On the leaves of a prefoliation of a $\mathbb{K}$-differential space

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March 7, 2001

Two years ago at the conference in Krynica, prof. W. Waliszewski has presented the definition of a $\mathbb{K}$-differential space which is a generalization of the notions of a differentiable manifold and a differential space in the sense of Sikorski.

I would like to define the notion of a prefoliation of a $\mathbb{K}$-differential space and to present theorems which describe some properties of leaves of the prefoliation.

First, I remind the definition of a $\mathbb{K}$-differential space.

Let $\mathbb{K}$ be an arbitrary field with a non-trivial norm (we can think here about $\mathbb{R}$ or $\mathbb{C}$). Let $M^{(0)} = \{\alpha : \quad \alpha : D_\alpha \to \mathbb{K}\}$ be a family of functions with an arbitrary family of sets $\{D_\alpha\}$ as domains. Define the set

$$.M^{(0)} = \bigcup_{\alpha \in M_0} D_\alpha$$

which will be called *the set of points of* $M^{(0)}$. In the set of points of $M^{(0)}$ define a topology $\operatorname{top} M^{(0)}$ as the weakest topology containing the family

$$\{\alpha^{-1}(B) : B \text{ is open in } \mathbb{K} \text{ and } \alpha \in M^{(0)}\}.$$

Next set

$$\begin{aligned} \operatorname{an} M^{(0)} := \quad & \{\varphi \circ (\alpha_1, \ldots \alpha_m) : m \in \mathbb{N} \text{ and } \alpha_1, \ldots, \alpha_m \in M^{(0)} \\ & \text{and } \varphi \text{ is an analytical function defined on an open} \\ & \text{set in } \mathbb{K}^m \text{ with values in } \mathbb{K}\}. \end{aligned}$$

If $A \subset .M^{(0)}$ then

$$M^{(0)}|A := \{\alpha|A \cap D_\alpha : \alpha \in M^{(0)}\}$$

and

$$M_A^{(0)} := \{\beta : \forall_{p \in D_\beta} \exists_{U \in \operatorname{top} M^{(0)}} \exists_{\alpha \in M^{(0)}} (p \in U \cap A \subset D_\beta \wedge U \subset D_\alpha \wedge \beta|U \cap A = \alpha|U \cap A)\}.$$

It is easy to see that $M^{(0)}|A \subset M_A^{(0)}$.

Definition 1 *The family M of functions with its values in $\mathbb{K}$ is called a $\mathbb{K}$-differential space, if the condition*

$$\operatorname{an} M = M = M_{.M}$$

is fulfilled.

Let $M^{(0)}$ be an arbitrary family of functions with its values in $\mathbb{K}$. One can prove the following

Proposition 2 *The family $M := (\operatorname{an} M^{(0)})_{.M^{(0)}}$ is the smallest $\mathbb{K}$-differential space with the set $.M^{(0)}$ as the set of points, containing $M^{(0)}$.*

Definition 3 *The $\mathbb{K}$-differential space M defined above is called the $\mathbb{K}$-differential space generated by a family $M^{(0)}$.*

Prof. W. Waliszewski has proved the following

Proposition 4 *Let M be a $\mathbb{K}$-differential space and $A \subset .M$. Then $M_A \subset M$ if and only if $A \in \operatorname{top} M$.*

The above proposition gives

Corollary 5 *Let M be a $\mathbb{K}$-differential space and $A \in \operatorname{top} M$. Then $M|A = M_A$.*

Let M, N be $\mathbb{K}$-differential spaces.

Definition 6 *The mapping $f : .M \to .N$ is said to be smooth if for each $\beta \in N$ we have $\beta \circ f \in M$.*

If the above condition holds then we write $f : M \to N$.

It is obvious that if $f : M \to N$ then $f : \operatorname{top} M \to \operatorname{top} N$ i.e. f is a continuous mapping respective to the topologies $\operatorname{top} M$ and $\operatorname{top} N$.

Let M be a $\mathbb{K}$-differential space and $p \in .M$. Define $M(p) := \{\alpha \in M : p \in D_\alpha\}$.

