Cohomology of flat connections

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Abstract

We study flat connections in spherical Lie algebroids (over oriented compact manifolds) defined everywhere but a finite number of points. Under some assumptions concerning dimensions with any such isolated singularity we join a real number called an index. For $\mathbb{R}$-spherical Lie algebroids, this index cannot be an integer. We prove the index theorem saying that the index sum is independent of the choice of a connection. Multiplying this index sum by the orientation class of M, we get the Euler class of this Lie algebroid. Some integral formulae for indexes are given.

1 Introduction

Lie algebroids arise in many subjects of differential geometry and play a role analogous to that of Lie algebras for Lie groups (i.e. compose an infinitesimal invariant). For example, they arise in the theory of differential groupoids, principal bundles and vector bundles [P], [L2], [K-S], [M1], [K2], [N1], [N2], transversally complete and transversally parallelizable foliations [MO1], [MO2], nonclosed Lie subgroups [MO2], [K4], Poisson manifolds [C-D-W], [D-S], [G1], [V1], [V2], [Ko1], [Ko2] and others (see the survey article by K.Mackenzie [M2], and also see J.Kubarski [K6]). Connections – splittings of the Atiyah sequences of Lie algebroids (in regular case) – correspond in the majority of the above categories to well known geometric objects such as distributions or differential systems.

On the ground of Lie algebroids, we observe an interesting analogy of the theory of sphere bundles, namely, it turns out that, in some sense, the roles of flat connections for Lie algebroids and of cross-sections for sphere bundles correspond mutually. The common ideas are the index at a singularity and the theorem of Euler-Poincaré-Hopf type as well as some technique methods. This analogy was first noticed for Lie algebroids with 1-dimensional isotropy Lie algebras in the geometry of regular Poisson manifolds over $\mathbb{R}$-Lie foliations [K8]. The main purpose of our work is to research this phenomenon in the domain of transitive Lie algebroids without a restriction of the dimensions of isotropy Lie algebras.

This paper is based on [K7] and [K10]. In [K7] the idea of the fibre integral is adopted to regular Lie algebroids: the integration operator over the adjoint bundle of Lie algebras is defined, the class of Lie algebroids for which this operator commutes with differentials (giving then a homomorphism on cohomology) is characterized and many families of examples coming from principal bundles, TC-foliations and Poisson manifolds are given. Paper [K10] deals with a subclass of the above class which contains so-called *s-Lie algebroids,* defined as the transitive ones with spherical isotropy Lie algebras i.e. cohomologically looking like a sphere (Lie algebras $\mathbb{R}$, $sk\left(3,\mathbb{R}\right)$, $sl\left(2,\mathbb{R}\right)$ are examples). For an s-Lie algebroid there is constructed a long exact sequence of cohomology (Gysin types) and the *Euler class.*

In this paper the cohomology theory of flat connections in s-Lie algebroids is developed. If the dimension of the base manifold is equal to $n+1$, where n is the dimension of the isotropy Lie algebras, the index at a isolated singularity is defined. A version of the Euler-Poincaré-Hopf theorem joining the sum of indexes to the Euler class is given. In the context of Lie algebroids coming from S^1 or $\mathrm{Spin}\,(3)$ principal bundles (over M^2 or M^4, respectively) the above theorem generalizes the classical E-P-H theorem since cross-sections of these bundles determine flat connections (but not every connection determines a cross-section). In the end, some integral formulae for indexes are obtained. We add that in general the index can not be an integer and the sum of indexes has nothing in common with the Euler-Poincaré characteristic of the Lie algebroid understood as an alternative sum of the dimensions of the cohomology groups of this Lie algebroid.

2 Fibre integral and Gysin sequence

By a *Lie algebroid on a manifold* M [P] we mean a system $A=(A,[\![\cdot,\cdot]\!],\gamma)$ consisting of a vector bundle A on M and mappings $[\![\cdot,\cdot]\!]:\mathrm{Sec}\,A\times\mathrm{Sec}\,A\to\mathrm{Sec}\,A$, $\gamma:A\to TM$, such that (1) $(\mathrm{Sec}\,A,[\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra, (2) γ, called the *anchor*, is a homomorphism of vector bundles, (3) $\mathrm{Sec}\,\gamma:\mathrm{Sec}\,A\to\mathfrak{X}(M)$, $\xi\mapsto\gamma\circ\xi$, is a homomorphism of Lie algebras, (4) $[\![\xi,f\cdot\eta]\!]=f\cdot[\![\xi,\eta]\!]+(\gamma\circ\xi)(f)\cdot\eta$, $\xi,\eta\in\mathrm{Sec}\,A$, $f\in C^\infty(M)$. The properties make the space of global cross-sections $\mathrm{Sec}\,A$ a Lie-Rinehart algebra [called also a Lie module, a Lie pseudoalgebra, etc, according to particular authors] over the commutative algebra $C^\infty(M)$ [H1]. A Lie algebroid A is said to be *transitive* if γ is an epimorphism of vector bundles, and *regular* if γ is of constant rank. When the isotropy Lie algebras $\mathfrak{g}_{|x}$ of A are isomorphic to a given Lie algebra $\mathfrak{g}$ then A is shortly called a $\mathfrak{g}$-Lie algebroid. In the sequel, the notions and the notations from [P], [M1], [K3], [M-R], [K1], [M1] are adopted. Among them, the adjoint bundle of Lie algebras $\boldsymbol{g}:=\mathrm{Ker}\,\gamma$, the Atiyah sequence $0\to\boldsymbol{g}\to A\xrightarrow{\gamma}F\to 0$, the notions of representations, connections, homomorphisms (strong or non-strong) of Lie algebroids as well as the exterior derivative d_A and the pullback of A-differential forms, are used.

