Geometry of Goursat flags and their singularities of codimension 2

Piotr Mormul
Institute of Mathematics, Warsaw University
Banacha 2 , 02-097 Warsaw, Poland
e-mail: mormul@mimuw.edu.pl

April 9, 2001

1 Basic geometry – geometric classes – Jean's strata.

With every Goursat distribution – a particular rank–2 subbundle D in the tangent bundle to an n–dimensional manifold M (C^∞ or analytic; $n \geq r+2$) such that the Lie square $[D,\, D]$ of D is everywhere of rank 3, the Lie square of $[D,\, D]$ is everywhere of rank 4, and so on until obtaining the full TM – associated is its *flag* of ascending induced subbundles $D^r = D,\ D^{r-1} = [D,\, D],\ D^{r-2} = [[D,\, D],\ [D,\, D]], \ldots,\ D^0 = TM$ indexed by their *coranks* assumed to be constant independently of a point in M. The *length* of such a flag is r.

This is a very restrictive condition and G. germs are (excepting $n=4$ and $r=2$ – the classical situation of Engel, 1889) of codimension ∞ among germs of all distributions of fixed rank and corank. Yet, there exists an interesting trade off – the absence of functional parameters in local preliminary normal forms.
In fact, Goursat distributions of corank r locally admit polynomial presentations of degrees $\leq r-1$ of Kumpera and Ruiz [KRu], using only real parameters, in numbers not exceeding $r-3$, many of them possibly redundant. They also admit a trigonometric presentation springing from the kinematic model of a car pulling $r-1$ passive trailers, developed by several authors in the 90s and refined to its limits by Jean [J] bringing in critical angles $a_1 = \frac{\pi}{2},\ a_{j+1} = \arctan(\sin a_j))$. As a matter of fact, Kumpera and Ruiz discovered *singularities* hidden in flags of Goursat distributions. First attempts at defining them in a coordinate-free way were made in [BH] (p. 455), then in [CMPRe]. In [MonZ] singularities of Goursat flags were described in a canonical way, with a consistent use of the associated subflag of Cauchy–characteristic subdistributions.

Recalling, the basic singular features in the car' presentation (attention: in that model, the last trailer has number 0, while trailer hooked to the car – number $r-2$, the car itself has number $r-1$) are possible right angles between neighbouring trailers No $k-1$ and k. They correspond to *coalescences*, at a point, of flag' member D^{k+1} with the Cauchy–characteristic directions of two-step bigger member D^{k-1}. But flags exhibit also higher order singularities, implicitly present already in [J] (and constituting its strength), explicitly called *tangent* in [MonZ].

After [J], [CM] (last chap. 6), [MonZ] it is known that germs of G. flags of length r (or: distributions of corank r) can be stratified into F_{2r-3} (Fibonacci number) *geometric classes* encoded by words of length r over the alphabet $\{\mathrm{G}, \mathrm{S}, \mathrm{T}\}$ s. t. two first letters are always G and never a T goes directly after a G.

A letter S is written in the code when a basic geometric coalescence takes place for the corresponding flag' member D^k, members being indexed, we recall, backwards (or else: the right angle a_1 occurs between two neighbouring trailers). A sequence STT...T inside a code corresponds to the next (i.e., closer to the car) neighbouring trailer making angle a_2 with the second in the couple having the right angle, still next making angle a_3 with the first next, and so as long as many T's is in the sequence. This singular behaviour geometrically means that D^{k+1} is tangent at the reference point to the *locus* of the previous singularity 'D^k in basic singular position', plus D^{k+2} is tangent at that point to the locus of the singularity 'ST', plus D^{k+3} is tangent at that point to 'STT', and so on.

At the one next step the angle rule, or consecutive tangencies rule, breaks down and now the letters G are written in row until some next S (related to a new right angle in the configuration of trailers) appears.

