Generalized Symmetric Spaces

Oldřich Kowalski

Charles University, Czech Republic

1 Introduction

Let (M, g) be a Riemannian symmetric space. Then for any $x \in M$, there exists an isometry $s_x : M \longrightarrow M$ such that x is an isolated fixed point of s_x and ${s_x}^2 = \mathrm{Id}$. Then we have

$$\begin{aligned} &(s_x)_{*x} = (-\mathrm{Id})_x \\ &v_x \mapsto -v_x \end{aligned}$$

Now we consider a generalization of the notion of Riemannian symmetric spaces: Let (M, g) be a Riemannian manifold. We consider only the following condition.

$$\forall x \in M, \exists s_x : M \longrightarrow M : \text{isometry} \quad \text{s.t. } x : \text{isolated fixed point}$$

Such an isometry s_x is called a **generalized symmetry**. And a set of generalized symmetries $\{s_x \mid x \in M\}$ is called a (Riemannian) **s-structure**.

Lemma 1. *If $\{s_x \mid x \in M\}$ on (M, g) is an s-structure, then the closure $Cl(\{s_x\})$ of the group generated by all s_x, $x \in M$, in the Lie group $I(M, g)$ (the full isometry group) acts transitively on (M, g).*

Theorem 1 (Brickel). *Each (M, g) admitting an s-structure is homogeneous and thus real analytic.*

Here, "homogeneous" means that $\forall x, y \in M$, $\exists \varphi : M \longrightarrow M$: isometry, such that $\varphi(x) = y$.

The proof is elementary and interesting, but not short.

Affine case analogue:

Theorem 2 (Ledger). *Let (M, ∇) be an affine manifold (with an affine connection ∇). Let there exists a family $\{s_x \mid x \in M\}$ of generalized affine symmetries. If the map $x \mapsto s_x$ is smooth, then (M, ∇) is a homogeneous affine manifold.*

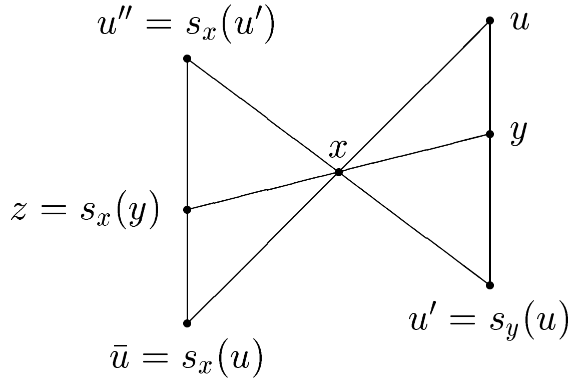
In general, it is an open problem whether (M, ∇) is still homogeneous if the map $x \mapsto s_x$ is not smooth. (Elementary but interesting question).

Regularity condition:

$$s_x \circ s_y = s_z \circ s_x \quad z = s_x(y) \quad \forall x, y \in (M, g)$$

This condition is satisfied for each symmetric space and its standard symmetries s_x!

Example. In the case $\mathbb{R}^2$, we can illustrate the above condition by a picture. (In general, it is not easy to check).



Definition 1. *Tangent symmetry field S of $\{s_x \mid x \in M\}$*

$$S_x := (s_x)_{*x} \quad \forall x \in (M, g)$$

Lemma 2. *The regularity condition for $\{s_x \mid x \in M\}$ is satisfied if and only if S is invariant with respect to each s_x:*

$$(s_x)_{*y} \circ S_y = S_{s_x(y)} \circ (s_x)_{*y} \quad \forall x, y \in (M, g)$$

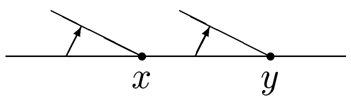
In the case of symmetric spaces,

$$S_y = (-\mathrm{Id})_y, \quad S_{s_x(y)} = (-\mathrm{Id})_{s_x(y)},$$

and the regularity follows.

An s-structure $\{s_x \mid x \in M\}$ which satisfies the regularity condition is called a **regular s-structure**. (It is a very strong condition).