In [K7] we introduce the notion of a *vertically oriented Lie algebroid* as a pair (A,ε) consisting of a regular Lie agebroid A and a non-singular cross-section ε of $\bigwedge^n\boldsymbol{g}$, $n=\mathrm{rank}\,\boldsymbol{g}$. By a *homomorphism of vertically oriented Lie algebroids* $(A,\varepsilon)\to(A',\varepsilon')$, assuming $\mathrm{rank}\,\boldsymbol{g}=\mathrm{rank}\,\boldsymbol{g}'=n$, we mean a non-strong, in general, homomorphism $T:A\to A'$, inducing $t:M\to M'$, of Lie algebroids, such that $(\bigwedge^n T^+)(\varepsilon_x)=\varepsilon'_{tx}$, $x\in M$ (where $T^+:\boldsymbol{g}\to\boldsymbol{g}'$ is the restriction of T to adjoint bundles). We write $(T,t):(A,\varepsilon)\to(A',\varepsilon')$.

An operator of the fibre integral $\fint_A$ in a *vertically oriented Lie algebroid* (A,ε) is introduced. For a transitive Lie algebroid (the case considered in this work), the fibre integral $\fint_A:\Omega_A^\star(M)\to\Omega^{\star-n}(M)$ is defined in the following way: $\fint_A\Phi=0$ if $\deg\Phi<n$ and $\gamma^\star\left(\fint_A\Phi\right)=(-1)^{nk}\,\iota_\varepsilon\Phi$ if $\deg\Phi=n+k$, $k\geqslant 0$. We recall that $\Omega_A(M)$ denotes the space of real A-differential forms, i.e. the space $\mathrm{Sec}\bigwedge A^\star$ of cross-sections of the bundle $\bigwedge A^\star$. The following are basic properties (a) $\fint_A\circ\gamma^\star=0$, (b) $\fint_A\gamma^\star\psi\wedge\Phi=\psi\wedge\fint_A\Phi$ for arbitrary forms $\psi\in\Omega(M)$ and $\Phi\in\Omega_A(M)$, (c) $\fint_A$ is an epimorphism [K7].

The operator $\fint_A$ commutes with the exterior derivatives d_A and d_M if and only if (a1) the isotropy Lie algebras $\boldsymbol{g}_{|x}$ are unimodular, and (a2) the cross-section ε is invariant with respect to the adjoint representation of A on $\bigwedge^n\boldsymbol{g}$. The transitive Lie algebroid A fulfilling properties (a1) and (a2) above is shortly called a *TUIO-Lie algebroid.* In [K7] and [K9] many examples can be found. The paper [K10] deal with the subcategory of TUIO-Lie algebroids for which isotropy Lie algebras $\mathfrak{g}$ are spherical, i.e. satisfy conditions $H^k(\mathfrak{g})=\mathbb{R}$ for $k=0,\dim\mathfrak{g}$, and $H^k(\mathfrak{g})=0$ for $1\leq k\leq\dim\mathfrak{g}-1$. Such Lie algebroids are called briefly *s-Lie algebroids.* Many examples for principal bundles and TC-foliations are given. For an s-Lie algebroid

(A,ε), there is constructed a long exact sequence of cohomology (Gysin sequence)

$$\cdots \longrightarrow H^{p}(M) \xrightarrow{D^{p}} H^{p+n+1}(M) \xrightarrow{\gamma^{\#}} H_{A}^{p+n+1}(M) \xrightarrow{\int_{A}^{\#}} H^{p+1}(M) \xrightarrow{D^{p+1}} \cdots \tag{2.1}$$

The class $\chi_{(A,\varepsilon)} := D^{0}(1) \in H^{n+1}(M)$ is called the *Euler class* of (A,ε) and $\chi_{(A,\varepsilon)} = [\Psi]$ where $\gamma^{*}\Psi = d_{A}\Phi$ for $\int_{A}\Phi = -1$. The equality $D(\alpha) = \alpha \wedge \chi_{(A,\varepsilon)}$ holds. The Euler class $\chi_{(A,\varepsilon)}$ can be computed via the Chern-Weil homomorphism of A (introduced in [K3]).

