Minimal nilpotent algebras in Goursat flags of lengths not exceeding 6

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1 Nilpotent approximation of a distribution at a point.

For any distribution D of rank d on an n–dimensional, smooth or real analytic, manifold M (i. e., rank–d subbundle in the tangent bundle TM) its *small flag* is the nested sequence

$$V_1 \subset V_2 \subset V_3 \subset \cdots$$

of modules (or: presheaves of modules) of vector fields (of the same category as M) tangent to M: $V_1 = D$, $V_{j+1} = V_j + [D, V_j]$ for $j = 1, 2, \dots$ The *small growth vector* at $p \in M$ is the sequence (n_j) of linear dimensions at p of the modules V_j: $n_j = \dim V_j(p)$. Naturally, $n_1 = d$ independently of p.

D is *completely nonholonomic* when at every point of M its small growth vector attains (sooner or later) the highest value $n = \dim M$. Once this value attained, we truncate the vector after the first appearance of n in it. The length $d_{\rm NH}$ of thus truncated vector is called the *nonholonomy degree* of D at p.

In the theory that we recall (cf. [H], [AGS], [BS], [B]; this list of references is not complete) important are the *weights* w_i related to the small flag at a point: $w_1 = \cdots = w_d = 1$, $w_{d+1} = \cdots = w_{n_2} = 2$ (no value 2 among them when $n_2 = d$), and generally $w_{n_j+1} = \cdots = w_{n_{j+1}} = j + 1$ (no value $j + 1$ among them when $n_j = n_{j+1}$) for $j = 1, 2, \dots$

Definition 1. For a completely nonholonomic distribution D on M, coordinates $z_1, z_2, \dots, z_n$ around $p \in M$ are *linearly adapted* at p when $D(p) = (\partial_1, \dots, \partial_d)$, $V_2(p) = (\partial_1, \dots, \partial_d, \dots, \partial_{n_2})$, and so on until $V_{d_{\rm NH}}(p) = (\partial_1, \dots, \partial_n) = T_pM$. Here and in the sequel we skip writing 'span' before a set of v. f. generators.

For such linearly adapted coordinates we define their *weights* $w(z_i) = w_i$, $i = 1, \dots, n$.

On the other hand, having a completely nonholonomic D, every smooth function f on M near p has its *nonholonomic order* wrt D at p ($+\infty$ is not excluded). It is the minimal number of differentiations of f along the local generators of D that give at p a nonzero result.

It follows directly from the above definitions that, for linearly adapted coordinates, their nonholonomic orders **do not exceed** their weights. Linearly adapted coordinates $z_1, \dots, z_n$ are *adapted* (or: *privileged*) when the nonholonomic order of z_i **equals** $w(z_i)$, $i = 1, \dots, n$. (In particular, adapted coordinates must vanish at p; one says that they are centered at p.)

It is not so quick, but true, that adapted coordinates always exist, and can even be algorithmically constructed from any à priori given coordinates, even in a polynomial way, as is done for inst. in [B]. They are not unique, there remains plenty of liberty behind the requirement that the nonholonomic orders (of linearly adapted coordinates) be maximal possible. In adapted coordinates it is purposeful to attach quasihomogeneous weights also to monomial vector fields (this definition goes back to the 1970's, to the theory of differential operators),

$$w(z_{i_1} \cdots z_{i_k} \partial_j) = w(z_{i_1}) + \cdots + w(z_{i_k}) - w(z_j)\,. \tag{1}$$

Proposition 1 *Every smooth vector field X with values in D has in its Taylor expansion in arbitrary coordinates adapted for D only terms of weights not smaller than -1 that can be grouped in homogeneous summands $X = X^{(-1)} + X^{(0)} + X^{(1)} + \cdots$*

(superscripts mean the weights defined by (1)). We denote by $\widehat{X}$ the lowest ('nilpotent') summand $X^{(-1)}$. That is, $\widehat{X} = X^{(-1)}$.

When a distribution D has around p local generators (vector fields) $X_1, \dots, X_d$, then

Definition 2. The distribution $\widehat{D} = \big(\widehat{X_1}, \dots, \widehat{X_d}\big)$ is called the *nilpotent approximation* of D at p.

This object $\widehat{D}$ is invariantly defined, independently of the used adapted coordinates, see Prop. 5.20 in [B]. Its basic properties are

Proposition 2. *The nilpotent approximation $\widehat{D}$ of D has at p the same small growth vector as D (and hence the same nonholonomy degree d_{NH}). Moreover, the real Lie algebra generated by $\widehat{X_1}, \dots, \widehat{X_d}$ is a nilpotent Lie algebra of nilpotency order d_{NH}*

(the *nilpotency order* of a Lie algebra is the minimal number of multiplications in that algebra yielding always zero).

Definition 3 ([AG]). Distribution D is *strongly nilpotent* at a point p when its germ at p is equivalent to its nilpotent approximation $\widehat{D}$ at p.