Jean's strata (the materializations of geometric classes on a given manifold carrying a G. flag) are, when non-empty, regular embedded submanifolds of codimensions that are easily computable. Namely, the codimension of a stratum having code $\mathcal{C}$ is equal to the number of letters S and T in $\mathcal{C}$ (cf. [M5], Sec. 1.4). The only codimension–0 stratum GGG...G, being open and dense, is non-empty on any n–dimensional manifold carrying a flag; G. germs at its points are equivalent to the classical *chained model* of von Weber (1898) – Cartan (1914) – Goursat (1922) featuring no extra parameters:

$$\begin{array}{rcl} dx^2 - x^3dx^1 & = & 0\,, \\ dx^3 - x^4dx^1 & = & 0\,, \\ dx^4 - x^5dx^1 & = & 0\,, \\ * & & * \\ dx^{r+1} - x^{r+2}dx^{k+1} & = & 0 \end{array}$$

(it should be understood as the germ at $0 \in \mathbb{R}^n\,(x^1, x^2, \ldots, x^n)$).

The germs at points of the codimension–1 strata GG...GSG...G have been classified (for arbitrary length) in [M2], although only in [M3] the *geometric* description – based on [MonZ] – of the conditions securing the singular models was given. These singularities are *simple* in the singularity theory sense; parameters of K-R can be eliminated. The only invariant is the position $3 \le k \le r$ of the unique letter S in the r-letter code (the place in the flag where the unique coalescence of linear spaces at a point occurs). As a representative of the relevant orbit on n–dimensional manifolds can be taken the germ at $0 \in \mathbb{R}^n\,(x^1, x^2, \ldots, x^n)$ of

$$\begin{array}{rcl} dx^2 - x^3dx^1 & = & 0\,, \\ dx^3 - x^4dx^1 & = & 0\,, \\ * & & * \\ dx^k - x^{k+1}dx^1 & = & 0\,, \\ dx^1 - x^{k+2}dx^{k+1} & = & 0\,, \\ dx^{k+2} - (1 + x^{k+3})dx^{k+1} & = & 0\,, \\ dx^{k+3} - x^{k+4}dx^{k+1} & = & 0\,, \\ * & & * \\ dx^{r+1} - x^{r+2}dx^{k+1} & = & 0\,. \end{array}$$

In the present report we want to itemize what is known concerning the classification question for singularities of flags of codimension 2. That is – singularities at points in Jean's strata having exactly *two* non–G letters in the code.

2 Strata having ST in their codes.

These strata have been recently classified in [M5] modulo one more singular (of codimension 3) feature of flags. In fact, for the whole strata GGSTG...G, G...GSTG, G...GST and for each GG...GSTG...G (at least three G's in the beginning, at least two G's in the end) *less* certain embedded submanifold of codimension 3 (not definable in the G, S, T language), we derive unique local models, the same in either of the categories C^∞ or analytic. Again, only the position of the sequence ST in the code counts, and these singularities appear to be simple for Goursat flags of arbitrary length. When the letter S is at the k-th place in the code, $4 \le k \le r-3$, the only local model, on a manifold of dimension n, is the germ at $0 \in \mathbb{R}^n\,(x^1,\, x^2, \dots,\, x^n)$ of the rank–2 distribution described by the following r Pfaffian equations

$$\begin{array}{rcl}
dx^2 - x^3 dx^1 & = & 0\,, \\
dx^3 - x^4 dx^1 & = & 0\,, \\
* & & * \\
dx^k - x^{k+1} dx^1 & = & 0\,, \\
dx^1 - x^{k+2} dx^{k+1} & = & 0\,, \\
dx^{k+2} - x^{k+3} dx^{k+1} & = & 0\,, \\
dx^{k+3} - (1 + x^{k+4}) dx^{k+1} & = & 0\,, \\
dx^{k+4} - (1 + x^{k+5}) dx^{k+1} & = & 0\,, \\
dx^{k+5} - x^{k+6} dx^{k+1} & = & 0\,, \\
* & & * \\
dx^{r+1} - x^{r+2} dx^{k+1} & = & 0\,.
\end{array}$$

The unique local model for the entire stratum GGSTGG...G is as the above for $k = 3$ except that its $(k+3)$-th, i.e., sixth equation reads $dx^7 - x^8 dx^4 = 0$ instead of $dx^7 - (1 + x^8) dx^4 = 0$. The models for G...GSTG and G...GST are the above for $k = r-2$ and $k = r-1$, respectively, with simplifications due to the fact that only the coordinates up to x^{r+2} enter the local description.