Example of regular s-structures:
$M = \mathbb{R}^2$, $s_x =$ rotation arround x with constant angle α.
It is obvious from Lemma 2 that this s-structure is regular because each tangent symmetry S_x is also a rotation. (This direct proof of regularity is a good excercise for high school students).



Proposition 1. *If $\{s_x \mid x \in M\}$ is a regular s-structure on (M,g), then the tangent field is analytic and the map $(x,y) \mapsto s_x(y)$ is also analytic.*

Proposition 2. *If (M,g) admits an s-structure (resp. regular s-structure) $\{s_x \mid x \in M\}$, then (M,g) admits an s-structure (resp. regular s-structure) $\{s'_x \mid x \in M\}$ of finite order. It means that $\exists k \in \mathbb{Z},\ k \geq 2$ s.t.*

$$(s'_x)^k = Id \quad \text{for } \forall x \in M.$$

Remark: It it not true for the affine case.

If there exists a regular s-structure $\{s_x \mid x \in M\}$ of order k on (M,g), then (M,g) is called a **k-symmetric space**. (Without regularity, a pointwise k-symmetric space).

Thus if there exists a regular s-structure on (M,g), then for some $\exists k$, (M,g) is a k-symmetric space.

Regularity condition is really an **additional** condition but in a nontrivial sense:
1) (R.A. Marinosci [4]) If $\dim(M,g) \leq 5$, then every (M,g) admitting an s-structure also admit a regular s-structure.
2) $S^9 = SU(5)/SU(4)$
S^9 is a geodesic sphere in (P^5, Fubini-Study metric) with the induced metric, which is not the standard sphere. This S^9 admits an s-structure of order 4 but no regular s-structure at all.

There are finitely many other examples $S^{13}, S^{17}, \cdots$.

In the following, we always assume regularity!

(M,g) is called a **generalized symmetric space** if it admits a regular s-structure.

(M,g) is called of **order** k $(k \geq 2)$ if it admits a regular s-structure of order k but not of any lower order $l < k$.

2 Examples and Classifications

Dimension $\boxed{n = 3}$

$$\begin{aligned} G &= \left\| \begin{matrix} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{matrix} \right\| \\ & \quad\ \text{(solvable)} \\ G &\cong \mathbb{R}^3(x,y,z) \end{aligned}$$

Special invariant metrics on G:

$$g = e^{2z}dx^2 + e^{-2z}dy^2 + \lambda^2 dz^2$$

where $\lambda > 0$ is an arbitrary parameter.
Symmetry of order 4 at the origin:

$$x' = -y, \quad y' = x, \quad z' = -z$$

(See B-Spaces by Takahashi).

Thses are all generalized symmetric spaces of dimension $n = 3$ which are not locally symmetric!

One kind of symmetric spaces:

G: compact connected Lie group

Consider the coset space $(G \times G)/\Delta(G \times G)$, where $\Delta(G \times G)$ is the **diagonal** of $G \times G$, i.e., $\Delta(G \times G) = \{(g, g) \mid g \in G\}$. $G \times G/\Delta(G \times G)$ is diffeomorphic to G via the map

$$(g_1, g_2) \mapsto g_1 {g_2}^{-1}.$$

$G \times G$ acts on G by

$$(g_1, g_2)(y) = g_1 y {g_2}^{-1}.$$

The isotropy group at the origin $e \in G$ is $\Delta(G \times G)$.

Now define $\sigma : G \times G \longrightarrow G \times G$ by

$$\sigma(g_1, g_2) = (g_2, g_2),$$

which is an involutive automorphism. The fixed point set is $(G \times G)^{\sigma} = \Delta(G \times G)$. σ induces a map $s : G \longrightarrow G$ defined by

$$s(g) = g^{-1} \quad \text{for } \forall g \in G.$$

Take a bi-invariant Riemannian metric Φ on G. Then Φ is **invariant** with respect to the action of $G \times G$ on G. It is also invariant with respect to $s : G \longrightarrow G$.

