SHORTLY ON NATURAL BUNDLES

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Abstract: In this year it is celebrated centenary of Polish mathematician Professor **Stanisław Gołąb** at Jagellonian University, who formulated the definition of geometric objects and created "Polish Shool" investigating these objects. On this occasion a short survey on so called natural bundles and bundle functors is presented. A new result is presented, too. Namely, a bundle functor on $Mf_m \times Mf$ of infinite order in the first factor is constructed.

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Introduction

The modern development of the differential geometry clarified that geometric objects (see [1]) are sections of fiber bundles over manifolds. (For example, the vector fields on a manifold M are sections of the tangent bundle $T(M)$ of M, the 1-forms on M are sections of the cotangent bundle $T^*(M)$, etc.) Each type of geometric objects can be interpreted as a rule F transforming every m-dimensional manifold M into a fibered manifold $F(M) \to M$ over M and every local diffeomorphism $f : M \to N$ into a fibered manifold morphism $F(f) : F(M) \to F(N)$ over f. The geometric character of F is then expressed by the functoriality condition $F(g \circ f) = F(g) \circ F(f)$. Hence the classical bundles of geometric objects are now studied in the form of the so called natural bundles on the category Mf_m of all m-dimensional manifolds and their local diffeomorphism.

All manifolds are assumed to be finite dimensional, paracompact, second countable, without boundaries and smooth, i.e. of class C^∞. Maps between manifolds are assumed to be smooth.

I. Natural bundles over m-manifolds

Let us present a definition of natural bundles over m-manifolds.

Definition 1. (Nijenhuis, [22]) A *natural bundle over m-manifolds* is a covariant functor $F : Mf_m \to FM$ from the category Mf_m of m-dimensional manifolds and local diffeomorphisms into the category FM of fibered manifolds and fibered maps satisfying the following conditions:

(i) (Prolongation) For every m-manifold M $F(M)$ is a fibered manifold over M with projection $p_M : F(M) \to M$ and for every local diffeomorphism $f : M \to N$ of two m-manifolds $F(f) : F(M) \to F(N)$ is a fibered map covering f;

(ii) (Locality) If $i : U \to M$ is an inclusion of an open submanifold, then $F(U) = p_M^{-1}(U)$ and $F(i)$ is the inclusion of $p_M^{-1}(U)$ into $F(M)$.

(iii) (Regularity) F transform smoothly parametrized systems of Mf_m-morphisms into smoothly parametrized systems of FM-morphisms.

Simple examples of natural bundles are following.

Example 1. *The tangent bundle.* $T(M)$ denotes the tangent bundle of M and $T(f) : T(M) \to T(N)$ is the tangent map of $f : M \to N$. The functor $T : Mf_m \to FM$ is a natural bundle over m-manifolds.

Example 2. *The cotangent bundle.* $T^*(M)$ denotes the cotangent bundle of M, and for a local diffeomorphism $f : M \to N$ and a point $x \in M$ let $(T_x(f))^* : T^*_{f(x)}(N) \to T^*_x(M)$ be the transposed mapping of the differential $T_x(f) : T_x(M) \to T_{f(x)}(N)$ of f at x. Then $T_x(f)$ is a linear isomorphism, and we define $T^*_x(f) = ((T_x(f))^*)^{-1} : T^*_x(M) \to T^*_{f(x)}(N)$ and the induced mapping $T^*(f) : T^*(M) \to T^*(N)$ by $(T^*(f))_{|T^*_x(M)} = T^*_x(f)$. The functor $T^* : Mf_m \to VB$ is a natural bundle over m-manifolds.

Example 3. *The bundle of frames of order r.* Let m and r be natural numbers. We denote by G^r_m *the differential group of order r in dimension m*, it means

$$G^r_m = \{\ j^r_0(\xi) \mid \xi : U \to \mathbf{R}^m,\ U \subset \mathbf{R}^m \text{ is open}, \xi \text{ is a local diffeom.}, \xi(0) = 0\ \},$$

where $j^r_0(\xi)$ is the r-jet of ξ at 0. G^r_m is a Lie group with respect to r-jets multiplication $j^r_0(\xi_1) \cdot j^r_0(\xi_2) = j^r_0(\xi_1 \circ \xi_2)$.

Let M be an m-manifold. We denote by $P^r(M)$ *the r-th order frame bundle of* M over M. It means

$$P^r(M) = \{\ j^r_0(\gamma) \mid \gamma : U \to M,\ U \subset \mathbf{R}^m \text{ is open}, \gamma \text{ is a local diffeomorphism}\ \}.$$

The group G^r_m acts on $P^r(M)$ on the right by the formula $j^r_0(\gamma) \cdot j^r_0(\xi) = j^r_0(\gamma \circ \xi)$. $P^r(M)$ is a principal fibre bundle with the structure group G^r_m. For a local diffeomorphism $f : M \to N$ of two m-manifolds the induced mapping $P^r(f) : P^r(M) \to P^r(N)$ is given by $P^r(f)(j^r_0(\gamma)) = j^r_0(f \circ \gamma)$. The functor $P^r : Mf_m \to FM$ is a natural bundle over m-manifolds.

The most general example of natural bundles is following.

Example 4. *The associated bundle.* Let S be an G^r_m-space, it means that S is a manifold and G^r_m acts on S on the left by an action α. For an m-manifold M we put

$$E^S(M) = P^r(M)[S, \alpha]\ ,$$

where $P^r(M)[S, \alpha]$ is the associated bundle with $P^r(M)$ and standard fibre S. We recall that $P^r(M)[S, \alpha]$ is the set of all orbits in $P^r(M) \times S$ with respect to the action of G^r_m given by $(j^r_0(\gamma), y) \cdot j^r_0(\xi) = (j^r_0(\gamma \circ \xi), j^r_0(\xi^{-1}) \cdot y)$. We denote by $< j^r_0(\gamma), y >$ the orbit of $(j^r_0(\gamma), y)$ and we define $\pi : E^S(M) \to M$, $\pi(< j^r_0(\gamma), y >) = \gamma(0)$.