Definition 7 *Any $\mathbb{K}$-linear mapping $v : M(p) \to \mathbb{K}$ such that*

$$v(\alpha \cdot \beta) = v(\alpha) \cdot \beta(p) + \alpha(p) \cdot v(\beta)$$

for $\alpha, \beta \in M(p)$ is called a vector tangent to M at p. The family of all vectors tangent to M at p forms a vector space. This vector space is said to be tangent to M at p. It is denoted by $\mathrm{T}_p M$.

It is easy to see that if $v \in \mathrm{T}_p M$, $U \in \operatorname{top} M$ and $\alpha \in M$ then $v(\alpha) = v(\alpha|U)$.

Let M, N be $\mathbb{K}$-differential spaces and $f : M \to N$. For each $p \in .M$, the mapping f determines a linear mapping $(f_*)_p : \mathrm{T}_p M \to \mathrm{T}_{f(p)} N$ called *a tangent mapping.* Namely, for $v \in \mathrm{T}_p M$ and $\beta \in N(f(p))$ we have

$$((f_*)_p v)(\beta) = v(\beta \circ f).$$

Definition 8 *The mapping $f : M \to N$ is called an immersion, if for each $p \in .M$ the tangent mapping $(f_*)_p$ is a monomorphism.*

Now we define a prefoliation of a $\mathbb{K}$-differential space. Let $M = \{\alpha : \quad \alpha : D_\alpha \to \mathbb{K}\}$ be a $\mathbb{K}$-differential space.

Definition 9 *A pair (M,F) of $\mathbb{K}$-differential spaces is called a prefoliation of M if*

$$\begin{array}{ll} 1) & .F = .M, \\ 2) & \operatorname{top} F \text{ is locally connected,} \\ 3) & \forall_{p\in .M}\exists_{U\in \operatorname{top} F}(p\in U \wedge F_U = M_U). \end{array}$$

Connected components of $\operatorname{top} F$ *are called leaves of (M,F).*

It is easy to see that the notion of a prefoliation is a generalization of the notion of the regular foliation, and of the Stefan foliation. Moreover, if (M,F) is a prefoliation of $\mathbb{K}$-differential space M then $(\operatorname{top} M, \operatorname{top} F)$ is a topological foliation in the sense of Ehresmann.

It is not difficult to prove the following

Theorem 10 *If (M,F) is a prefoliation of a $\mathbb{K}$-differential space M then for the mapping $f = \mathrm{id}_{.M}$ we have $f : F \to M$.*

Proof. If $\beta \in M$ then for each $p \in .M$ denote by U_p such a neighbourhood of p respective to $\operatorname{top} F$ for which

$$F_{U_p} = M_{U_p}.$$

Thus we get an open covering $\{U_p\}_{p\in .M}$ of the set $.M$ respective to $\operatorname{top} F$ with

$$\beta|U_p \in M|U_p \subset M_{U_p} = F_{U_p}.$$

Therefore $\beta \in F$, since F is a $\mathbb{K}$-differential space. ■

From the theorem we get

Corollary 11 *Let (M,F) be a prefoliation of a $\mathbb{K}$-differential space M. Then* $\operatorname{top} M \subset \operatorname{top} F$ *and $M \subset F$.*

We also have

Theorem 12 *If (M,F) is a prefoliation of a $\mathbb{K}$-differential space M then the mapping $f = \mathrm{id}_{.M}$ is an immersion.*

Proof. Suppose that $(f_*)_p(v) = 0$ for some $v \in \mathrm{T}_p F$. Then for each $\beta \in M(p)$ we have $v(\beta \circ f) = 0$, i.e. for each $G \in \operatorname{top} F$ with $p \in G$ we have

$$v((\beta \circ f)|G) = 0 \tag{1}$$

Let $\alpha \in F(p)$. There exists $U \in \operatorname{top} F$ such that $p \in U$ and $F_U = M_U$ by the definition of a prefoliation. Therefore, $\alpha|D_\alpha \cap U \in F|U = F_U = M_U$ by Corollary 5. Thus there exist $V \in \operatorname{top} M$ and $\gamma \in M$ such that $p \in V \cap U \subset D_\alpha$ and $V \subset D_\gamma$ and $\gamma|V \cap U = \alpha|V \cap U$. Obviously $\gamma \in M(p)$ and because of Corollary 11, $V \cap U \in \operatorname{top} F$. Consequently,