3 Difference class

By a *connection* in a transitive Lie algebroid A we mean a splitting $\lambda : TM \to A$ of the Atiyah sequence $0 \to \boldsymbol{g} \hookrightarrow A \xrightarrow{\gamma} TM \to 0$, i.e. a homomorphism of vector bundles such that $\gamma \circ \lambda = \mathrm{id}$. If λ is a homomorphism of Lie algebroids $\lambda \circ [X,Y] = [\![\lambda \circ X, \lambda \circ Y]\!]$, $X, Y \in \mathfrak{X}(M)$, then λ is called *flat.* In this situation, the pullback of differential forms $\lambda^{\star} : \Omega_{A}(M) \to \Omega(M)$ commutes with differentials $\lambda^{\star} \circ d_{A} = d_{M} \circ \lambda^{\star}$, giving — on cohomology — a homomorphism of algebras $\lambda^{\#} : H_{A}(M) \to H(M)$. Let λ be a flat connection in an s-Lie algebroid (A,ε). According to the exactness of sequence (2.1), [K7, Prop. 4.2.1 (b)] and [K10, Cor. 3.4] we can easily show that

$$H_{A}(M) = \ker \int_{A}^{\#} \bigoplus \ker \lambda^{\#}, \tag{3.1}$$

$$\int_{A}^{\#} \Big| \ker \lambda^{\#} : \ker \lambda^{\#} \xrightarrow{\cong} H(M). \tag{3.2}$$

By the above, there exists a uniquely determined cohomology class

$$\omega_{\lambda} \in \ker \lambda^{\# n} \subset H_{A}^{n}(M)$$

such that $\int_{A}^{\#}\omega_{\lambda} = 1 \in H^{0}(M)$. ω_{λ} depends on the mapping $\lambda^{\#} : H_{A}(M) \to H(M)$ only. The class ω_{λ} is called the *cohomology class* of a flat connection λ.

For two flat connections $\lambda, \sigma : TM \to A$, their cohomology classes $\omega_{\lambda}, \omega_{\sigma} \in H_{A}^{n}(M)$ satisfy the equality $\int_{A}^{\#}(\omega_{\lambda} - \omega_{\sigma}) = 0$. Thanks to the exactness of sequence (2.1) there exists a cohomology class $[\lambda, \sigma] \in H^{n}(M)$ such that

$$\omega_{\lambda} - \omega_{\sigma} = \gamma^{\#}[\lambda, \sigma].$$

Definition 3.1 *The class* $[\lambda,\sigma]$ is called the *difference class* of flat connections λ and σ in the s-Lie algebroid (A,ε).

Proposition 3.2 *For flat connections λ and σ in an s-Lie algebroid (A,ε), we have* $\lambda^{\#}\alpha - \sigma^{\#}\alpha = -\left(\int_{A}^{\#}\alpha\right) \wedge [\lambda,\sigma]$, $\alpha \in H_{A}(M)$.

Since $\gamma^{\#}$ is a monomorphism for a flat Lie algebroid A, we obtain

Corollary 3.3 *Let λ and σ be two flat connections in an s-Lie algebroid A. Then the following conditions are equivalent: (a) $\lambda^{\#} = \sigma^{\#}$, (b) $\omega_{\lambda} = \omega_{\sigma}$, (c) $[\lambda,\sigma] = 0$. If $H^{n}(M) = 0$, then, clearly, $H^{n}(M) \ni [\lambda,\sigma] = 0$, therefore $\lambda^{\#} = \sigma^{\#}$.*

Lemma 3.4 *Let (A,ε) be an arbitrary s-Lie algebroid with* $\operatorname{rank}\boldsymbol{g} = n$. *For a representative Ψ of the Euler class $\chi_{(A,\varepsilon)}$ and an n-form $\Phi \in \Omega_{A}^{n}(M)$ such that $\int_{A}\Phi = -1$ and $d_{A}\Phi = \gamma^{\star}\Psi$, for any open subset $U \subset M$ and two flat connections $\lambda, \sigma : TU \to A_{|U}$, the following equalities hold:*

(1) $\omega_\sigma = [\gamma^\star_{|U}\sigma^\star(\Phi_{|U}) - \Phi_{|U}]$,

(2) $[\lambda,\sigma] = [\lambda^\star\Phi_{|U} - \sigma^\star\Phi_{|U}]$.

Theorem 3.5 (The naturality of the difference class) *Let* $(T,t):(A,\varepsilon)\to(A',\varepsilon')$ *be a homomorphism of s-Lie algebroids such that* $T_x : A_{|x} \to A'_{|tx}$, $x\in M$, *are isomorphisms.*

(a) *Assume that* σ,σ' *are flat connections in* A *and* A', *respectively, such that* $T\circ\sigma = \sigma'\circ t_*$. *Then* $T^\#\omega_{\sigma'} = \omega_\sigma$.