For a local diffeomorphism $f : M \to M$ of two m-manifolds we define $E^S(f) : E^S(M) \to E^S(N)$ by

$$E^S(f)(< j_0^r(\gamma), y >) =< j_0^r(f \circ \gamma), y > .$$

The functor $E^S : Mf_m \to FM$ is a natural bundle over m-manifolds.

Definition 2. Let F be a natural bundle over m-manifolds. We say that F is *of finite order* if there is a number r such that from $j_x^r(f) = j_x^r(g)$ it follows $F_x(f) = F_x(g)$ for every local diffeomorphisms $f, g : M \to N$ between m-manifolds and every point $x \in M$, where (obviously) $F_x(f) : F_x(M) \to F_{f(x)}(N)$ is the restriction and corestriction of $F(f) : F(M) \to F(N)$ to the fibres $F_x(M)$ and $F_{f(x)}(N)$ of $F(M)$ and $F(N)$ over $x \in M$ and $f(x) \in N$. The smallest number r satisfying the above property is called *the order of* F.

The tangent bundle T is of order 1, the cotangent bundle T^* is of order 1, the bundle P^r of frames of order r is of order r and the associated (with P^r) bundle E^S is at most of order r.

Theorem 1. (Palais and Terng, [23]) *Every natural bundle F over m-manifolds has a finite order. The order of F is less than or equal to $2^f + 1$, where $f = dim(F_0(\mathbf{R}^m))$.*

In [16], Masuda investigated homomorphisms of Lie algebras of vector fields. From the results of Masuda one can very easily deduce the above order theorem (even with a better estimation). Namely, from the results of [16] it follows that the order of a natural bundle F over m-manifolds is less than or equal $2(f^2 + f) + 1$, see [17]. The paper of Masuda was published one year ealier than the paper of Palais and Terng. However Masuda has not formulated the order theorem explicitly.

Theorem 2. (Epstein and Thurston, [5]) *If $F : Mf_m \to FM$ is a covariant functor from Definition 1 satisfying the conditions (i) and (ii), then F satisfies also (iii) and F has a finite order less than or equal to $2f+1$, where $f = dim(F_0(\mathbf{R}^m))$. If $m = 1$, the estimation of the order is sharp.*

Theorem 3. (Zajtz, [26]) *Let F be a natural bundle over m-dimensional manifolds, $m \geq 2$, with a standard fibre of dimension f. Then the order r of F satisfies the inequality*

$$r \leq \max(\frac{f}{m-1}, \frac{f}{m} + 1) .$$

This estimation of the order is sharp.

Let F be a natural bundle over m-manifolds. According to Theorem 1 F has a finite order r. Let $S_F = F_0(\mathbf{R}^m)$ be the standard fibre of F. We can define an action α_F on the left of the r-th order differential group G_m^r on S_F by the formula

$$j_0^r(\xi) \cdot y = F(\xi)(y) ,$$

where $y \in S_F$ and $j_0^r(\xi) \in G_m^r$. This action is well-defined because of the order argument and it is smooth because of the regularity condition (iii) of Definition 1.

The above action is called *the standard action of G_m^r on the standard fibre S_F of F.*

Now we can consider the associated bundle $E^{S_F} : Mf_m \to FM$.

Theorem 4. (Palais and Terng, [23]) *Every natural bundle F over m-manifolds is equivalent with the associated bundle E^{S_F}, where $S_F = F_0(\mathbf{R}^m)$ is the standard fibre of F with the standard action of G_m^r on S_F, where r is the order of F.*

Proof. (see [6]) A canonical diffeomorphism $I_M : E^{S_F}(M) \longrightarrow F(M)$ is given by $I_M(< j_0^r(\gamma), y >) = F(\gamma)(y)$. $\square$

In particular case when $F(M)$ is a principal bundle Theorem 4 was proved independently by Krupka, [13].

II. Bundle functors on manifolds

Some "natural bundles" (as T) can be defined on the whole category Mf of all manifolds and maps.

Definition 3. A *bundle functor on Mf* is a covariant functor $F : Mf \to FM$ satisfying the following conditions (i)—(iii) of the definition of natural bundles with Mf instead of Mf_m.

We have the following examples of bundle functors on Mf:

Example 5. *The tangent bundle $T : Mf \to FM$* is a bundle functor on Mf.

Example 6. *The bundle T_p^r of p^r-velocities. The r-tangent bundle T^r.* Let M be a manifold. Let $T_p^r(M) = J_0^r(\mathbf{R}^p, M)$ be the bundle of r-jets at $0 \in \mathbf{R}^p$ of smooth functions $\mathbf{R}^p \to M$. $T_p^r(M)$ is called the bundle of p^r-velocities over M. Every mapping $f : M \to N$ of two manifolds is extended to a fibre bundle mapping $T_p^r(f) : T_p^r(M) \to T_p^r(N)$, $j_0^r(\gamma) \to j_0^r(f \circ \gamma)$. The functor $T_k^r : Mf \to FM$ is a bundle functor.

For $p = 1$ the functor $T^r = T_1^r$ is called the r-tangent bundle functor.

Example 7. *The r-th order vector tangent bundle $T^{(r)}$.* Let M be a manifold. Let $T^{(r)}(M) = (T^{r*}(M))^*$ be the dual vector bundle, where $T^{r*}(M) = J^r(M, \mathbf{R})_0$. Vector bundle $T^{(r)}(M)$ is called *the r-th order vector tangent bundle of M.* Every smooth mapping $f : M \to N$ of two manifolds induces vector bundle mapping $T^{(r)}(f) : T^{(r)}(M) \to T^{(r)}(N)$ as follows. We denote the fibres of $T^{(r)}(M)$ and $T^{r*}(M)$ over a point $x \in M$ by $T_x^{(r)}(M)$ and $T_x^{r*}(M)$ respectively. For any $x \in M$ we have the linear mapping $f_{|x}^{r*} : T_{f(x)}^{r*}(N) \to T_x^{r*}(M)$ defined by $f_{|x}^{r*}(j_{f(x)}^r(\gamma)) = j_x^r(\gamma \circ f)$. We define $T^{(r)}(f) : T^{(r)}(M) \to T^{(r)}(N)$ by the condition that the restriction $T^{(r)}(f)_{|T_x^{(r)}(M)} : T_x^{(r)}(M) \to T_{f(x)}^{(r)}(N)$ is the transposed mapping to $f^{r*}_{|x}$. The functor $T^{(r)} : \mathcal{M}f \to \mathcal{VB}$ is a vector bundle functor on $\mathcal{M}f$.