([HLS], [M4]) Distribution D is *weakly nilpotent* at a point p when D possesses, locally around p, a basis generating over $\mathbb{R}$ a nilpotent (real) Lie algebra of vector fields.

In the founding work [HLS], and posterior control theory literature, weakly nilpotent distributions are called *nilpotentizable.* Nilpotent approximations of distributions, and in particular strongly nilpotent distributions play a growing role in sub-riemannian geometry ([B], [AG], [S]).

2 Kumpera-Ruiz algebras of Goursat distributions.

In the sequel we deal with Goursat distributions – a rather restricted class of objects for which preliminary (local) polynomial normal forms of [KR] exist with real parameters only, and no functional moduli. Their definition requires that the sequence of consecutive Lie squares of the original rank–2 subbundle of TM consist of regular distributions of ranks $3, 4, \ldots$ until $n = \dim M$. In the abstract [M3] we recalled a basic partition of Goursat germs into disjoint *geometric classes* encoded by words of length $n-2$ over the alphabet $\mathrm{G}, \mathrm{S}, \mathrm{T}$, with two first letters always G and such that never a T goes directly after a G. Their construction was done by Montgomery and Zhitomirskii; it is reproduced in Sec. 1.3 of [M2]. (Implicitly these classes are already present in a pioneering work [J], in which the author uses a trigonometric presentation of Goursat objects. Another way of constructing them has been proposed in [PR1] where they are called *singularity types.*)

In dimension 4 there is but one class GG, in dimension 5 – only GGG and GGS, in dimension 6 – GGGG, GGSG, GGST, GGSS, GGGS.

The union of all geometric classes of fixed length with letters S in *fixed* positions in the codes is called, after [MZ], a *Kumpera-Ruiz class* of Goursat germs of that corank. For inst., in dimension 6 the two geometric classes GGSG and GGST build one KR class $**\mathrm{S}*$.

Passing to nonholonomy degrees, Jean was able to compute them for objects showing up in his trigonometric presentation. That, in view of Thm. 4.1 of [BH] (see also [MZ], [PR1]) and after putting geometric classes in relief in Jean's approach, does for all G. germs.

Proposition 3 ([J]). *In dimension n, the nonholonomy degree d_{NH} of any Goursat germ in the geometric class $\mathcal{C}$, equals the last term b_{n-2} in the sequence $b_1, b_2, \ldots, b_{n-2}$ defined only in terms of $\mathcal{C}$: $b_1 = 2,\ \ b_2 = 3,$*

$b_{j+2} = b_j + b_{j+1}$ when the $(j+2)$-th letter in $\mathcal{C}$ is S,

$b_{j+2} = 2b_{j+1} - b_j$ when the $(j+2)$-th letter in $\mathcal{C}$ is T,

$b_{j+2} = 1 + b_{j+1}$ when the $(j+2)$-th letter in $\mathcal{C}$ is G.

The mentioned polynomial (local) presentations of G. objects were not used in [J], but what are they? The essence of the contribution [KR], given in the notation of vector fields and taking into account posterior works, is as follows. We construct a (not unique) rank–2 distribution on $(\mathbb{R}^n(x^1, \ldots, x^n), 0)$ departing from the code of a geometric class $\mathcal{C}$.

When the code starts with precisely s letters G, one puts $\overset{1}{Y} = \partial_1$, $\overset{2}{Y} = \overset{1}{Y} + x^3\partial_2$, ..., $\overset{s+1}{Y} = \overset{s}{Y} + x^{s+2}\partial_{s+1}$. When $s < n-2$, then the $(s+1)$th letter in $\mathcal{C}$ is S. More generally, if the mth letter in $\mathcal{C}$ is S, and $\overset{m}{Y}$ is already defined, then

$$\overset{m+1}{Y} = x^{m+2}\overset{m}{Y} + \partial_{m+1} \,.$$

But there can also be T's or G's after an S. If the mth letter in $\mathcal{C}$ is not S, and $\overset{m}{Y}$ is already defined, then

$$\overset{m+1}{Y} = \overset{m}{Y} + \left(c^{m+2} + x^{m+2}\right)\partial_{m+1} \,,$$

where a real constant c^{m+2} is not absolutely free but

- equal to 0 when the mth letter in $\mathcal{C}$ is T,
- not equal to 0 when the mth letter is G going directly after a string ST... T (or after a short string S).

Now, on putting $\mathbf{X} = \partial_n$ and $\mathbf{Y} = \overset{n-1}{Y}$, and understanding $(\mathbf{X}, \mathbf{Y})$ as the germ at $0 \in \mathbb{R}^n$,

Theorem 1 ([KR]). *Any Goursat germ D on a manifold of dimension n, sitting in a geometric class $\mathcal{C}$, can be put (in certain local coordinates) in a form $D = (\mathbf{X}, \mathbf{Y})$, with certain constants in the field $\mathbf{Y}$ corresponding to* G*'s past the first* S *in $\mathcal{C}$.*

Definition 4. The real Lie algebra generated by $\mathbf{X}$ and $\mathbf{Y}$ is called the *KR algebra* of the germ D. This algebra does not depend on the choice of coordinates in Thm. 1, although so do the constants in $\mathbf{Y}$. Its nilpotency order is denoted by O_{KR}.
(A short analysis shows that this algebra depends solely on the Kumpera-Ruiz class of D.)