Then (G, Φ) is a Riemannian symmetric space in which the symmetry with respect to e is just the map

$$s : g \mapsto g^{-1}.$$

The symmetry s_x at general $x \in G$ is given by

$$g \mapsto x g^{-1} x.$$

Generalization(Ledger + Obata)

$$G^{k+1}/\Delta G^{k+1} \cong G^k$$

via $\pi : G^{k+1} \longrightarrow G^k$, where

$$\pi(g_1, \cdots, g_{k+1}) = (g_1 {g_{k+1}}^{-1}, \cdots, g_k {g_{k+1}}^{-1}).$$

Further define $\sigma : G^{k+1} \longrightarrow G^{k+1}$ by

$$\sigma(g_1, \cdots, g_{k+1}) = (g_{k+1}, g_1, \cdots, g_k).$$

Then σ is an automorphism of order $k + 1$. It induces a map $s : G^k \longrightarrow G^k$ defined by

$$s(g_1, \cdots, g_k) = ({g_k}^{-1}, g_1 {g_k}^{-1}, \cdots, g_{k-1} {g_k}^{-1}).$$

Let Φ be a bi-invariant metric on G. Then Φ generates a bi-invariant metric Φ^{k+1} on G^{k+1} such that

$$(G^{k+1}, \Phi^{k+1}) \cong \underbrace{(G, \Phi) \times \cdots \times (G, \Phi)}_{k+1}.$$

Then Φ^{k+1} induces a G^{k+1}-invariant metric $\Phi^{[k]}$ on G^k. The Riemannian manifold $(G^k, \Phi^{[k]})$ is a $(k+1)$-symmetric space. (Here $\Phi^{[k]} \neq \Phi^k$). $(G^k, \Phi^{[k]})$ is not a symmetric space.

Proposition 3. *Let*
a) G be compact and simple,
b) $\Phi = -$(Killing form).
Assume that $\tau : G^{k+1} \longrightarrow I(G^k, \Phi^{[k]})$ has as image the full identity component of $I(G^k, \Phi^{[k]})$. Then $(G^k, \Phi^{[k]})$ is not l-symmetric for any $l < k+1$.

Example. If we choose $G = SO(3)$, $\Phi = -(\text{Killing form})$, then the assumption of the Proposition 3 can be shown to be satisfied. Hence we get

Theorem 3. *For each $k \geq 2$, there is a generalized symmetric space (M, g) of order k, i.e., k-symmetric but not l-symmetric for $\forall l < k$. $M = G/K$ with semisimple group G.*

Classifications:
1) All generalized symmetric spaces of order 3 with semisimple (or reductive group) G by A. Gray [1].
2) All compact generalized symmetric spaces of order 4 by J.A. Jiménez [2].

An example of generalized symmetric Riemannian spaces of **solvable type**:

$$G = \left\| \begin{array}{ccccc} e^{u_0} & 0 & \cdots & 0 & x_0 \\ 0 & e^{u_1} & \cdots & 0 & x_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{u_n} & x_n \\ 0 & 0 & \cdots & 0 & 1 \end{array} \right\| \cong \mathbb{R}^{2n+1}$$

$u_0, u_1, \cdots, u_n, x_1, \cdots, x_n$ are variables, where

$$u_0 + u_1 + \cdots u_n = 0.$$

We define a Riemannian metric

$$g = \sum_{i=0}^{n} e^{-2u_i}(dx_i)^2 + a \sum_{\alpha,\beta=1}^{n} du_\alpha du_\beta, \quad a > 0.$$

Then
1) (G, g) is generalized symmetric of order $2n+2$ (even).
2) The group of all isometries preserving the origin is finite and isometric to

$$(\mathbb{Z}_2)^{n+1} \times S_{n+1}$$

where S_{n+1} is the permutation group pf $n+1$ elements. Thus, the group is of order $2^{n+1}(n+1)!$.

Remark. There are also examples of generalized symmetric spaces of solvable type and odd order. For any order k, there exists an example of solvable type. This is the main obstacle to the classification of **all** generalized symmetric spaces (in contrary to the ordinary symmetric spaces, where the classification is known).