$$v(\alpha) = v(\alpha|V\cap U) = v(\gamma|V \cap U) = v(\gamma \circ f|V\cap U) = 0$$

by (1). Thus $v = 0$. ■

Corollary 13 *If L is a leaf of a prefoliation (M, F) and $\varphi : L \ni q \mapsto q \in .M$ then $\varphi : F_L \to M$ and φ is an immersion.*

Corollary 14 *If L is a leaf of a prefoliation (M, F) then $M_L \subset F|L$.*

We show that even if $\mathrm{id} : F \to M$ is an immersion and $(\mathrm{top}\, M, \mathrm{top}\, F)$ is a topological foliation then (M, F) has not to be a prefoliation.

Example 15 *Let M be the $\mathbb{R}$-differential space generated by the family of all continuous functions defined on $\mathbb{R}$ with values in $\mathbb{R}$ and let F be the $\mathbb{R}$-differential space generated by the family of all C^∞ functions defined on $\mathbb{R}$ with values in $\mathbb{R}$. Then $(\mathrm{top}\, M, \mathrm{top}\, F)$ is the trivial topological foliation in the sense of Ehresmann of $\mathbb{R}$. Remark that $f = \mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is a smooth mapping $F \to M$ since $M \subset F$. One can prove that $\mathrm{T}_p F$ is a vector space of dimension 0 for each $p \in \mathbb{R}$. Therefore, $(f_*)_p$ is a monomorphism for each $p \in \mathbb{R}$.*

It is obvious that for each $U \in \mathrm{top}\, F = \mathrm{top}\, M$ we have $M_U = M|U \neq F|U = F_U$ which means that (M, F) is not a prefoliation of an $\mathbb{R}$-differential space M.

Using the definition of a prefoliation, one can prove

Lemma 16 *If (M, F) is a prefoliation of a $\mathbb{K}$-differential space, then for each $\beta \in F$ and for each $p \in D_\beta$ there exist $U \in \mathrm{top}\, F$ and $\alpha \in M$ such that $p \in U \subset D_\beta$ and $\beta|U = \alpha|U$.*

Definition 17 *The leaf L of a prefoliation (M, F) is said to be proper if $(\mathrm{top}\, F)|L = (\mathrm{top}\, M)|L$.*

We have the following

Theorem 18 *Let (M, F) be a prefoliation of a $\mathbb{K}$-differential space M and L be a proper leaf of (M, F). Then*

$$M_L = F|L.$$

Proof. The inclusion $M_L \subset F|L$ holds for each L by Corollary 14.

Let $\gamma \in F|L$. By a local connectedness of $\mathrm{top}\, F$ we have $L \in \mathrm{top}\, F$ and consequently $\gamma \in F|L \subset F$ by Proposition 4 and Corollary 5. By Lemma 16, for each $p \in D_\gamma$ there exists $V \in \mathrm{top}\, F$ such that $p \in V \subset D_\gamma \subset L$ and there exists $\alpha \in M$ such that $\gamma|V = \alpha|V$. Since L is proper, there exists $W \in \mathrm{top}\, M$ such that $V = W \cap L$. Let $p \in D_\gamma$ and define $U := W \cap D_\alpha \in \mathrm{top}\, M$ and $\overline{\alpha} := \alpha|U \in M$. Then

1) $p \in U \cap L \subset D_\gamma$ since $U \cap L \subset W \cap L = V \subset D_\gamma$;

2) $U \subset D_{\overline{\alpha}}$ (in fact, the equality holds);

3) $\overline{\alpha}|U \cap L = \alpha|U \cap L = \alpha|W \cap L \cap D_\alpha = \gamma|W \cap L \cap D_\alpha = \gamma|U \cap L$ since $W \cap L \cap D_\alpha \subset V$.

By 1)-3) we have $\gamma \in M_L$. ■