Spaces of differential operators as modules over the Lie algebra of vectorfields

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The space $\mathcal{D}_\lambda$ of differential operators acting on the λ-densities over a smooth manifold M is filtered by the order of differentiation. The associated graded space is the space $\mathcal{S}(M)$ of smooth functions on T^*M that are polynomial on the fibers. Both $\mathcal{D}_\lambda$ and $\mathcal{S}(M)$ are modules over the Lie algebra of vector fields $Vect(M)$ of M and the projection map $\mathcal{D}_\lambda \to \mathcal{S}(M)$, called the *principal symbol map*, is equivariant. It is onto. Moreover, although as vector spaces, $\mathcal{D}_\lambda$ and $\mathcal{S}(M)$ are isomorphic, there is no equivariant bijection between them.

Viewing T^*M as the phase space of some mechanical system, $\mathcal{S}(M)$ is then the Poisson algebra of the classical observables. A bijection from $\mathcal{S}(M)$ onto $\mathcal{D}_\lambda$ that preserves the principal symbol could then be interpreted as a quantification procedure. From the infinitesimal point of view, symmetries of the system lead to vector fields leaving that procedure equivariant.

It is known that there is no natural quantization, that is no quantization procedure that is equivariant under $Vect(M)$. On the other hand, quantization procedures have been constructed on $\mathbb{R}^m$ that are equivariant under the projective embedding sl_{m+1} of $sl(m+1, \mathbb{R})$ and the conformal embedding of $so(p+1, q+1), p+q=m$. Moreover some uniqueness properties have been shown for these quantizations. In particular, the sl_{m+1}-equivariant quantization is unique. This has been used to study various questions about $Vect(M)$-modules of differential operators over arbitrary manifolds, the strategy being as follows : first study them over $\mathbb{R}^m$, filtering $Vect(\mathbb{R}^m)$ by sl_{m+1} then glue the local informations collected on the various domains of chart by this means to get a global result. The power of the method comes from the fact that the filtering algebra is finite dimensional and simple, simplifying for instance cohomological considerations.

The above algebras of symmetries turn out to be *maximal subalgebras* of the Lie algebra of polynomial vector fields of $\mathbb{R}^m$. Because of that the family of these maximal subalgebras has been studied since then as well as the corresponding existence and uniqueness problem for the corresponding quantization procedures. The main result about the maximal subalgebras is that they coincide with the well known filtered algebras studied by Kobayashi and Nagano, that are related to geometries of order 2. Besides, algorithms have been found to decide wether or not the corresponding quantization procedures exist and are unique, using the ressources of the represntation theory of the semisimple Lie algebras.

The existence problem of quantization procedures is a particular case of the more general problem of classification of the spaces of differential operators as modules over Lie subalgebras of the algebra of vector fields. These questions involved some cohomological considerations that have also been in vestigated, leading to some nice universal cocycles.

On the other hand, going from vector space to curved manifold in order to get coordinate free expression of these quantization procedures also poses nice questions that one has started to study.

My goal would be to present a landscape of all that stuff, presenting the main results, the main methods and tools and the main contributors in the field.