To present a next example we need a preparation.

Let $L^r = \{L^r_{m,n}\}_{m,n=0,1,2,...}$, where

$$L^r_{m,n} = J^r_0(\mathbf{R}^m, \mathbf{R}^n)_0$$

is the space of r-jets at $0 \in \mathbf{R}^m$ of maps $\mathbf{R}^m \to \mathbf{R}^n$ sending 0 into 0.

Let $S = \{S_0, S_1, ...\}$ be a system of manifolds. An action α of L^r on S is a system of smooth maps $\alpha_{m,n} : L^r_{m,n} \times S_m \to S_n$ satisfying

$$\alpha_{m,p}(B \circ A, s) = \alpha_{n,p}(B, \alpha_{m,n}(A, s))$$

for every $A \in L^r_{m,n}$, $B \in L^r_{n,p}$ and $s \in S_m$.

Example 8. *The associated bundle.* Consider a system of manifolds $S = \{S_0, S_1, ...\}$ and an action α of L^r on S. We shall construct a bundle functor E^α determined by this action. The restriction α_m of the maps $\alpha_{m,m}$ to invertible jets form actions of the jet group G^r_m on S_m. For every m-manifold M we put

$$E^\alpha(M) = P^r(M)[S_m, \alpha_m] \ .$$

For every map $f : M \to N$ we define $E^\alpha(f) : E^\alpha(M) \to E^\alpha(N)$ by

$$E^\alpha(f)(< u, s >) = < v, \alpha_{m,n}(v^{-1} \circ A \circ u, s) > \ ,$$

where $m = dim(M)$, $n = dim(n)$, $u \in P^r_x(M)$, $A = j^r_x(f)$, $s \in S_m$ and $v \in P^r_{f(x)}(N)$ is an arbitrary element. One can easily seen that this is a corect definition and the correspondence $E^\alpha : \mathcal{M}f \to \mathcal{FM}$ is a bundle functor on $\mathcal{M}f$.

Definition 4. Let r be a non-negative integer or infinity. Let F be a bundle functor on $\mathcal{M}f$. We say that F is *of order less or equal to* r if from $j^r_x(f) = j^r_x(g)$ it follows $F_x(f) = F_x(g)$ for every mappings $f, g : M \to N$ and every point $x \in M$. The smallest number r satisfying the above property is called *the order of* F.

Theorem 5. (Mikulski, [19]) *Every bundle functor F on $\mathcal{M}f$ has locally a finite order. More precisely, for any manifolds M, N, any point $x \in M$ and any mappings $f, g : M \to N$ the condition $j^{r(dim(M)+1)}_x(f) = j^{r(dim(M)+1)}_x(g)$ implies $F_x(f) = F_x(g)$, where $r(m)$ denotes the order of the natural bundle obtained by the restriction of F to the category of m-dimensional manifolds and their local diffeomorphisms.*

An open problem. Whether the condition $j^{r(dim(M))}_x(f) = j^{r(dim(M))}_x(g)$ implies $F_x(f) = F_x(g)$ for any manifolds M, N, any point $x \in M$ and any mappings $f, g : M \to N$, i.e. is the estimation in Theorem 5 sharp

Example 9. (Mikulski, [18]) *A bundle functor on Mf of infinite order.* We recall that $T^{(r)}(M)$ denotes the r-th order vector tangent bundle (see Example 7). Let $d_r = dim(T_0^{(r)}(\mathbf{R}^r))$. We set

$$F(M) = \bigoplus_{k=1}^{+\infty} (\bigwedge^{d_k} T^{(k)}(M)) \ .$$

For every manifold M, $F(M)$ is a finite dimensional manifold because for $k > dim(M)$ the bundle $\bigwedge^{d_k} T^{(k)}(M)$ is zero-bundle. Hence the direct sum in the definition of $F(M)$ is in reality a finite sum. For a mapping $f : M \to N$ the induced mapping $F(f) : F(M) \to F(N)$ is defined in the natural way from $\bigwedge^{d_k} T^{(k)}(f)$.

The bundle functor F is of infinite order because its restriction to the category of k-dimensional manifolds is at least of order k.

Theorem 6. (Kolář and Slovák, [12]) *If F is a covariant functor from Definition 3 satisfying the conditions (i) and (ii), then F satisfies also (iii).*

Let F be a bundle functor on Mf of finite order r. Put $S^F = \{S_0^F, S_1^F, ..., \}$, where $S_m^F = F_0(\mathbf{R}^m)$ for $m = 0, 1, ...$ are the standard fibers of F. We can define an action $\alpha^F = \{\alpha_{m,n}^F\}$ of L^r on S^F by the formula

$$j_0^r(h) \cdot s = F(h)(s) \, ,$$

where $s \in S_m^F$ and $j_0^r(h) \in J_0^r(\mathbf{R}^m, \mathbf{R}^n)_0$. This action is well defined because of the order argument, and it is smooth because of the regularity condition (iii).

The above action is called *the standard action of L^r on the standard fibres S^F of F.*

Now we can consider the associated bundle $E^{\alpha^F} : Mf \to FM$.

Theorem 7. (Janyška, [7]) *Every bundle functor F on Mf of finite order r is equivalent with the associated bundle E^{α^F}.*

Proof. A canonical diffeomorphism $I_M : E^{\alpha^F}(M) \longrightarrow F(M)$ is given by $I_M(< j_0^r(\gamma), s >) = F(\gamma)(s)$. □

III. Product preserving bundle functors on manifolds

Many bundle functors on Mf are product preserving.

Definition 5. Let $F : Mf \to FM$ be a bundle functor. For two manifolds M_1, M_2 we denote the standard projection onto i-th factor by $pr_i : M_1 \times M_2 \to M_i$, where $i = 1, 2$. F is called *product preserving* if the mapping

$$(F(pr_1), F(pr_2)) : F(M_1 \times M_2) \to F(M_1) \times F(M_2)$$

is a diffeomorphism for all manifolds M_1, M_2.