This series of smooth, or analytic, local models consists of certain mentioned in the beginning (but highly specified in the course of a long proof) polynomial presentations of [KRu], of the G. germs in the respective geometric classes. After passing to the (dual) vector fields' writing, polynomials are only of degree 2, because only one flag's member is in a basic singular position, but there is plenty of constants in a preliminary presentation which should either be normalized (to 1 in the occurrence) or annihilated. Among the latter, certain are (much) more resistant. Surprisingly, the reason for that boils down to the arithmetical fact that – only for $k \ge 4$ – there exist natural i's such that $3k - 2 + i$ does not sit in the *additive semigroup* generated by 3 and $3k-5$. As one can easily check, these values of i are precisely $1,\, 4, \dots,\, 1 + 3(k-4)$ (for inst., $i = 1$ and 4 for $k = 5$). They are just instances of the *interesting distances* put forward in [M2] (Def. 4.2 there for $j = 1$; note that that-time-k is now $k - 3$ [1]).

[1] in Thm. 4.1 in [M2] the condition '$(i - 1 \ge k(j+2)$ and $i - 1 \in \mathcal{Z}_{jk})$' should be replaced by '$(i - 1 - k(j+2) \in \mathcal{Z}_{jk})$'

3 Strata having SS in their codes.

The work on these singularities is in progress. There are rather strong indications that the whole strata GGSSG…G are just single orbits of the local classification, with possible representatives

$$\begin{aligned}
dx^2 - x^3 dx^1 &= 0\,,\\
dx^3 - x^4 dx^1 &= 0\,,\\
dx^1 - x^5 dx^4 &= 0\,,\\
dx^4 - x^6 dx^5 &= 0\,,\\
dx^6 - (1 + x^7) dx^5 &= 0\,,\\
dx^7 - x^8 dx^5 &= 0\,,\\
dx^8 - x^9 dx^5 &= 0\,,\\
* \qquad & \quad *\\
dx^{r+1} - x^{r+2} dx^5 &= 0\,.
\end{aligned}$$

Concerning the strata GG…GSSGG…G with at least three G's in the beginning and at least three G's in the end, when the letters S are at the k-th and $(k+1)$-th places in the code, $4 \le k \le r-4$, we **conjecture** that, on an n–dimensional manifold, excepting certain embedded submanifold of codimension 3 (again, not definable in the G, S, T language) sitting in the stratum and cutting it into two disjoint parts, the germs at points of either part are equivalent to precisely one of the couple of germs at $0 \in \mathbb{R}^n\,(x^1,\, x^2, \dots,\, x^n)$ of

$$\begin{aligned}
dx^2 - x^3 dx^1 &= 0\,,\\
dx^3 - x^4 dx^1 &= 0\,,\\
* \qquad & \quad *\\
dx^k - x^{k+1} dx^1 &= 0\,,\\
dx^1 - x^{k+2} dx^{k+1} &= 0\,,\\
dx^{k+1} - x^{k+3} dx^{k+2} &= 0\,,\\
dx^{k+3} - (1 + x^{k+4}) dx^{k+2} &= 0\,,\\
dx^{k+4} - x^{k+5} dx^{k+2} &= 0\,,\\
dx^{k+5} - (\pm 1 + x^{k+6}) dx^{k+2} &= 0\,,\\
dx^{k+6} - x^{k+7} dx^{k+2} &= 0\,,\\
* \qquad & \quad *\\
dx^{r+1} - x^{r+2} dx^{k+2} &= 0\,.
\end{aligned}$$

This should hold in both categories C^∞ and analytic. As of now, this is, we repeat, a conjecture with work on it being in progress.

