Poincare Duality of Topological manifolds

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The problem of writing the Hirzebruch formula (see [1]) for topological manifolds for families of representations of the fundamental group collides with two difficulties. The first is related to difinition of rational Pontryagin characteristic classes that demands in particular the Novikov theorem about topological invariance of the rational Pontryagin classes. The second difficulty consists of that the construction of the signature needs a modification of classical construction for topological manifolds. Really, it is impossible to define the Poincare duality as a homomorphism of finite generated differential module since homology by itself is defined using either singular chains or spectrum of open coverings of manifolds. In both cases one should deal with infinite generated modules although homology here turns out to be finite generated spaces.

Here the following construction is possible. Let $\mathcal{U} = \{U_\alpha\}$ be a finite covering of the compact manifold X. Let $N_{\mathcal{U}}$ be the nerve of the covering $\mathcal{U}$. The nerve $N_{\mathcal{U}}$ detemines a finite simplicial polyhedron and hence determines chain and cochain complexes with local system of coefficients defined by a finite dimensional representation ρ of the fundamental group $\pi_1(X)$.

Consider a refining sequence of covering $\mathcal{U}_n = \{U_\alpha^n\}, \mathcal{U}_{n+1} \succ \mathcal{U}_n$.

This means that $U_\alpha^{n+1} \subset U_\beta^n$ for a proper $\beta = \beta(\alpha)$. Hence the function $\beta = \beta(\alpha)$ defines a simplicial mapping

$$\pi_n^{n+1} : N_{\mathcal{U}_{n+1}} \to N_{\mathcal{U}_n},$$

and defines the homomorphism of the homology and cohomology groups

$$\begin{aligned}(\pi_n^{n+1})_* &: H_*\left(N_{\mathcal{U}_{n+1}}\right) \to H_*\left(N_{\mathcal{U}_n}\right),\\ (\pi_n^{n+1})^* &: H^*\left(N_{\mathcal{U}_n}\right) \to H_*\left(N_{\mathcal{U}_{n+1}}\right).\end{aligned}$$

Here homology and cohomology of the manifold X are defined as limits of homology and cohomology of nerves of the coverings:

$$\begin{aligned}H^*(X) &= \lim_{\to} H^*\left(N_{\mathcal{U}_n}\right),\\ H_*(X) &= \lim_{\leftarrow} H_*\left(N_{\mathcal{U}_n}\right).\end{aligned}$$

Then the Poincare duality should defined as a homomorphism $D : H^*(X) \to H_*(X)$, generated by the intersection operator with the open cycle of dimension n.

Notice that one can choose the refining sequence of covering in such way that each covering had multiplicity equal to $N+1$, $N = \dim(X)$. The the Poincare duality can be defined as the intersection operator with the cycle $D_n \in C_n\left(\tilde{N}_{\mathcal{U}_n}\right)$, where $(\pi_n^{n+1})_*(D_{n+1}) = D_n$. There is the following simple argument due to the Mittag-Leffler condition: let $C_*^{\infty}(X) = \lim\limits_{\leftarrow} C_*\left(N_{\mathcal{U}_n}\right)$, $C^*_{\infty}(X) = \lim\limits_{\rightarrow} C^*\left(N_{\mathcal{U}_n}\right)$. Then $H_*(X) = H\left(C_*^{\infty}(X)\right)$, $H^*(X) = H\left(C_*^{\infty}(X)\right)$, where the Poincare homomorphism D is induced by the intersection operator with the cycle $D_\infty = \lim\limits_{\leftarrow}(D_n)$, that is $D = H(\pitchfork D_\infty)$.

On the other hand

$$C_*^{\infty}(X) = \hom\left(C^*_{\infty}(X); R\right),$$

that allows to define the signature on the level of a quadratic form defined on the cycle group.

To this end let consider the category $\mathcal{C}_0$ of countable generated vector spaces over real numbers. Each space V can be considered as a direct limit of countable sequence of finite dimensinal spaces with topology of the direct limit. Then the vector space $V^* = \hom(V, R)$ can be represented as a inverse limit of finite dimensional spaces with topology of inverse limit. Denote by $\hom^t(V, R)$ the space of continuous linear functionals. Then for $V \in \mathcal{C}_0$ one has $\hom(V, R) = \hom^t(V, R)$ where $\hom^t(V^*, R) = V$. Let $\mathcal{C}$ denote the category with objects being topological space of the form $V = V_1 \oplus V_2^*$ and morphisms being continuous linear mappings. A non degenerated quadratic form on the object of the category $\mathcal{C}$ is defined as a continuous isomorphism

$$\varphi : V^* \to V, \quad \varphi^* = \varphi.$$

If $V = V_1 \oplus V_1^*$ and the isomorphism φ defined by a hyperbolic matrix then the quadratic form is called trivial.

Theorem 1 *For non degenerated quadratic form there exists correctly defined invariant* $\mathrm{sign}\varphi$ *which is additive with respect to direct sum, equals to zero on trivial quadratic forms and coincides with classical signature on finite dimensional vector spaces.*

The invariant defined above is admissible for constructing of signature of topological manifold with local system of coefficients ([2]).

References

[1] A.S.Mishchenko, *The Hirzebruch formula: 45 years of history and the present state of the art* St. Petersburg Math.J., Vol **12**, No. **4** (2001), p. 519–533

[2] M.Gromov, *Positive curvature, macroscopic mimension, spectral gaps and higher signatures,* Functional Anal. on the Eve og the 21st Century, v. II. Progress in Math., Basel–Boston: Birkhauser,Vol. **132,**(1995).