Roughly speaking, F is product preserving if $F(M_1 \times M_2) = F(M_1) \times F(M_2)$ for any manifolds M_1 and M_2.

Example 10. The tangent bundle $T : Mf \to FM$ is a product preserving bundle functor.

Example 11. The p^r-velocities bundle $T^r_p : Mf \to FM$ is a product preserving bundle functor.

To present a next example of product preserving bundle functors on Mf we need a preparation.

Definition 6. A finite dimensional real commutative associative unital algebra A is called a *Weil algebra* if it is of the form $A = \mathbf{R} \cdot 1 \oplus N$, where N is a nilpotent ideal.

Remark 1. Let $C^\infty_0(\mathbf{R}^p)$ be the algebra of all germs at 0 of maps $\mathbf{R}^p \to \mathbf{R}$. It is a local algebra and the ideal m of all germs $\mathbf{R}^p \to \mathbf{R}$ vanishing at 0 is its maximal ideal. Let $\underline{A} \subset C^\infty_0(\mathbf{R}^p)$ be an ideal such that $m \supset \underline{A} \supset m^{\kappa+1}$ for some natural number κ, where $m^{\kappa+1}$ is the algebraic $\kappa+1$ power of m. The factor algebra $A = C^\infty_0(\mathbf{R}^p)/\underline{A}$ is a Weil algebra. Any Weil algebra is isomoprhic to some $A = C^\infty_0(\mathbf{R}^p)/\underline{A}$.

Example 12. (Weil, [25]) *The Weil bundle $T^A = Hom(.,A)$ of near A-points.* Let A be a Weil algebra. Let M be a manifold. Let $C^\infty_x(M)$ be the algebra of all germs at $x \in M$ of mappings $M \to \mathbf{R}$. An unital algebra homomorphism $v : C^\infty_x(M) \to A$ is called *a near A-point over $x \in M$*. The space of all near A-points over x will be denoted by $Hom(C^\infty_x(M), A)$. Let

$$T^A(M) = \bigcup_{x \in M} Hom(C^\infty_x(M), A) \ .$$

$T^A(M)$ is a smooth fiber bundle over M with the obvious projection. Every map $f : M \to N$ of two manifolds is extended to a fibre bundle map $T^A(f) : T^A(M) \to T^A(N)$ over f by

$$T^A(f)(v)(germ_{f(x)}(g)) = v(germ_x(g \circ f)) \ ,$$

$x \in M$, $v \in Hom(C^\infty_x(M), A)$, $g : N \to \mathbf{R}$. The functor $T^A : Mf \to FM$ is a product preserving bundle functor. It is of finite order less or equal to the nilpotency order of A.

Let us denote the addition and the multiplication on $\mathbf{R}$ by $+, \cdot : \mathbf{R}^2 \to \mathbf{R}$ and for $\lambda \in \mathbf{R}$ denote $m_\lambda : \mathbf{R} \to \mathbf{R}$ the scalar multiplication by λ. If we apply a product preserving bundle functor F we obtain $F(+) : F(\mathbf{R}) \times F(\mathbf{R}) = F(\mathbf{R} \times \mathbf{R}) \to F(\mathbf{R})$, $F(\cdot) : F(\mathbf{R}) \times F(\mathbf{R}) \to F(\mathbf{R})$ and $F(m_\lambda) : F(\mathbf{R}) \to F(\mathbf{R})$. Now, we can formulate the key lemma.

Lemma 1 (Kainz and Michor, [8], Eck, [3], Luciano, [15]) *If F is a product preserving bundle functor on Mf, then $A^F = F(\mathbf{R})$ is a Weil algebra with operations $F(+)$, $F(\cdot)$, $F(m_\lambda)$, zero $F(0)$ and unit $F(1)$, and $F_0(\mathbf{R})$ is its nilpotent ideal.*

Proof. Let r be the order of the restriction of F to the category Mf_1. Then r is finite because of Theorem 1. The map $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x + x^{r+1}$, satisfies

$j^r_0(f) = j^r_0(id_{\mathbf{R}})$. Then $F_0(f) = id_{F_0(\mathbf{R})}$. Hence $a + a^{r+1} = F_0(f)(a) = a$, i.e. $a^{r+1} = 0$ for any $a \in F_0(\mathbf{R})$. $\square$

We call A^F *the Weil algebra of* F.

Then we have Weil bundle T^{A^F} of near A^F-points.

Theorem 8. (Kainz and Michor, [8], Eck, [3], Luciano, [15]) *Every product preserving bundle functor* F *on* Mf *is equivalent to the Weil bundle* T^{A^F} *of near* A^F *point, where* $A^F = F(\mathbf{R})$ *is the Weil algebra of* F.

Proof. A canonical diffeomorphism $I_M : F(M) \to T^{A^F}(M)$ can be constructed as follows. If $v \in F_x(M)$ we have an algebra homomorphism $\tilde{v} : C^\infty_x(M) \to A^F$ by $\tilde{v}(germ_x(g)) = F(g)(v)$ for any $g : M \to \mathbf{R}$, and we put $I_M(v) = \tilde{v}$. $\square$

Very interesting is also the following characterization:

Theorem 9. (Kolář and Slovák, [12]) *A bundle functor* F *on* Mf *is product preserving if and only if* $card(F(\mathbf{R}^0)) = 1$ *and* $dim(F(\mathbf{R}^n)) = n \cdot dim(F(\mathbf{R}))$ *for any* $n = 0, 1, 2, \ldots$.

IV. Bundle functors on fibered manifolds and on some other local categories over manifolds

Some importrant "bundle functors" are defined on the category $FM_{m,n}$ of fibered manifolds with m-dimensional basis and n-dimensional fibers and their local fibered diffeomorphisms. Some of them are defined on a larger category FM_m of fibered manifolds with m-dimensional bases and fibered maps covering local diffeomorphisms. Some are defined on the whole category FM of all fibered manifolds and fibered maps. The definitions of such bundle functors are obvious modifications of the definition of natural bundles or bundle functors. Similarly we define the order of such bundle functors.