4 Strata having two not neighbouring S in their codes.

One should *not*, however, suppose that all codimension–2 singularities of Goursat flags are simple (admit only discrete local models). It is already not so in the geometric classes having the sequence SGS in the code, and at least three G's in the beginning, as shown in [M4].

Before reviewing it in more detail, we want to note that first examples of continuous moduli in the Goursat world were found in geometric classes of codimension 3:

GGGSTTGGG — [PRe], length 9, and, slightly later, GGSGSGSG — [M1], length 8.[2] The latter example extends naturally and easily – see [M1], Rem. 4 – to the series of geometric classes GGSGSG...SG (when r is even) and GGGSGSG...SG (when r is odd) having precisely modality $m = \left[\frac{r}{2}\right] - 3$ wrt the classification of germs by diffeos acting in the base manifold: the orbits in these classes are exactly parametrized by m different real parameters.
A far-reaching [but, it should be admitted, not yet sufficiently verified on various examples] **conjecture** of 1997 says that $\left[\frac{r}{2}\right] - 3$ is the maximal modality of germs of Goursat flags of length r.

Since the previous (2nd) Krynica Conference, an extensive work has been done on the geometric classes having the sequence SGS in their codes, and qualitatively new features, wrt the basic geometries ST and SS, revealed. In fact, excepting the class GGSGSGGG, a module of local C^∞ or analytic classification was found not later than 3 steps after the behaviour SGS in the flag, see Thm. 1 in [M4]. Any germ in the class GG...GSGSGGG, with the first S being at the place $k \geq 4$, appears equivalent to precisely one of the germs at $0 \in \mathbb{R}^n\,(x^1, \dots, x^{k+7};\, x^{k+8}, \dots, x^n)$ of

$$\begin{aligned}
dx^2 - x^3 dx^1 &= 0\,,\\
dx^3 - x^4 dx^1 &= 0\,,\\
* \qquad & \quad *\\
dx^k - x^{k+1} dx^1 &= 0\,,\\
dx^1 - x^{k+2} dx^{k+1} &= 0\,,\\
dx^{k+2} - (1 + x^{k+3})\, dx^{k+1} &= 0\,,\\
dx^{k+1} - x^{k+4} dx^{k+3} &= 0\,,\\
dx^{k+4} - (1 + x^{k+5})\, dx^{k+3} &= 0\,,\\
dx^{k+5} - x^{k+6} dx^{k+3} &= 0\,,\\
dx^{k+6} - (c + x^{k+7})\, dx^{k+3} &= 0
\end{aligned}$$

parametrized by $c \in \mathbb{R}$. This invariant parameter can be better exemplified by *not* reducing to 0 the constant in the one before last Pfaffian equation above. Quoting from Rem. 1 in [M4], in the family of KR pseudo-normal forms (for germs in the geometric class under consideration)

$$\begin{aligned}
dx^2 - x^3 dx^1 &= 0\,,\\
dx^3 - x^4 dx^1 &= 0\,,\\
* \qquad & \quad *\\
dx^k - x^{k+1} dx^1 &= 0\,,\\
dx^1 - x^{k+2} dx^{k+1} &= 0\,,\\
dx^{k+2} - (1 + x^{k+3})\, dx^{k+1} &= 0\,,\\
dx^{k+1} - x^{k+4} dx^{k+3} &= 0\,,\\
dx^{k+4} - (1 + x^{k+5})\, dx^{k+3} &= 0\,,\\
dx^{k+5} - (b + x^{k+6}) dx^{k+3} &= 0\,,\\
dx^{k+6} - (c + x^{k+7})\, dx^{k+3} &= 0\,,
\end{aligned}$$

[2] Those findings were also briefly reported in [CMPRe].

the quantity $c - 7b - \frac{5}{3}b^2$ is an invariant of the local smooth or analytic conjugacies preserving $0 \in \mathbb{R}^n$.
The analysis of the classes GG... GSGSG... G, but now with at least 4 letters G in the beginning and more letters G in the end, was being continued after closing [M4].