In 2000 we proved (cf. Rem. 1 in [M1])

Theorem 2 ([M4]). *The KR algebra of any Goursat distribution in dimension n, locally around a point belonging to a geometric class $\mathcal{C}$, is nilpotent of nilpotency order d_{n-2}, where d_{n-2} is the last term in the sequence $d_1, d_2, \ldots, d_{n-2}$ defined only in function of the Kumpera-Ruiz class subsuming $\mathcal{C}$: $d_1 = 2$, $d_2 = 3$, $d_{j+2} = d_j + d_{j+1}$ when the $(j+2)$-th letter in $\mathcal{C}$ is* S*, and otherwise $d_{j+2} = 2d_{j+1} - d_j$.*

Note that in [PR2] a result stating just the nilpotency of the same algebras (the *existence* of finite nilpotency orders for them) is given.

Definition 5. We call *tangential* geometric classes whose codes possess letters G only in the beginning, before the first S (if any) in the code. Tangential Goursat germs are those sitting in the tangential classes.

Example 1. Up to dimension 5 all geometric classes are tangential. The first, and unique non-tangential class in dimension 6 is GGSG. In dimension 7 there are eight tangential classes and five non-tangential: GGSGG, GGSTG, GGSGS, GGGSG, GGSSG. In dimension 8 there are sixteen tangential and eighteen non-tangential geometric classes (cf. Thm. 3 below).

Tangential classes become clearly visible when one uses the polynomial presentation.

Observation 1. *A Goursat germ D is tangential $\Longleftrightarrow$ in any KR presentation $(\mathbf{X}, \mathbf{Y})$ of D there is no non-zero constant*

(so tangential germs are easily given local models with no parameters).

3 Minimality of KR algebras in several geometric classes.

The comparison of the formulas for the nonholonomy degree $d_{\rm NH}$ (Prop. 3) with the ones for nilpotency order $O_{\rm KR}$ (Thm. 2) yields that in each fixed geometric class $\mathcal{C}$ a). $d_{\rm NH} \le O_{\rm KR}$, and b). $d_{\rm NH} = O_{\rm KR}$ **only** when $\mathcal{C}$ is tangential. Thus within tangential classes the KR algebras cannot be improved in the sense of lowering nilpotency orders. (Also – as is visible in the proof of Thm. 2 in [M4] – the germs in tangential classes are all strongly nilpotent, and Kumpera-Ruiz coordinates of Thm. 1 are already adapted.)

How is it in non-tangential classes? Do there exist there Goursat germs with better nilpotent bases – with lower nilpotency orders?

This question makes sense from dimension 6 onwards (see Ex. 1). We announce below the full answer in dimensions 6 and 7 (addressing all non-tangential classes in these dimensions), and a partial one in dimension 8 (addressing eight out of eighteen non-tangential classes).

Theorem 3.
A. *In dimension* 6*, for germs in the class* GGSG *the nilpotency order* $O_{\rm KR} = 7$ *is minimal among all possible local nilpotent bases, despite the fact that* $d_{\rm NH} = 6$ *for these germs.*

B. *In dimension* 7*, for germs in the classes:*

geometric class	$d_{\rm NH}$	$O_{\rm KR}$
GGSGG	7	9
GGSTG	8	9
GGSSG	9	11
GGSGS	11	12
GGGSG	8	10

their respective KR algebras are of minimal possible nilpotency order.

C. *In dimension* 8*, for germs in the non-tangential classes:*

geometric class	d_{NH}	O_{KR}
GGSGGG	8	11
GGSTGG	9	11
GGSTTG	10	11
GGSSGS	17	19
GGSGSG	12	17
GGSTSG	13	17
GGSGST	16	17
GGGSGS	15	17

their respective KR algebras are of minimal possible nilpotency order, too.

We do not yet know the answer in the remaining ten non-tangential classes in dimension 8.

Corollary 1. *In dimension* 6 *and* 7*, for all Goursat germs in the non-tangential geometric classes, their nilpotent approximations are not equivalent to the departure germs. The same concerns the germs in dimension* 8 *sitting in the eight non-tangential classes listed in* Thm. 3, C. *It is so because of the second property of nilpotent approximations recalled in* Prop. 2. *Thus those germs are not strongly nilpotent.*

The weak form of nilpotency – possession of a nilpotent basis – appears thus much weaker than the strong form of nilpotency of a distribution germ. In [M4] it is conjectured that the pattern emerging from Thm. 3 is valid for all non-tangential geometric classes in all dimensions. Implying that, conjecturally, the notions 'tangential' and 'strongly nilpotent' simply coincide.

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