Example 13. *The* r*-jet prolongation bundle functor.* For an FM_m-object $Y \to M$ we set

$$J^r(Y) = \left\{\, j^r_x(\sigma) \mid \sigma \text{ is a (locally defined) section of } Y \ ,\ x \in M \right\}$$

Then $J^r(Y)$ is a fibered manifold over Y with respect to the target projection. $J^r(Y)$ is called *the* r*-jet prolongation* of Y. For an FM_m-morphism $f : Y \to Z$ covering a local diffeomorphism $\underline{f} : M \to N$ we have a fibered map $J^r(f) : J^r(Y) \to J^r(Z)$ covering f by the formula $J^r(f)(j^r_x(\sigma)) = j^r_{\underline{f}(x)}(f \circ \sigma \circ \underline{f}^{-1})$. Functor $J^r : FM_m \to FM$ is a bundle functor on FM_m.

Let r, s, q be natural numbers such that $q \geq r \leq s$. Let $Y \to M$ and $Z \to N$ be fibered manifolds. Two fibered maps $f, g : Y \to Z$ with base maps $\underline{f}, \underline{g} : M \to N$ determine the same (r, s, q)-jet $j^{r,s,q}_y(f) = j^{r,s,q}_y(g)$ at $y \in Y_x$, $x \in M$, if

$$j^r_y(f) = j^r_y(g) \ ,\ j^s_y(f|Y_x) = j^s_y(g|Y_x) \text{ and } j^q_x(\underline{f}) = j^q_x(\underline{g}) \ .$$

The space of all (r,s,q)-jets of Y into Z is denoted by $J^{r,s,q}(Y,Z)$, see [10].

Example 14. *The bundle of $(k,l)^{r,s,q}$-velocities.* Let r,s,q,k,l be natural numbers with $q \geq r \leq s$. For any fibered manifold Y we define $T^{r,s,q}_{k,l}(Y) = J^{r,s,q}_0(\mathbf{R}^{k,l}, Y)$. It is a fiber bundle over Y with respect to the target projection. If $f : Y \to Z$ is a fibered map we have fibered map $T^{r,s,q}_{k,l}(f) : T^{r,s,q}_{k,l}(Y) \to T^{r,s,q}_{k,l}(Z)$ covering f by $T^{r,s,q}_{k,l}(f)(j^{r,s,q}_0(\gamma)) = j^{r,s,q}_0(f \circ \gamma)$. The correspondence $T^{r,s,q}_{k,l} : FM \to FM$ is a bundle functor on FM.

For bundle functors on $FM_{m,n}$ we have a similar characterization as in Theorem 4 (we construct the associated bundle in similar way as for m-manifolds by using $FM_{m,n}$-maps instead of Mf_m-maps). By Example 9, one can construct bundle functors on FM_m (or FM) of infinite order. Every bundle functor of FM_m is of locally finite order. For finite order bundle functors on FM_m (or FM) one can obtain similar characterization as in Theorem 7. In particular we have the following order theorem similar to Theorems 2, 3 and 5.

Theorem 10. (Slovák, [24]) (i) *Let $F : FM_{m,n} \to FM$ be a functor from the definition of bundle functors satisfying conditions (i) and (ii) of the definition and let $m \geq 1$ and $n \geq 0$. Then F satisfies the regularity condition (iii) and it is of finite order $r \leq 2f + 1$, where f is the dimension of the standad fiber $S_F = F_0(\mathbf{R}^{m,n})$ of F. If moreover $m > 1$, $n = 0$, then*

$$r \leq max(\frac{f}{m-1}, \frac{f}{m} + 1) \ ,$$

and if $m > 1$ and $n > 1$, then

$$r \leq max(\frac{f}{m-1}, \frac{f}{m} + 1, \frac{f}{n-1}, \frac{f}{n} + 1) \ .$$

All these estimations are sharp.

(ii) *Every bundle functor F on FM_m has locally a finite order. More precisely, for any fibered $FM_{m,n}$-manifold Y, any FM_m-object Z, any point $y \in Y$ and any FM_m-maps $f, g : Y \to Z$ the condition $j^{r(n+1)}_y(f) = j^{r(n+1)}_y(g)$ implies $F_y(f) = F_y(g)$, where $r(n)$ denotes the order of the bundle functor obtained by the restriction of F to the category $FM_{m,n}$.*

Some bundle functors F (as $T^{r,s,q}_{k,l}$) on FM are product preserving, i.e. $(F(Y_1 \times Y_2) = F(Y_1) \times F(Y_2)$. We have the following characterization of such functors.

Theorem 11. (Mikulski, [20]) (i) *Every algebra homomorphism $\mu : A \to B$ of Weil algebras determines (explicitely) a product preserving bundle functor $T^\mu : FM \to FM$.* (ii) *Conversely, any product preserving bundle functor $F : FM \to FM$ determines (explicitely) an algebra homomorphism $\mu^F : A^F \to B^F$ of Weil algebras, and $F \tilde{=} T^{\mu^F}$.*

A simple proof of Theorem 11 one can find also in [2].

Some bundle functors F (as J^r) on FM_m are fiber product preserving, i.e. $(F(Y_1 \times_M Y_2) = F(Y_1) \times_M F(Y_2)$. We have the following characterization of such functors.

Theorem 11. (Kolář and Mikulski, [11]) (i) *Every fiber product preserving bundle functor $F : FM_m \to FM$ is of finite order.* (ii) *Every triple (A, H, t), where A is a Weil algebra of nipotency order less or equal to a finite number r, H is a group homomorphism from the differential group G^r_m into the group $Aut(A)$ of all automorphisms of A and t is a G^r_m-invariant algebra homomorphism from the Weil algebra $D^r_m = J^r_0(\mathbf{R}^m, \mathbf{R})$ into A, determines (explicitely) a fiber product preserving bundle functor $T^{(A,H,t)} : FM_m \to FM$.* (iii) *Conversely, every fiber product preserving bundle functor $F : FM_m \to FM$ of order r determines (explicitely) a triple (A^F, H^F, t^F), where A^F is a Weil algebra of nipotency order less or equal to a finite number r, H^F is a group homomorphism from the differential group G^r_m into the group $Aut(A^F)$ of all automorphisms of A^F and t^F is a G^r_m-invariant algebra homomorphism from the Weil algebra $D^r_m = J^r_0(\mathbf{R}^m, \mathbf{R})$ into A^F, and $F \tilde{=} T^{(A^F, H^F, t^F)}$.*

Remark 2. A fiber product preserving bundle functors $F : FM_m \to FM$ such that $J^r(Y) \subset F(Y) \subset \tilde{J}^r(Y)$ (the non holonomic r-jet prolongation of Y) is called of jet type. In [9], Kolář described completely all such bundle functors of jet type.