(b) *For any two pairs of such flat connections* (λ,λ'), (σ,σ'), *we obtain* $t^\#([\lambda',\sigma']) = [\lambda,\sigma]$.

Proof. (a): To check (a), it is sufficient to notice that $\sigma^\#(T^\#\omega_{\sigma'}) = (T\circ\sigma)^\#\omega_{\sigma'} = (\sigma'\circ t_\star)^\#\omega_{\sigma'} = t^\#\sigma'^\#\omega_{\sigma'} = 0$ and (see [K7]) $\oint_A^\# T^\#\omega_{\sigma'} = t^\#\oint_{A'}^\#\omega_{\sigma'} = 1$.

(b): According to the fact that $\gamma^\#$ is a monomorphism, we only need to notice

$$\begin{aligned}\gamma^\# t^\#[\lambda',\sigma'] &= (t_\star\circ\gamma)^\#[\lambda',\sigma'] = (\gamma'\circ T)^\#([\lambda',\sigma']) = T^\#(\omega_{\lambda'} - \omega_{\sigma'})\\ &= \omega_\lambda - \omega_\sigma = \gamma^\#[\lambda,\sigma].\end{aligned}$$

■

The folowing theorem gives a relationship between the Euler class and the difference class (compare with the classical theorem for sphere bundles, for example [G-H-V]).

Theorem 3.6 *Let* $\{U,V\}$ *be an open covering of* M *and let* $\lambda_U : TU \to A_{|U}$ *and* $\sigma_V : TV \to A_{|V}$ *be flat connections in* (A,ε) *over* U *and* V, *respectively* (U, V *need not be connected*). *Consider the Mayer-Vietoris sequence of the triad* $\{M,U,V\}$ *for the usual real de Rham cohomology and let* $\partial : H(U\cap V)\to H(M)$ *be the connecting homomorphism. Then*

$$\chi_{(A,\varepsilon)} = \partial[\lambda,\sigma]$$

where $\lambda = \lambda_U|_{U\cap V}$ *and* $\sigma = \sigma_V|_{U\cap V}$.

Proof. For the inclusions $j_1 : U\cap V \hookrightarrow U$ and $j_2 : U\cap V\hookrightarrow V$ according to Lemma 3.4, $[\lambda,\sigma] = [\lambda^\star\Phi_{|U\cap V} - \sigma^\star\Phi_{|U\cap V}] = [j_1^\star(\lambda_U^\star\Phi_{|U}) - j_2^\star(\sigma_V^\star\Phi_{|V})]$. Since $d(\lambda_U^\star\Phi_{|U}) = \lambda_U^\star d_{A_{|U}}\Phi_{|U} = \lambda_U^\star\gamma^\star_{|U}\Psi_{|U} = \Psi_{|U}$, analogously $d(\sigma_V^\star\Phi_{|V}) = \Psi_{|V}$, we get — via the construction of ∂ — $\partial[\lambda,\sigma] = [\Psi] = \chi_{(A,\varepsilon)}$. ■

4 The index of a flat connection at an isolated singular point and the Euler number

By a *local connection with singularity at a point* $a\in M$ in a Lie algebroid A we mean the connection

$$\sigma : T\dot{U}\to A_{|\dot{U}},\ a\in U\subset M\ \ (U \text{ is open}),\ \dot{U} = U\setminus\{a\}.$$

Let (A,ε) be an arbitrary s-Lie algebroid over an $n+1$-dimensional oriented manifold M ($n\geqslant 1$) with $n = \operatorname{rank}\boldsymbol{g}$ and let $\sigma : T\dot{U}\to A_{|\dot{U}}$ be a local connection with singularity at $a\in U\subset M$. Take additionally a neighbourhood $V\ni a$ such that $V\subset U$ and $V\cong\mathbb{R}^{n+1}$. $A_{|V}$ possesses, [M1], a global flat connection $\lambda : TV\to A_{|V}$. Denote $\lambda|_{\dot{V}}$ ($\dot{V} = V\setminus\{a\}$) by $\dot{\lambda}$ and consider the difference class $[\dot{\lambda},\sigma|_{\dot{V}}]\in H^n(\dot{V})$. Let $\alpha_V : H^n(\dot{V})\xrightarrow{\cong}\mathbb{R}$ be the canonical mapping [G-H-V, Vol.I] ($\dot{V}$ has the orientation induced from M). By analogous reasoning as in the theory of sphere bundles [G-H-V, Vol.I] and due to Corollary 3.3 we check that the number $\alpha_V([\dot{\lambda},\sigma|_{\dot{V}}])$ is independent of the auxiliary flat connection λ and of the neighbourhood V. This means $\alpha_V([\dot{\lambda},\sigma|_{\dot{V}}])$ depends only on the choice of σ.