3 Canonical connection

Let (M, g) be a Riemannian manifold and $\{s_x \mid x \in M\}$ a regular s-structure on M. Then there is a unique affine conncetion $\widetilde{\nabla}$ on M such that
(i) $\widetilde{\nabla}$ is invariant under all s_x,
(ii) $\widetilde{\nabla} S = 0$ (where S is the tangent symmetry field defined by $S_x := (s_x)_{*x} \quad x \in M$).

In the explicit form, $\widetilde{\nabla}$ is given by the Ledger's formula

$$\widetilde{\nabla}_X Y = \nabla_X Y - (\nabla_{(I-S)^{-1}X} S)(S^{-1}Y) \qquad \text{for } X, Y \in \mathfrak{X}(M)$$

where ∇ denotes the Riemannian connection of (M, g).

Note that since S has no engenvalue equal to 1 (s_x has only isolated fixed point), $(I-S)^{-1}$ exists.

$\widetilde{\nabla}$ is always complete.

Each tensor field P on M which is invariant by all s_x is parallel with respect to $\widetilde{\nabla}$, i.e., $\widetilde{\nabla} P = 0$. Therefore, the torsion tensor field $\widetilde{T}$ and the curvature tensor field $\widetilde{R}$ of $\widetilde{\nabla}$ are parallel:

$$\widetilde{\nabla}\widetilde{T} = 0, \quad \widetilde{\nabla}\widetilde{R} = 0.(*)$$

Remark that if M is a symmetric space with standard $\{s_x\}$, $\widetilde{\nabla} = \nabla$.

Because of $(*)$, we have

$$\widetilde{\nabla}\widetilde{R} = \widetilde{\nabla}\widetilde{T} = \widetilde{\nabla} S = \widetilde{\nabla} g = 0.$$

Therefore we can linearize Riemannian manifolds $((M, g), \{s_x \mid x \in M\})$ with specific regular s-structure. We get algebraic objects $(V, <, >, S, \widetilde{R}, \widetilde{T})$, where S is a nonsingular linear isometry without fixed directions and $\widetilde{R}, \widetilde{T}$ are tensors of types $(1,3)$, $(1,2)$ respectively satisfying standard algebraic identities for the curvature and torsion. These objects are called **infinitesimal models**.

We have one-to-one correspondence between the set of simply connected Riemannian manifolds with regular s-structures and the set of infinitesimal models. Using the methods of linear algebra, we can classify regular s-structures and hence also generalized symmetric spaces (at least locally and for small dimensions).

If (M, g) is a generalized symmetric space and simply connected, then we have the de Rham decomposition

$$M = M_1 \times \cdots \times M_r,$$

here if $\{s_x\}$ is an s-tructure of order k of M, each M_i is a generalized symmetric space of order k_i and k_i divides k $(i = 1, \cdots, r)$.

4 Theory of eigenvalues for generalized symmetric spaces

This theory is useful for the classification procedure of generalized symmetric spaces.

Let $\{s_x \mid x \in M\}$ be a regular s-structure. Then $S_x : T_xM \longrightarrow T_xM$ has the same eigenvalues for $\forall x \in M$. S_x is real, orthogonal, without fixed vectors. Hence its system $(\theta_1, \cdots, \theta_n)$ of eigenvalues must satisfy

$$|\theta_i| = 1, \quad \theta_i \neq 1 \quad \text{for } i = 1, \cdots, n$$

and

$$(\bar{\theta_1}, \cdots, \bar{\theta_n}) = (\theta_1, \cdots, \theta_n) \qquad \text{up to numeration.}$$

Every n-tuple with these properties is called an **admissible** n-tuple. Denote by $\mathcal{P}_n$ the set of all admissible n-tuples.