Remark 3. Roughly speaking, a local category over manifolds is a category C, whose objects can be interpreted as manifolds with "structures" and morphisms as "maps preserving structures", such that "open subsets of C-objects" are C-object and "restrictions of C-morphisms to open subsets" are C-morphisms. For example, Mf_m, Mf, $FM_{m,n}$, FM_m and FM are local categories over manifolds. (For example FM is local because an open subset of a fibered manifold is again a fibered manifold and the restriction of a fibered map to an open subset is again a fibered map). In contrast, the category VB of vector bundles and their maps is not local (not every open subset $U \subset E$ of a vector bundle E is a vector bundle). The concept of bundle functors $F : C \to FM$ on an arbitrary local category C over manifolds is presented in [10]. Roughly speaking a bundle functor on a local category C over manifolds is a covariant functor $F : C \to FM$ satisfying the conditions (i)—(iii) of Definition 1 (with C instad of Mf_m). In [10], under some assumptions on C the authors proved that condition (iii) for $F : C \to FM$ is a consequence of functoriality of F and conditions (i) and (ii) for F (i.e. a theorem similar to the first part of Theorem 2), and under some assumptions on C the authors presented a characterization of bundle functors $F : C \to FM$ of finite order (similar to the one given in Theorem 7). The mentioned above assumptions are satisfied for $C = Mf_m,\ Mf,\ FM,\ FM_{m,n},\ FM_m$.

The category $C = SympMf_{2m}$ of symplectic $2m$-manifolds and their local symplectic diffeomorphisms is a local category over manifolds. The category $C = VolMf_m$ of m-manifolds with locally integrable volume forms and their local diffeomorphisms preserving volume forms is local, too.

Theorem 12. (Lubaś and Zajtz, [14]) (i) *Let* $F : SympMf_{2m} \to FM$ *be a covariant functor satisfying (i), (ii) and some stronger than (iii) regularity condition. If* $m \geq 2$*, then* F *is of finite order*

$$r \leq \frac{f}{m} ,$$

where $f = dim(F_0(\mathbf{R}^{2m})$ *and* $\mathbf{R}^{2m}$ *is with the standard symplectic structure. The estimation is sharp.*

(ii) *Let* $F : VolMf_m \to FM$ *be a covariant functor satisfying (i), (ii) and some stronger than (iii) regularity condition. If* $m \geq 2$*, then* F *is of finite order*

$$r \leq \frac{f}{m-1} ,$$

where $f = dim(F_0(\mathbf{R}^{m})$ *and* $\mathbf{R}^{m}$ *is with the standard volume form. The estimation is sharp.*

V. Bundle functors on some not local categories over manifolds

Fix a Lie group G and consider the category $PB_m(G)$, whose objects are principal G-bundles over m-manifolds and whose morphisms are the morphisms of principal G-bundles $f : P \to \overline{P}$ with the base map lying in Mf_m and with homomorphisms of G equal to id_G. Since $PB_m(G)$ is not local (not every open subset in a principal G-bundle P is again a principal G-bundle), there is a problem with the locality condition (ii) of bundle functors. To introduce the locality condition (ii) we must modify the prolongation condition (i).

Definition 7. (Eck, [4]) A *gauge natural bundle on* $PB_m(G)$ is a covariant functor $F : PB_m(G) \to FM$ satisfying the following conditions:

(i) (Prolongation) For every $PB_m(G)$-object $P \to M$ $F(P)$ is a fibered manifold over M (not over P) with projection $p_P : F(P) \to M$ and for every $PB_m(G)$-map $f : P \to \overline{P}$ covering $\underline{f} : M \to \overline{M}$ $F(f) : F(P) \to F(\overline{P})$ is a fibered map covering $\underline{f}$;

(ii) (Locality) If $i : P|U \to P$ is an inclusion, where $U \subset M$ is open, then $F(P|U) = p_P^{-1}(U)$ and $F(i)$ is the inclusion of $p_P^{-1}(U)$ into $F(P)$.

(iii) (Regularity) F transform smoothly parametrized systems of $PB_m(G)$-morphisms into smoothly parametrized systems of FM-morphisms.

Example 15. *The bundle of connection.* Let $P \to M$ be an $PB_m(G)$-object with the right action $r : P \times G \to P$. We shall also denote by r the canonical right action $r : J^1(P) \times G \to J^1(P)$ given by $r^g(j^1_x(s)) = j^1_x(r^g \circ s)$ for all $g \in G$ and $j^1_x(s) \in J^1(P)$. Then

$$Q(P) = J^1(P)/G ,$$

where $J^1(P)/G$ is the set of orbits of the action r, is a bundle over M with projection $[j^1_x(s)] \to x$. It is called *the bundle of principal connections on* P.

(The smooth sections $M \to Q(P)$ are in bijection with principal connections on P.) Every $PB_m(G)$-map $f : P \to \overline{P}$ covering $\underline{f} : M \to \overline{M}$ induces a fibered map $Q(f) : Q(P) \to Q(\overline{P})$ over $\underline{f}$ by

$$Q(f)([j^1_x(s)]) = [j^1_{\underline{f}(x)}(f \circ s \circ \underline{f}^{-1})] \ .$$

The correspondence $Q : PB_m(G) \to FM$ is a gauge natural bundle.

If two $PB_m(G)$-maps $f, g : P \to \overline{P}$ satisfy $j^r_y(f) = j^r_y(g)$ at a point $x \in P_x$ of the fiber of P over $x \in M$, then the fact that the raight translations of principal bundles are diffeomorphisms implies $j^r_z(f) = j^r_z(g)$ for every $z \in P_x$. In this case we write $j^r_x(f) = j^r_x(g)$.

