} + g^a(x)\partial_{x^a} \tag{90}$$

where $D(e_i) = f_i^j(x)e_j$ and $\widehat{D} = g_a(x)\partial_{x^a}$. It is a matter of simple calculations to check that the formula (89) is satisfied. On the other hand, (89) defines $\overline{D}$ by D (and vice versa) uniquely. ■

PROOF OF THE THEOREM: Let us choose a linear section σ of ρ and let $\{\,,\,\}_0$ be the linear Poisson bracket on $\mathsf{V}(A)^*$ corresponding to the Lie algebroid bracket $[\,,\,]_{\mathsf{v}}$ on $\mathsf{V}(A)$. Let a_σ be the section of A corresponding to σ and let $D = [a_\sigma, \cdot]^{\mathsf{v}}$ be the quasi derivation on $\mathsf{V}(A)$ corresponding to a_σ, and let $\overline{D}$ be the linear vector field on $\mathsf{V}(A^*)$ corresponding to D, according to Lemma 2.

Define a bracket on sections of ρ by

$$\{\sigma + f, \sigma + g\} = \overline{D}(g - f) + \{f, g\}_0 \tag{91}$$

for all $f, g \in C^\infty(M)$. It is obvious that this is an affine derivative with respect to each argument separately. Moreover, if f and g are linear functions, i.e. $f = \imath_{X_f}$, $g = \imath_{X_g}$ for certain $X_f, X_g \in \mathsf{Sec}(\mathsf{V}(A))$, we have

$$\begin{aligned}\{\sigma + \imath_{X_f}, \sigma + \imath_{X_g}\} &= \overline{D}(\imath_{X_f} - \imath_{X_g}) + \{\imath_{X_f}, \imath_{X_g}\}_0\\ &= \imath_{D(X_f - X_g)} + \imath_{[X_f, X_g]_{\mathsf{v}}}\\ &= \imath_{[a_\sigma + X_f, a_\sigma + X_g]}\\ &= \imath_{[a_{\sigma+f}, a_{\sigma+g}]}.\end{aligned} \tag{92}$$

It is clear that (87) determines $\{\, ,\}$ by $[\, ,]$ uniquely and vice-versa. ■

Remark. The Poisson structure on $A^\dagger/\langle 1_A\rangle$ can be regarded as a reduction of the Poisson structure on $A^\dagger$ with respect to the canonical projection $\rho\colon A^\dagger \to A^\dagger/\langle 1_A\rangle$ (cf. Theorem 11 (4)). This means that

$$\{\rho^*(f), \rho^*(g)\}^0 = \rho^*\{f, g\}_{\mathsf{v}}, \tag{93}$$

where $\{\, ,\}^0$ is the Poisson bracket on $A^\dagger$ corresponding to the Lie algebroid structure on $\widehat{A}$. Also the aff-Poisson bracket (87) can be obtained as a reduced bracket, but this time with respect to the morphism of the special affine bundles (cf. Theorem 18):

$$\begin{tikzcd} A^\dagger \times \overline{\mathbb{R}} \arrow[r, "\overline{\rho}"] \arrow[d, "pr_1"'] & A^\dagger \arrow[d, "\rho"] \\ A^\dagger \arrow[r, "\rho"] & \mathsf{V}(A)^* \end{tikzcd} . \tag{94}$$

This morphism defines the pull-back $\overline{\rho}^*\sigma$ of a section of ρ to a function on $A^\dagger$. Then

$$\rho^*\{\sigma_1, \sigma_2\} = \{\overline{\rho}^*\sigma_1, \overline{\rho}^*\sigma_2\}^0. \tag{95}$$

We can say that $\{\, ,\}$ is obtained from $\{\, ,\}^0$ by an *affine reduction*, in contrast to the standard reduction (93).

EXAMPLE 6. ([19]) Let us consider the case important for the classical mechanics when the base manifold M is the space-time fibered over the time represented by the real line $t\colon M \to \mathbb{R}$. We parameterize first-jets of motions by the affine subbundle A of $\mathsf{T}M$: $A = \{v \in \mathsf{T}M\colon \mathsf{T}t(v) = \frac{\partial}{\partial t}\}$. We have then $\widehat{A} = \mathsf{T}M$, $A^\dagger = \mathsf{T}^*M$, and $\varphi = \mathrm{d}t$. The Lie affgebroid structure on A is given by the canonical Lie algebroid structure on $\mathsf{T}M$. The quotient manifold $A^\dagger/\langle\varphi\rangle = \mathsf{V}^*M$ is the dual to the bundle of vertical vectors on M and represents the phase manifold for a non-relativistic particle. It carries a canonical Poisson structure obtained by the linear reduction (93). Hamiltonian vector fields related to this structure are t-vertical and cannot represent the dynamics of a particle.

Similarly, we can associate a vector field X_σ to a section of $\rho\colon \mathsf{T}^*M \to \mathsf{V}^*M$ by the formula

$$\{\sigma, f\}^{\mathsf{v}} = X_\sigma(f), \quad \text{for } f \in C^\infty(\mathsf{V}^*M), \tag{96}$$

where $\{\, ,\}$ is the aff-Poisson bracket (87). It is easy to check that X_σ projects to $\frac{\partial}{\partial t}$ and can represent the dynamics. The image of the generating section is the energy surface of the system.

9. Concluding remarks.

In this paper we have discussed some basic aspects of Lie-like structures on affine bundles. Such structures arise in a natural way in differential geometry and classical mechanics. The next step would be a more extended analysis as one can do in the case of Lie algebroids (se e.g. [4]). One needs also a similar theory for Lie algebroids on special vector bundles with the affine dual structures as well a theory for Lie affgebroids on special affine bundles. This would include the duality between the affgebroid structure on $\overline{\mathsf{T}}Z$ and the contact structure on the first jets bundle $\mathsf{J}^1(Z)$. Another open question is a groupoid version of Lie affgebroids. We postpone these problems to a separate paper.

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