If the first S is at the place $k \geq 5$, then (if the ambient dimension n is big enough) in the next group of 4 letters G there hides itself a new invariant independent of the one discussed above. Thus the modality of the SGS singularities is in many cases at least 2. One can say that the first module is concealed, provided $k \geq 4$, in the group of three generic positions in the flag directly after the SGS behaviour, while the second module – provided $k \geq 5$ – in the group of the following four generic positions. It is plausible that, when k is at least 6 and n is big enough, in certain next group of G's a third module is located. And more, it is not even excluded that in *geometric classes of unbounded length*[3] [GG... GSGS], with l letters G in the beginning, modality can be at least $l - 2$.

The geometric classes with sequences SG... GS in their codes have been little investigated yet. Nevertheless, the arguments pinpointing moduli in generic prolongations of the geometry SGS seem to guarantee the abundance of numeric invariants among germs featuring the geometries SG... GS, too.

References

[BH] R. Bryant, L. Hsu; Rigidity of integral curves of rank 2 distributions, *Inventiones math.* ***114*** (1993), 435 - 461.

[CM] [CM] M. Cheaito, P. Mormul; Rank–2 distributions satisfying the Goursat condition: all their local models in dimension 7 and 8, *ESAIM: Control, Optimisation and Calculus of Variations* (URL: http://www.emath.fr/cocv/) **4** (1999), 137 – 158.

[CMPRe] [CMPRe] ——, ——, W. Pasillas-Lépine, W. Respondek; On local classification of Goursat structures, *C. R. Acad. Sci. Paris* ***327*** (1998), 503 – 508.

[J] [J] F. Jean; The car with n trailers: characterisation of the singular configurations, *ESAIM: Control, Optimisation and Calculus of Variations* (URL: http://www.emath.fr/cocv/) **1** (1996), 241 – 266.

[KRu] [KRu] A. Kumpera, C. Ruiz; Sur l'équivalence locale des systèmes de Pfaff en drapeau, in: *F. Gherardelli (ed.), Monge–Ampère Equations and Related Topics*, Inst. Alta Math., Rome 1982, 201 – 248.

[MonZ] R. Montgomery, M. Zhitomirskii; Geometric approach to Goursat flags, preprint 1999, available at http://orca.ucsc.edu/˜rmont.

[M1] P. Mormul; Local classification of rank–2 distributions satisfying the Goursat condition in dimension 9, in: *P. Orro et F. Pelletier (eds.), Singularités et Géométrie sous-riemannienne*, Travaux en cours **62**, Hermann, Paris 2000,89 – 119.

[3] a notion put forward in 1999 by Zhitomirskii

[M2] ——; Goursat distributions with one singular hypersurface – constants important in their Kumpera-Ruiz pseudo-normal forms, preprint 185, Labo Topologie, Université de Dijon 1999.

[M3] ——; Goursat flags: classification of codimension–one singularities, *J. Dyn. Control Syst.* ***6*** (2000), 311 – 330.

[M4] ——; Real moduli in codimension 2 in local classification of Goursat flags, preprint, Warsaw 2000.

[M5] P. Mormul; Simple codimension-two singularities of Goursat flags I: one flag's member in singular position, preprint 01-01 (39) Inst. of Math., Warsaw University, available at http://www.mimuw.edu.pl/wydzial/raport_mat.html

[PRe] W. Pasillas-Lépine, W. Respondek, On the geometry of Goursat structures, *ESAIM: Control, Optimisation and Calculus of Variations* (URL: http://www.emath.fr/cocv/) **6** (2001), 119 – 181.