Definition 8. (Eck, [4]) Let F be a gauge natural bundle on $PB_m(G)$. We say that F is *of finite order* if there is a number r such that from $j^r_x(f) = j^r_x(g)$ it follows $F_x(f) = F_x(g)$ for every $PB_m(G)$-maps $f, g : P \to \overline{P}$ covering $\underline{f} : M \to \overline{M}$ and every point $x \in M$. The smallest number r satisfying the above property is called *the order of* F.

We have the following order theorem similar to Theorems 2 and 3. This theorem was firstly proved by Eck, [4], but without a sharp estimation.

Theorem 13. (Eck, [4], Kolář, Michor, Slovák, [10]) *Let $F : PB_m(G) \to FM$ be a functor from Definition 7 satisfying condition (i) and (ii), $m \geq 1$. Then F satisfies condition (iii) and it is of finite order $r \leq 2f + 1$, where f is the dimension of the standad fiber $S_F = F_0(\mathbf{R}^m \times G)$ of F. If moreover $m > 1$, then*

$$r \leq max(\frac{f}{m-1}, \frac{f}{m} + 1) \ .$$

All these estimations are sharp.

Theorem 13 gives a complete characterization of gauge natural bundles on $PB_m(G)$ similar to Theorem 4 (we construct the associated bundle in similar way as for m-manifolds by using $PB_m(G)$-maps instead of Mf_m-maps). This characterization was obtained by Eck, [4].

Another not local category over manifolds is the category VB of all vector bundles and their vector bundle maps. Gauge bundle functors on VB we define similarly as gauge natural bundles on $PB_m(G)$. We have the following complete characterization of product preserving gauge bundle functors on VB.

Theorem 14. (Mikulski, [21]) (i) *Every pair (A, V) of a Weil algebra A and of an A-module V with $dim_{\mathbf{R}}(V) < \infty$ determines (explicitely) a product preserving gauge bundle functor $T^{(A,V)} : VB \to FM$.* (ii) *Conversely, any product preserving gauge bundle functor $F : VB \to FM$ determines (explicitely) a pair (A^F, V^F) of a Weil algebra A^F and of an A^F-module V^F with $dim_{\mathbf{R}}(V^F) < \infty$, and $F \tilde{=} T^{(A^F, V^F)}$.*

Remark 4. Another not local category over manifolds is the product $\mathcal{M}f_m \times \mathcal{M}f$ of categories $\mathcal{M}f_m$ and $\mathcal{M}f$. The objects of $\mathcal{M}f_m \times \mathcal{M}f$ are pairs (M, N), where M is an m-manifold and N is a manifold. The morphisms $(M, N) \to (\overline{M}, \overline{N})$ are the pairs (φ, f), where $\varphi : M \to \overline{M}$ is a local diffeomorphism between m-manifolds and $f : N \to \overline{N}$ is a map. An example of a bundle functor F on $\mathcal{M}f_m \times \mathcal{M}f$ is the bundle functor $J^r(M, N)$ of holonomic r-jets. Every pair of a local diffeomorphism $\varphi : M \to \overline{M}$ and a map $f : N \to \overline{N}$ induces fibered map $J^r(\varphi, f) : J^r(M, N) \to J^r(\overline{M}, \overline{N})$ over $\varphi \times f$ by $J^r(\varphi, f)(j^r_x(g)) = j^r_{\varphi(x)}(f \circ g \circ \varphi^{-1})$. Taking a bundle functor $G : \mathcal{M}f \to \mathcal{FM}$ of infinite order we produce a bundle functor $F(M, N) = M \times G(N)$, $F(\varphi, f) = \varphi \times G(f)$ of infinite order in the second factor. In [11], we present a complete characterization of all bundle functors F on $\mathcal{M}f_m \times \mathcal{M}f$ which are of finite order in the first factor.

We have the following new unpublished example.

Example 16. *A bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$ of infinite order in the first factor.* For any $r = 1, 2, 3, ...$ we define a vector bundle functor $G^r : \mathcal{M}f \to \mathcal{VB}$ by $G^r(N) = T(N) \times D^r$ and $G^r(f) = T(f) \times id_{D^r} : G^r(N) \to G^r(\overline{N})$ for any map $f : N \to \overline{N}$, where $D^r = J^r_0(\mathbf{R}^m, \mathbf{R})$ is the vector space of r-jets at 0 of maps $\mathbf{R}^m \to \mathbf{R}$.

Let $d_r = dim(G^r_0(\mathbf{R}^r))$. We set

$$G(N) = \bigoplus_{k=1}^{+\infty} (\bigwedge^{d_k} G^k(N)) \ .$$

For every manifold N, $G(N)$ is a finite dimensional manifold because for $k > dim(N)$ the bundle $\bigwedge^{d_k} G^k(N)$ is zero-bundle. Hence the direct sum in the definition of $G(N)$ is in reality a finite sum. For a mapping $f : N \to \overline{N}$ the induced mapping $G(f) : G(N) \to G(\overline{N})$ is defined in the natural way from $\bigwedge^{d_k} G^k(f)$. In this way we obtain vector bundle functor G on $\mathcal{M}f$ of order 1.

The group $G^\infty_m = invJ^\infty_0(\mathbf{R}^m, \mathbf{R}^m)_0$ of invertible ∞-jets acts on D^r by the pull-back. Hence for $r = 1, 2, ...$ we have a group homomorphism $H^r : G^\infty_m \to NE_{vb}(G^r)$ into the group of vector bundle natural equivalences of G by $H^r(\xi)_N : G^r(N) \to G^r(N)$, $H^r(\xi)_N(v, \eta) = (v, \xi \cdot \eta)$, $v \in T(N)$, $\xi \in G^\infty_m$, $\eta \in D^r$, where the dot is the action. Hence for any $\xi \in G^\infty_m$ and a manifold N we have a vector natural equivalence $H(\xi)_N : G(N) \to G(N)$ which is defined in natural way from $\bigwedge^{d_k} H^k(\xi)_N : \bigwedge^{d_k} G^k(N) \to \bigwedge^{d_k} G^k(N)$. This defines a group homomorphism $H : G^\infty_m \to NE_{vb}(G)$.