Definition 4.1 The number $\alpha_V([\dot{\lambda},\sigma|_{\dot{V}}])$ is called the *index of* σ *at* a and denoted by

$$j_a(\sigma).$$

Proposition 4.2 (Naturality of the index) *Let* $(\hat{A},\hat{\varepsilon})$ *be another s-Lie algebroid over an oriented* $n+1$*-dimensional manifold* $\hat{M}$ *and* $(T,t):(\hat{A},\hat{\varepsilon})\to(A,\varepsilon)$ *be a homomorphism of s-Lie algebroids fulfilling conditions*

$$\begin{aligned}&T_x:\hat{A}_{|x}\to A_{|tx},\ x\in M,\ \text{is an isomorphism},\\ &t:\hat{M}\to M \text{ is a diffeomorphism onto an open subset.}\end{aligned}$$

Let $a\in M$, $\hat{a}\in\hat{M}$, $t(\hat{a})=a$. *Take a local flat connection* $\sigma:T\dot{U}\to A_{|\dot{U}}$ *with singularity at* a. *Then the mapping* $T^{\#}\sigma:T\dot{W}\to\hat{A}_{|\dot{W}}$, $W=t^{-1}[U]$, $\dot{W}=W\setminus\{\hat{a}\}$, *defined by* $\left(T^{\#}\sigma\right)(v)=T_{|pv}^{-1}(\sigma(t_\star v))$ *is a flat connection in* $\hat{A}$ *with singularity at* $\hat{a}$, *and* $j_{\hat{a}}\left(T^{\#}\sigma\right)=j_a(\sigma)$.

The main goal of this article is a theorem joining the index sum $\sum j_{a_v}(\sigma)$ of any flat connection with a finite number of singularities $\{a_1,\dots,a_k\}$ to the Euler class of Lie algebroid.

Theorem 4.3 (The Euler-Poincaré-Hopf theorem for flat connections) *Let* (A,ε) *be an s-Lie algebroid of rank* n *over an oriented compact manifold* M *of dimension* $n+1$ *and let* $\sigma:T(M\setminus\{a_1,\dots,a_k\})\to A$ *be a flat connection with singularities at points* $a_1,\dots,a_k$. *Then the Euler class* $\chi_{(A,\varepsilon)}\in H^{n+1}(M)$ *is given by the formula*

$$\chi_{(A,\varepsilon)}=\left(\sum_{v=1}^{k}j_{a_v}(\sigma)\right)\cdot\omega_M$$

where ω_M *is the orientation class of* M; *equivalently,* $\int_M^{\#}\chi_{(A,\varepsilon)}=\sum_{v=1}^{k}j_{a_v}(\sigma)$. *In particular, the index sum* $\sum_{v=1}^{k}j_{a_v}(\sigma)$ *is independent of the choice of the connection.*

Proof. For each $v=1,\dots,k$, choose a neighbourhood $U_v\ni a_v$ diffeomorphic to $\mathbb{R}^{n+1}$ and such that the sets U_v are pairwise disjoint. Put $U=\bigcup U_v$, $V=M\setminus\{a,\dots,a_k\}$. Then $M=U\cup V$ and $U\cap V=\bigcup\dot{U}_v$ where $\dot{U}_v=U_v\setminus\{a_v\}$. Take arbitrary flat connections $\tilde{\lambda}_v:TU_v\to A_{|U_v}$, $v=1,\dots,k$. The family $\{\tilde{\lambda}_v\}$ determines one flat connection $\tilde{\lambda}:TU\to A_{|U}$ such that $\tilde{\lambda}|_{U_v}=\tilde{\lambda}_v$. Define $\check{\lambda}=\tilde{\lambda}|_{U\cap V}$ and $\check{\sigma}=\sigma|_{U\cap V}$. According to Theorem 3.6, $\chi_{(A,\varepsilon)}=\partial[\check{\lambda},\check{\sigma}]$. In the sequel, put $\lambda_v=\tilde{\lambda}_v|_{\dot{U}_v}$ and $\sigma_v=\sigma|_{\dot{U}_v}$. Then $[\check{\lambda},\check{\sigma}]=\oplus_v[\lambda_v,\sigma_v]$. By [G-H-V, Prop.VII Chap.VI, Vol.I] $\int_M^{\#}\circ\partial=\alpha$, where $\alpha:\bigoplus_v H^n(\dot{U}_v)\to\mathbb{R}$ is equal to $\oplus\beta_v\mapsto\sum\alpha_{U_v}(\beta_v)$. Therefore we get

$$\begin{aligned}\int_M^{\#}\chi_{(A,\varepsilon)A}&=\int_M^{\#}\partial[\check{\lambda},\check{\sigma}]=\int_M^{\#}\partial\left(\oplus_\nu[\lambda_v,\sigma_v]\right)=\alpha\left(\oplus_\nu[\lambda_v,\sigma_v]\right)\\&=\sum_{v=1}^{k}\alpha_{U_v}[\lambda_v,\sigma_v]=\sum_{v=1}^{k}j_{a_v}(\sigma).\end{aligned}$$

