ON THE SET OF GEODESIC VECTORS OF A LEFT-INVARIANT METRIC

János SZENTHE

If $(M, <,>)$ is a Riemannian manifold its geodesic $\gamma : R \to M$ is said to be homogeneous if there is a 1-parameter group of isometries $\Phi : R \times M \to M$ of the Riemannian manifold such that

$$\gamma(\tau) = \Phi(\tau, \gamma(0)),\ \tau \in \mathbb{R}$$

holds. As the comprehensive paper of C. S. Gordon shows the existence of homogeneous geodesics in homogeneous Riemannian manifolds has essential geometric consequences [G]. Several results have been obtained recently concerning the existence of homogeneous geodesics. First it has been shown by V. V. Kajzer that if G is a Lie group and $<,>$ a left-invariant Riemannian metric on G then the Riemannian manifold $(G, <,>)$ has at least 1 homogeneous geodesic [Ka]. Generalizing this result of Kajzer it has been shown by O. Kowalski and J. Szenthe that if $M = G/H$ is a homogeneous manifold and $<,>$ an invariant metric on G/H then the homogeneous Riemannian manifold $(G/H, <,>)$ has at least 1 homogeneous geodesic [Ko-Sz]. Moreover, it has been shown by Szenthe that if G is a compact semi-simple Lie group of $rank \geq 2$ and $<,>$ is a left-invariant Riemannian metric on G then the Riemannian manifold has infinitely many homogeneous geodesics [Sz]. In the study of the set of homogenous geodesics of a homogeneous Riemannian manifold $(G/H, <,>)$ the concept of geodesic vector proved to be convenient [Ko-V]. Let $\Phi : G \times (G/H) \to G/H$ be the canonical action, g the Lie algebra of G and $Exp : g \to G$ its exponential map. Put $o = H \in G/H$, fix a tangent vector $v \in T_o(G/H)$ and consider the geodesic $\gamma : R \to G/H$ defined by $v = \dot{\gamma}(0)$. It is said that v is a geodesic vector if γ is a homogeneous geodesic of $(G/H, <,>)$; in other words if

$$\gamma(\tau) = \Phi(Exp(\tau X), o),\ \tau \in \mathbb{R}$$

holds with some $X \in g$. The study of the set of homogeneous geodesics of a homogenous Riemannian manifold is obviously reducible to the study of the set of its homogeneous vectors. However, it seems that the set of the geodesic vectors of a homogeneous Riemannian manifold does not admit a simple description in general. Namely, O. Kowalski, S. Nikčević and Z. Vlašek in a joint paper have given several examples where the set of geodesic vectors have essentially different structure. In the lecture results are presented concerning the set of geodesic vectors of a homogeneous Riemannian manifold $(G, <,>)$, where G is a compact semi-simple Lie group and $<,>$ is a left-invariant Riemannian metric on G.

References

[G] Gordon, C. S., Homogeneous Riemannian manifolds whose geodesics are orbits, *Topics in Geometry in Memory of Joseph D'Atri*, 1996, pp155-174.

[Ka] Kajzer, V. V., Conjugate points of left-invariant metrics on Lie groups, *Sov. Math.* **34** (1990), translation from *Izv. Vyssh. Uchebn. Zaved. Mat.* **342** (1990), pp 27-37.

[Ko-N-V] Kowalski, O., Nikčević, S., Vlášek, Z., Homogeneous geodesics in homogenous Riemannian manifolds, *to appear.*

[Ko-Sz] Kowalski, O., Szenthe, J., On the existence of homogeneous geodesics in homogeneous Riemannian manifolds, *Geom. Dedicata* **81** (2000), pp 209-214.

[Ko-V] Kowalski, O., Vanhecke, L., Homogeneous Riemannian manifolds with homogeneous geodesics, *Bull. Un. Mat. Ital.*, **5** (1991), pp 189-246.

[Sz] Szenthe, J., Homogeneous geodesics of left-invariant metrics, *Univ. Iagel. Acta Math.* **38** (2000), pp 99-103.

Address: J. Szenthe, Dept. of Geometry, Eötvös Univ. Kecskeméti u. 10-12, Budapest, H-1053, Hungary, e-mail: szenthe@ludens.elte.hu