For any manifold N the corresponding action H of G^∞_m on $G(N)$ can be factorized by a smooth action of $G^{l_N}_m$ on $G(N)$ for some finite l_N. Hence we can proced similarly as in [11] for above G and H and obtain bundle functor (G, H) on $\mathcal{M}f_m \times \mathcal{M}f$. More precisely, we set

$$(G, H)(M, N) = P^\infty(M)[G(N), H^N] \ ,$$

the associated bundle with $P^\infty(M)$ and fibre $G(N)$. Then $(G, H)(M, N)$ is a finite dimensional manifold diffeomorphic to $P^{l_N}(M)[G(N)]$. We define the source and the target projections $\alpha : (G,H)(M,N) \to M$ by $\alpha(< j_0^\infty(\gamma), v >) = \gamma(0)$ and $\beta : (G,H)(M,N) \to N$ by $\beta(< j_0^\infty(\gamma), v >) = y$ for all $j_0^\infty(\gamma) \in P^\infty(M)$ and $v \in G_y(N)$, $y \in N$. Every pair of a local diffeomorphism $\varphi : M \to \overline{M}$ and a map $f : N \to \overline{N}$ induce a fibered map $(G,H)(\varphi, f) = (G,H)(M,N) \to (G,H)(\overline{M}, \overline{N})$ by

$$(G,H)(\varphi, f)(< j_0^\infty(\gamma), v >) = < j_0^\infty(\varphi \circ \gamma), G(f)(v) > \ .$$

The bundle functor (G,H) is of order 1 in the second factor. The bundle functor (G,H) is of infinite order in the first factor because the homomorphism H can not be factorized by a group homomorphism $G_m^l \to NE(G)$ with finite l.

Similarly, starting from a bundle functor $\tilde{T} : Mf \to FM$ of infinite order instead of T we can produce a bundle functor on $Mf_m \times Mf$ of infinite order in both factors.

REFERENCES

[1] J. Aczel, S. Goł, *Funktionalgleichungen der Theorie der geometrischen Objekte*, Monografie Matematyczne, Warszawa 1960.

[2] M. Doupovec, I. Kolář, *On the jets of fibered manifold morphisms,* Cahiers Topo. Géom Diff. Cat. 40(1999), 21-30.

[3] D. J. Eck, *Product-precerving functors on smooth manifolds*, J. Pure Appl. Algebra 42(1986), 133 — 140.

[4] D. J. Eck, *Gauge natural bundles and generalized gauge theories,* Mem. Amer. Math. Soc. 247(1981).

[5] D. B. A. Epstein, W. P. Thurston, *Transformation groups and natural bundles*, Proc. London Math. Soc. (3) 38 (1979), 219 — 236.

[6] J. Gancarzewicz, W. Mikulski, Z. Pogoda, *Natural bundles and natural liftings. Prolongation of geometric structures*, Proc. Conf. on Diff. Geometry and Its Applications, Opava, August 24 - 28, 1992, Silesian University, Opava (1993), 281 — 320, Ed. by O. Kowalski and D. Krupka.

[7] J. Janyška, *Geometric properties of prolongation functors,* Čas. Pěst. Mat. 108(1983).

[8] G. Kainz, P. Michor, *Natural transformations in differential geometry*, Czech. Math. J. 37 (112), 1987, 584 — 607.

[9] I. Kolář, *Bundle functors of jet type,* Differential Geom. and Appl. Proc of the 7th International conference Brno (1998), 231-237.

[10] I. Kolář, P. W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, Springer Verlag Berlin, 1993.

[11] I. Kolář , W. M. Mikulski, *On the fiber product preserving bundle functors,* Differential Geom. and Its. Appl. 11(1999), 105-115.

[12] I. Kolář, J. Slovák, *On the geometric functors on manifolds*, Suppl. ai Rendiconti del Cir. Math. di Palermo, 21 (1989), 223 — 233.

[13] D. Krupka, *Reducibility theorems for differentiable liftings in fibre bundles*, Arch. Math. Scripta Fac. Sci. Nat. Ujep Brunensis 17 (1979), 93 — 106.

[14] K. Lubaś, A. Zajtz, *Estimates of the action order of some jet groups,* Proc. Conf. Differential Geometry And Appl. Brno (1998), Masaryk University in Brno (1999), 247-257.

[15] O. O. Luciano, *Categories of multiplicative functors and Morimoto's conjecture*, Preprint 46, Inst. Fourier, Laboratoire des Mathématiques, Grenoble, 1986.

[16] K. Masuda, *Homomorphism of the Lie algebras of vector fields*, J. Math. Soc. Japan, 28 (1976), 506 — 528.

[17] W. M. Mikulski, *Continuity of projections of natural bundles*, Ann. Pol. Math. LVII.2 (1992), 105 — 120.

[18] W. M. Mikulski, *There exists a prolongation functor of infinite order*, Čas. pest. mat. 114 (1989), 57 — 59.

[19] W. M. Mikulski, *Natural transformations of Weil functors into bundle functors*, Suppl. ai Rendiconti del Circolo Matematico di Palermo, Serie II, 1989, 177 — 191.

[20] W. M. Mikulski, *Product preserving bundle functors on fibered manifolds,* Arch. Math. Brno 32(4)(1996), 307-316.

[21] W. M. Mikulski *Product preserving gauge bundle functors on vector bundles,* Colloq. Math. 90(2)(2001), 277-285.

[22] A. Nijenhuis, *Natural bundles and their general properties*, Diff. Geom. in honor of K. Yano, Tokyo 1972, 317 — 334.

[23] R. S. Palais and C.-L. Terng, *Natural bundles have finite order*, Topology 16 (1977), 271 — 277.

[24] J. Slovák, *Bundle functors on fibered manifolds,* Annals of Global Anal. and Geom. (1990).

[25] A. Weil, *Thèorie des points proches sur les variétés differentiables*, Coloque Geom. Diff. Strasbourg 1953, 111 — 117.

[26] A. Zajtz, *The sharp upper bound on the order of natural bundles of given dimension*, Bull. Soc. Math. Belgique, 39 (1987), 347 — 357.

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