■

The sum

$$j_{(A,\varepsilon)}=\sum_{v=1}^{k}j_{a_v}(\sigma)$$

is called the *Euler number of the s-Lie algebroid* (A,ε). According to Theorem 5.4 from [K10], the Euler number $j_{(A,\varepsilon)}$ is not – in general – an invariant of the cohomology algebra of A and has nothing in common with the Euler-Poincaré characteristic of A. The last, when considered for TUIO-Lie algebroids ($\dim M+\operatorname{rank}\boldsymbol{g}$ is odd), always is 0 [K9].

5 Integral formulae

Proposition 5.1 *For a trivial s-Lie algebroid $A = TM \times \mathfrak{g}$ vertically oriented by a tensor $0 \neq \varepsilon_o \in \bigwedge^n \mathfrak{g}$ ($n = \dim \mathfrak{g}$) and equipped with a "standard" flat connection $\tau_0 : TM \to TM \times \mathfrak{g}$, $v \mapsto (v, 0)$, we have*

(a) *if $\sigma : TM \to A$ is a flat connection, then its cohomology class ω_σ is given by $\omega_\sigma = -\hat{\sigma}^{\#}[\varphi_o] \times 1 + 1 \times [\varphi_o]$ where $\hat{\sigma} = \mathrm{pr}_2 \circ \sigma : TM \to \mathfrak{g}$ and $\varphi_o \in \bigwedge^n \mathfrak{g}^\star$ is a tensor such that $\iota_{\varepsilon_o}\varphi_o = 1$. In particular $\omega_{\tau_0} = 1 \times [\varphi_o]$.*

(b) *The difference class $[\tau_0, \sigma]$ is equal to $[\tau_0, \sigma] = \hat{\sigma}^{\#}[\varphi_o]$.*

Proof. Using the fact that the projection $\mathrm{pr}_2 : TM \to \mathfrak{g}$ is a nonstrong homomorphism of Lie algebroids [K5], we can easily see that

$$\sigma^{\#}\left(-\hat{\sigma}^{\#}[\varphi_o] \times 1 + 1 \times [\varphi_o]\right) = 0,$$

$$\int_A^{\#}\left(-\hat{\sigma}^{\#}[\varphi_o] \times 1 + 1 \times [\varphi_o]\right) = 1.$$

Now (a) follows from the definition of the cohomology class ω_σ whereas (b) may be obtained from the definition of the difference class and the equality

$$\omega_{\tau_0} - \omega_\sigma = 1 \times [\varphi_o] - \left(-\hat{\sigma}^{\#}[\varphi_o] \times 1 + 1 \times [\varphi_o]\right) = \gamma^{\#}\hat{\sigma}^{\#}[\varphi_o].$$

■

Corollary 5.2 *For arbitrary flat connections $\lambda, \sigma : TM \to TM \times \mathfrak{g}$,*

$$[\lambda, \sigma] = (\hat{\sigma}^{\#} - \hat{\lambda}^{\#})[\varphi_o].$$

Put $\dot{\mathbb{R}}^{n+1} = \mathbb{R}^{n+1} \setminus \{0\}$ and let $\mathfrak{g}$ be any n-dimensional unimodular Lie algebra $\mathfrak{g}$. Take tensors $0 \neq \varepsilon_o \in \bigwedge^n \mathfrak{g}$, $\varphi_o \in \bigwedge^n \mathfrak{g}^\star$ joined by the relation $\iota_{\varepsilon_o}(\varphi_o) = 1$. Fix a flat connection $\sigma : T\dot{\mathbb{R}}^{n+1} \to T\dot{\mathbb{R}}^{n+1} \times \mathfrak{g}$ in the trivial Lie algebroid $A = T\mathbb{R}^{n+1} \times \mathfrak{g}$ (oriented by the tensor ε_o). Let $i : S^n \hookrightarrow \dot{\mathbb{R}}^{n+1}$ be the inclusion.

Proposition 5.3 *The index $j_0(\sigma)$ of σ is given by the formula*

$$j_0(\sigma) = \int_{S^n} \sigma_S^\star(\varphi_o) \tag{5.1}$$

where σ_S is a nonstrong homomorphism of Lie algebroids defined as the composition $\sigma_S : TS^n \overset{i_\star}{\hookrightarrow} T\dot{\mathbb{R}}^{n+1} \xrightarrow{\hat{\sigma}} \mathfrak{g}$.