We now make $\mathcal{P}_n$ a **partially ordered** set. We introduce **characteristic equations** (of two classes):
a) $X_iX_j = 1, \quad i, j = 1, \cdots, n,$
b) $X_iX_j = X_k, \quad i \neq j \neq k \neq i, \quad i, j, k = 1, \cdots, n.$

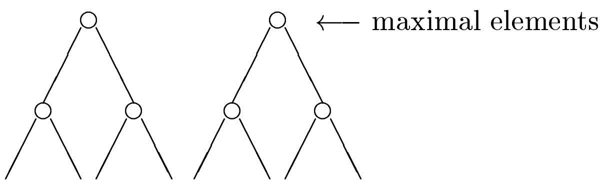
Each element $(\theta_1, \cdots, \theta_n) \in \mathcal{P}_n$ satisfies at least one equation of type a) $(n \geq 2)$. (If the eigenvalues of S_x on (M, g) do not satisfy any equation of type b), then (M, g) is locally symmetric.)

We introduce the **partial ordering** "$\preceq$" on $\mathcal{P}_n$ as follows: $(\theta_i) \preceq (\theta'_i)$ if all characteristic equations satisfied by (θ_i) are also satisfied by (θ'_i) after possible re-numeration.

(Here $(\theta_i) \preceq (\theta'_i)$ and $(\theta'_i) \preceq (\theta_i)$ is possible, e.g., if (θ'_i) is a renumeration of (θ_i).)

Proposition 4. *Let $\{s_x \mid x \in M\}$ be a regular s-structure on (M, g) with the corresponding eigenvalues (θ_i). Let $(\theta'_i) \in \mathcal{P}_n$ and $(\theta'_i) \succeq (\theta_i)$. Then there is a regular s-structure $\{s'_x \mid x \in M\}$ on (M, g) with the eigenvalues (θ'_i).*

Theorem 4. *In the partially ordered set $(\mathcal{P}_n, \preceq)$, there is only a finite set of maximal elements, which are all of finite order.*



This set will be denoted by $\mathcal{D}_n$.

The same procedure works without regularity condition.

We only have to add new type of characteristic equations
c) $X_iX_jX_k = X_l$.
The corresponding set $\mathcal{D}'_n$ of maximal elements contains $\mathcal{D}_n$, but it is bigger.

Now we go back to the regular s-structures. We have the following theorem.

Theorem 5. *For every dimension $n \geq 2$, there is a finite set $\mathcal{D}_n = \{(\theta_1^\alpha, \cdots, \theta_n^\alpha) \mid \alpha = 1, \cdots, r(n)\}$ with the following properties:*
a) all elements of $\mathcal{D}_n$ are of finite order,
b) if (M, g) admits a regular s-structure, then it also admits a regular s-structure with the system of eigenvalues contained in $\mathcal{D}_n$.

(Warning: the same (M, g) can admit regular s-structures corresponding to more elements of $\mathcal{D}_n$).

This theorem is important for the classification of generalized symmetric spaces in low dimensions.

4.1 Basic systems of eigenvalues in small dimensions

$\mathcal{D}_2$:	$(-1,-1)$	(symmetric space)
$\mathcal{D}_3$:	$(-1,-1,-1)$	(symmetric space)
	$(i,-i,-1)$	(4-symmetric space of dimension 3)
$\mathcal{D}_4$:	$(-1,-1,-1,-1)$	(symmetric space)
	$(\theta,\theta,\theta^2,\theta^2), \quad \theta = e^{\frac{2\pi i}{3}}$	($\exists$ new example of order 3)
	$(i,-i,-1,-1)$	(direct product or space of order 3)
	$(\theta,\theta^2,\theta^3,\theta^4), \quad \theta = e^{\frac{2\pi i}{5}}$	(only space of order 3)
$\mathcal{D}_5$:	$(-1,-1,-1,-1,-1)$	(symmetric space)
	$(i,-i,-1,-1,-1)$	(direct product)
	$(i,-i,i,-i,-1)$	($\exists$ new examples of order 4)
	$(\theta,\theta^2,\theta^3,\theta^4,\theta^5), \quad \theta = e^{\frac{2\pi i}{6}}$	($\exists$ new examples of order 6)
	$(e^{\frac{\pi i}{4}}, e^{-\frac{\pi i}{4}}, i, -i, -1)$	(no additional examples)

An estimate: Let $k(n)$ denote the maximum of all orders of the elements in $\mathcal{D}_n$. Then

$$\begin{aligned} k(n) &\leq 5^{\frac{n}{4}} \qquad && n : \text{even}, \\ k(n) &\leq 2 \cdot 5^{\frac{n-1}{4}} \qquad && n : \text{odd}. \end{aligned}$$

We see that $k(4) \leq 5$ (exact estimate) and $k(5) \leq 10$ (not exact, in fact $k(5) \leq 8$). It is a natural question whether there is an estimate to improve the above inequality or not.

