Combinatorics of the first neighbourhood of the diagonal

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The consideration of the k'th neighbourhood of the diagonal of a manifold M

$$M_{(k)} \subseteq M \times M$$

was initiated by Grothendieck to import notions from differential geometry into the realm of algebraic geometry. These notions were re-imported into differential geometry by Malgrange. Grothendieck and Malgrange utilzed the notion of *ringed* space (a space equipped with a *structure sheaf* of functions). The only points of the (underlying space of) $M_{(k)}$ are the diagonal points (x, x) with $x \in M$. But it is worthwhile to describe mappings to and from $M_{(k)}$ *as if* it consisted of "pairs of k-neighbour points (x, y)" (write $x \sim_k y$ for such a pair; such x and y are "point proches" in the terminology of A. Weil). The introduction of topos theoretic methods has put this "synthetic" way of speaking onto a rigourous basis, and we shall freely use it.

A differential 1-form ω on M is thus defined to be a map $\omega : M_{(1)} \to \mathbf{R}$, vanishing on the diagonal. So $\omega(x, y)$ makes sense whenever $x \sim_1 y$; and $\omega(x, x) = 0$ for all $x \in M$. Unravelling the definition of $M_{(1)}$ in terms of its structure sheaf almost immediately reveals that such an ω is an element of the Kähler differentials $\Omega^1(M) = I/I^2$. There is no linearity requirement on ω; and $\omega(x, y) = -\omega(y, x)$ is automatic.

One can go on and define a k-form on M as an element of the k'th exterior power of Ω^1. This is the classical approach in algebraic geometry. But there is an alternative, more geometric/simplicial approach to the theory of differential forms, which we shall expound.

It is based on the consideration of the space

$$M_{[k]} \subseteq M^{k+1}$$

of *"infinitesimal k-simplices"*. It is the "set" of $k+1$-tuples $(x_0, \ldots, x_k)$ with $x_i \sim_1 x_j$ for all $i, j = 0, \ldots, k$. We shall call the simplex *degenerate* if two of its vertices x_i and x_j are equal. Then the geometric/synthetic/combinatorial approach to differential forms is based on

Theorem *There is a bijective correspondence between functions $\omega : M_{[k]} \to \mathbf{R}$ vanishing on degenarate simplices, and classical differential k-forms on M.*

Note that there is no multilinearity or alternating requirement on ω. – Since the $M_{[k]}$'s jointly form a simplicial "set", a differential k-form may be seen as a k-cochain, and there is therefore a formula for its coboundary; for instance, if ω is a 1-form, $d\omega$ is the 2-form given by

$$d\omega(x, y, z) = \omega(x, y) + \omega(y, z) + \omega(z, x). \tag{1}$$

It can be proved to correspond to the classical exterior derivative, under the correspondence of the theorem. But it generalizes in a more seamless way to differential forms with values in Lie groups G more general than $\mathbf{R}$. For instance, (1) is replaced by

$$d\omega(x, y, z) = \omega(x, y) \cdot \omega(y, z) \cdot \omega(z, x). \tag{2}$$

This aspect of the theory is well suited for a treatment of the theory of *connections* in principal G-bundles P or groupoids PP^{-1}. A *connection* in a bundle $E \to M$ is, in the synthetic theory, a law ∇ which to a pair of 1-neighbour points $x \sim_1 y$ in M associates a "parallel transport" map ∇_{yx} from the x-fibre of E to its y-fibre.

We shall describe the relationship between gauge-P valued forms, connections, and curvature, in synthetic/combinatorial terms. The curvature R_∇ of ∇ is a 2-form (in the above sense) on M with values in a suitable group bundle on M, and is given by

$$R_\nabla(x, y, z) = \nabla(x, y) \circ \nabla(y, z) \circ \nabla(z, x),$$

– We shall utilize these descriptions to provide an explicit construction of a connection in a principal G-bundle $P \to M$, out of the data of a G-valued Čech-cocycle for the bundle P, and an $\mathbf{R}$-valued partition of unity on M; this is based on the possibility of forming arbitrary affine combinations of the vertices of an infinitesimal simplex.