Treat now $\hat{\sigma} : T\dot{\mathbb{R}}^{n+1} \to \mathfrak{g}$ as a 1-form on $\dot{\mathbb{R}}^{n+1}$ with values in $\mathfrak{g}$ and take the exterior n-product $\hat{\sigma} \wedge \ldots \wedge \hat{\sigma} \in \Omega^n(\dot{\mathbb{R}}^{n+1}; \bigwedge^n \mathfrak{g})$. We have $\hat{\sigma}^\star(\varphi_o) = \frac{1}{n!}(\varphi_o)_\star(\hat{\sigma} \wedge \ldots \wedge \hat{\sigma})$. Therefore

$$j_0(\sigma) = \frac{1}{n!}\varphi_o\left(\int_{S^n} i^\star(\hat{\sigma} \wedge \ldots \wedge \hat{\sigma})\right). \tag{5.2}$$

Example 5.4 Each trivial Lie algebroid $A = T\mathbb{R}^{n+1} \times \mathfrak{g}$ is integrable: $A = A(P)$ for $P = \mathbb{R}^{n+1} \times G$ where G is an arbitrary Lie group with the Lie algebra $\mathfrak{g}$. A connection $\sigma : T\dot{\mathbb{R}}^{n+1} \to A$ induces a connection $H \subset T(\dot{\mathbb{R}}^{n+1} \times G)$ in the principal bundle $\dot{\mathbb{R}}^{n+1} \times G$, and the flatness of σ means the integrability of H. Assume a leaf L of the foliation H is the graph of some function $f : \dot{\mathbb{R}}^{n+1} \to G$. (If $n \geq 2$, then such a function always exists which follows from the simple connectedness of $\dot{\mathbb{R}}^{n+1}$ and the reduction theorem [K-N]). Therefore $\hat{\sigma}(v) = R^{-1}_{f(x)\star}(f_\star v)$, $R_{f(x)}$ *which is the* right translation on G by $f(x)$, and $f^\star(\Delta_R) = \langle \varphi_o, \frac{1}{n!}(\hat{\sigma} \wedge \ldots \wedge \hat{\sigma}) \rangle$ for the right-invariant n-form $\Delta_R \in \Omega^n_R(G)$ equalling φ_o at the unity e of G.

(A) If G is compact, n-dimensional, oriented by Δ_R and the Lie algebra of G is spherical, then as a consequence of (5.1) and (5.2) we have

$$j_0(\sigma) = \int_{S^n} (f_{|S^n})^\star \Delta_R = \deg(f_{|S^n}) \cdot \int_G \Delta_R \,. \tag{5.3}$$

As a corollary (taking any mapping $f : \dot{\mathbb{R}}^{n+1} \to S^n$ such that $f_{|S^n} = \mathrm{id}_{S^n}$), we obtain the existence of a local, flat singular connection having a nonzero index at the singularity.

Formula (5.3) yields that the set of real numbers being the indexes at a given point of singular *local,* flat connections coming from functions is discrete (more exactly, is equal to the set of multiples of $\int_G \Delta_R$). Such a situation takes place, for example, for all flat connections in any $sk(3,\mathbb{R})$-Lie algebroid over M^4 (since we can take $G = SO(3)$).

(B) If G is not compact, then $\Delta_R = d(\Theta)$ for some Θ and

$$j_0(\sigma) = \int_{S^n} (f_{|S^n})^\star \Delta_R = \int_{S^n} d\left(f^*_{|S^n}\Theta\right) = 0.$$

Such a situation takes place, for example, in any $sl(2,\mathbb{R})$-Lie algebroid over M^4 (since we can take $G = SL(2,\mathbb{R})$). Clearly, this fact can be noticed immediately by using a base e, f, g of $sl(2,\mathbb{R})$ such that $[e,f] = g$, $[f,g] = 2f$, $[g,e] = 2e$.

For an $\mathbb{R}$-Lie algebroid, not every local, singular and flat connection comes from a function, see the below example.

Example 5.5 In any $\mathbb{R}$-Lie algebroid over M^2 we can construct a local, flat and singular connection whose index is a preassigned real number. Indeed, since $\mathbb{R}$ is abelian, therefore the flatness of σ is equivalent to the closedness of the 1-form $\hat{\sigma}$ on M^2. In this case, the product $k \cdot \hat{\sigma}$, $k \in \mathbb{R}$, also gives a flat connection. Therefore, if $\sigma : T\dot{\mathbb{R}}^2 \to T\dot{\mathbb{R}}^2 \times \mathbb{R}$, $v \mapsto (v, \hat{\sigma}(v))$, is a flat connection with a nonzero index at 0, $j_0(\sigma) \neq 0$, then, for an arbitrary real number $k \in \mathbb{R}$, the mapping

$$\tau : T\dot{\mathbb{R}}^2 \longrightarrow T\dot{\mathbb{R}}^2 \times \mathbb{R}, \quad v \longmapsto \left(v, \frac{k}{j_0(\sigma)} \cdot \hat{\sigma}(v)\right),$$

is a flat connection with $j_0(\tau) = k$. Except for a discrete set of real numbers, this connection does not come from a function. More implicitly, considering $\varepsilon_0 = 1 \in \mathbb{R}$ and taking $\hat{\tau} = \frac{k}{2\pi}\left(\frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx\right)$, $k \in \mathbb{R}$, we have $j_0(\tau) = \int_{S^1} \hat{\tau} = k$.