4.2 Classification in dimension $n = 4$

All generalized symmetric spaces of dimension 4 are symmetric spaces or spaces of order 3 isometric to one of the following forms:

$$M = \left\| \begin{matrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{matrix} \right\| \Big/ \left\| \begin{matrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{matrix} \right\|$$

$$\text{where} \quad \det \left\| \begin{matrix} a & b \\ c & d \end{matrix} \right\| = 1$$

$$M = \begin{pmatrix} \text{group of all equiaffine trans-} \\ \text{formations of the Euclidean} \\ \text{plane} \end{pmatrix} \Big/ \begin{pmatrix} \text{subgroup of all rotations ar-} \\ \text{round the origin} \end{pmatrix}$$

$M \cong \mathbb{R}^4[x, y, u, v]$ and then

$$\begin{aligned} g &= (-x + \sqrt{x^2 + y^2 + 1})du^2 + (x + \sqrt{x^2 + y^2 + 1})dv^2 \\ &\quad -2ydudv + \lambda^2 \left[\frac{(1 + y^2)dx^2 + (1 + x^2)dy^2 - 2xydxdy}{1 + x^2 + y^2} \right] \end{aligned}$$

where $\lambda > 0$ is a parameter.

A typical symmetry of order 3 at the origin is given by

$$\begin{aligned} u' &= \cos \frac{2\pi}{3} \cdot u - \sin \frac{2\pi}{3} \cdot v, \\ v' &= \sin \frac{4\pi}{3} \cdot u + \cos \frac{4\pi}{3} \cdot v, \\ x' &= \cos \frac{2\pi}{3} \cdot x - \sin \frac{2\pi}{3} \cdot y, \\ y' &= \sin \frac{4\pi}{3} \cdot x + \cos \frac{4\pi}{3} \cdot y. \end{aligned}$$

4.3 Classification in dimension $n = 5$

A generalized symmetric space of dimension 5 is locally isometric to one of the followings:

- a symmetric space,
- 11 families of order 4,
- 1 family of order 6.

4.3.1 Examples of order 4 – selection:

Example 1. Nilpotent matrix group

$$\left\| \begin{array}{cccc} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ u & v & 1 & z \\ 0 & 0 & 0 & 1 \end{array} \right\| \cong \mathbb{R}^5[x, y, z, u, v],$$

$$g = dx^2 + dy^2 + du^2 + dv^2 + \lambda^2(xdu - ydv + dz)^2,$$
$$\text{where} \quad \lambda > 0 \quad \text{arbitrary parameter.}$$

Typical symmetry of order 4 at the origin is given by

$$x' = -y, \ y' = x, \ z' = -z, \ u' = -v, \ v' = u.$$

Example 2. $M = SO(3,)/SO(2)$ with the invariant Riemannian metrics depending on 3 real parameters. The symmetry at the origin of M is as follows

$$\left\| \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right\| \mapsto \left\| \begin{array}{ccc} \bar{b}_2 & -\bar{b}_1 & \bar{b}_3 \\ -\bar{a}_2 & \bar{a}_1 & -\bar{a}_3 \\ \bar{c}_2 & -\bar{c}_1 & \bar{c}_3 \end{array} \right\|.$$

Example 3. Solvable complex matrix group

$$M = \left\| \begin{matrix} e^{\lambda t} & 0 & z \\ 0 & e^{-\lambda t} & w \\ 0 & 0 & 1 \end{matrix} \right\| \cong \mathbb{R}^2[z, w] \times [t]$$

where z, w: complex variables, t: real variable, λ: complex parameter, $\lambda \neq 0$.