Example 5.6 In the Hopf S^1-bundle $P = (S^3 \to S^2)$, for two different points $p_1, p_2 \in S^2$ and for any real number $k \in \mathbb{R}$, there exists a global flat connection σ_k with two singularities at $\{p_1, p_2\}$, such that the index $j_{p_1}(\sigma_k)$ is equal to k. Indeed, since the Euler class of P is equal to the orientation class of S^2, any flat connection λ with a singularity at $\{p_1\}$ has the index at p_1 equal to 1 (assuming $\int_G \Delta_R = 1$). Take $p_2 \neq p_1$ and $M = S^2 \setminus \{p_2\}$. Since M is contractible, $P_{|M}$ is trivial $P_{|M} \cong M \times S^1$. The connection λ determines a connection $\bar{\lambda} : T(M \setminus \{p_1\}) \to A\left((M \setminus \{p_1\}) \times S^1\right) = T(M \setminus \{p_1\}) \times \mathbb{R}$. Take $\hat{\sigma} = \mathrm{pr}_2 \circ \bar{\lambda} : T(M \setminus \{p_1\}) \to \mathbb{R}$. For an arbitrary real number $k \in \mathbb{R}$,

$$\bar{\sigma}_k : T(M \setminus \{p_1\}) \longrightarrow T(M \setminus \{p_1\}) \times \mathbb{R}, \quad v \longmapsto (v, k \cdot \hat{\sigma}(v)),$$

is a flat connection. $\bar{\sigma}_k$ determines a flat connection σ_k in P with a singularity at $\{p_1, p_2\}$, such that $j_{p_1}(\sigma_k) = k$.

In the end we give some remarks concerning the existence of a connection with a finite number of singularities. We start with the case $\mathfrak{g} = \mathbb{R}$.

Proposition 5.7 *In each invariantly oriented $\mathbb{R}$-Lie algebroid over an arbitrary manifold M for which $H^2(M) = 0$ there exists a flat connection, in particular, when M is 2-dimensional non-compact.*

Proof. According to [K7], each invariantly oriented $\mathbb{R}$-Lie algebroid A over M is isomorphic to $(M \times \mathbb{R}) \bigoplus TM$ with $\mathrm{pr}_2 : (M \times \mathbb{R}) \bigoplus TM \to TM$ as the anchor and the bracket $[\![\cdot,\cdot]\!]$ defined via some closed real 2-form Ω in the following way: $[\![(f, X), (g, Y)]\!] = (-\Omega(X, Y) + \partial_X g - \partial_Y f, [X, Y]), X, Y \in \mathfrak{X}(M), f, g \in C^\infty(M)$. Each connection $\lambda : TM \to (M \times \mathbb{R}) \bigoplus TM$ has the form $\lambda(v) = \left(\bar{\lambda}(v), v\right)$ for a 1-differential form $\bar{\lambda} \in \Omega^1(M)$. A simple calculation shows that λ is flat if and only if $d\left(\bar{\lambda}\right) = \Omega$. If $H^2(M) = 0$, such a 1-form exists. ■

As a corollary we get

Corollary 5.8 *In each s-Lie algebroid of rank 1 over a compact 2-manifold M there exists a flat connection with a beforehand finite non-empty set of isolated singularities.*

If a $sk(3, R)$-Lie algebroid over a compact 4-manifold comes from a Spin(3)-principal bundle, then - of course - it possesses a flat connection with one singularity (since such a cross-section of the sphere bundle exists [G-H-V, Vol. I]). In the general case, the problem is open.

The problem for $sl(2, \mathbb{R})$-Lie algebroids looks differently. Namely, by the main theorem (4.3) and Example 5.4 (B) we have that $\chi_{(A,\varepsilon)} = 0$ for any invariantly oriented $sl(2, \mathbb{R})$-Lie algebroid (A, ε) over a compact connected oriented manifold M^4, admitting a flat connection with a finite number of isolated singularities. Really, such a Lie algebroid is flat since locally we can remove a singularity: if σ is a flat connection in $T\dot{\mathbb{R}}^4 \times sl(2, \mathbb{R})$ then σ is given by a function $f : \dot{\mathbb{R}}^4 \to SL(2, \mathbb{R})$. Using the fact that the third group of homotopy of $SL(2, \mathbb{R})$ is zero, $\pi_3(SL(2, \mathbb{R})) = 0$, we can find $\bar{f} : \mathbb{R}^4 \to SL(2, \mathbb{R})$ such that $f(x) = \bar{f}(x)$ for $\|x\| \geq \varepsilon$ for a given small ε. This implies that we may remove the singularity at 0.

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