A family of invariant metrics g such that (M, g) is irreducible (i.e., not a product of Riemannian manifolds).

Symmetry at the origin is given by

$$z' = iw, \quad w' = iz, \quad t' = -t.$$

Example 4. $M = SO(3) \times SO(3)/SO(2)$

$$SO(2) = \left\{ \left(\left\| \begin{matrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{matrix} \right\|, \left\| \begin{matrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{matrix} \right\| \right), t \in \right\}$$

Symmetry of order 4 at the origin is induced by the following automorphisms of the group $GL(3,) \times GL(3,)$:

$$\left(\left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \right\|, \left\| \begin{matrix} \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 \\ \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 \\ \tilde{c}_1 & \tilde{c}_2 & \tilde{c}_3 \end{matrix} \right\| \right) \mapsto \left(\left\| \begin{matrix} \tilde{a}_1 & -\tilde{a}_2 & -\tilde{a}_3 \\ -\tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 \\ -\tilde{c}_1 & \tilde{c}_2 & \tilde{c}_3 \end{matrix} \right\|, \left\| \begin{matrix} a_1 & -a_2 & a_3 \\ -b_1 & b_2 & -b_3 \\ c_1 & -c_2 & c_3 \end{matrix} \right\| \right)$$

Example 5. $M = SU(3)/SU(2) \cong S^5$ ((M, g) is not a standard sphere). For a special choice of the invariant metric, S^5 has the metric induced from the Fubini–Study metric in P^3 as a geodesic sphere in P^3.

4.3.2 A family of spaces of order 6

$$M = \left\| \begin{matrix} e^{-(u+v)} & 0 & 0 & x \\ 0 & e^u & 0 & y \\ 0 & 0 & e^v & z \\ 0 & 0 & 0 & 1 \end{matrix} \right\| \cong \mathbb{R}^5[x, y, z, u, v] \qquad \text{(A solvable Lie group)}$$

We consider invarinat Riemannian metrics depending on two real parameters $a > 0, b > 0$:

$$\begin{aligned} g &= a^2(du^2 + dv^2 + dudv) + (b^2 + 1)(e^{2(u+v)}dx^2 + e^{-2u}dy^2 + e^{-2v}dz^2) \\ &\quad +(b^2 - 2)(e^v dxdy + e^u dxdz - e^{-(u+v)}dydz). \end{aligned}$$

The typical symmetry at the origin is given by

$$x' = y, \; y' = -z, \; z' = x, \; u' = v, \; v' = -(u + v).$$

This symmetry is of order 6!

The isotropy group of isometries at the origin consists of 8 elements.

Remark. Let (M, g) be not necessarily homogeneous in general. Let there exist a point $p \in M$ such that p is an isolated fixed point for some isometry $\exists s_p : M \longrightarrow M$. Then there exists an isometry $s'_p : M \longrightarrow M$ of finite order k:

$$k \leq (4 + s(p))^{\frac{\dim M + 1}{2}}$$

where $s(p)$ denotes the Singer number defined as follows: Put

$$G_p^{(l)} = \{A \in \mathrm{Aut}(T_pM) \mid A(g_p) = g_p, A(R_p) = R_p, \cdots, A((\nabla^{(l)} R)_p) = (\nabla^{(l)} R)_p\},$$

$l = 0, 1, 2, \cdots$. Then there exists the minimal $k \geq 0$ such that the sequence $\{G_p^{(l)}\}$ stabilizes, i.e.,

$$G_p^{(0)} \supset G_p^{(1)} \supset \cdots \supset G_p^{(k)} = G_p^{(k+1)} = \cdots$$

Such a number $k = s(p)$ is called the **Singer number**.

References

[1] A. Gray, Riemannian manifolds with geodesic symmetries of order 3, *J. Differential Geometry*, **7**(1972), 343–369.

[2] J.A. Jiménez, Riemannian 4-symmetric spaces, *Trans. Amer. Math. Soc.*, **306**(1988), 715–734.

[3] O. Kowalski, *Generalized symmetric spaces*, Lecture Notes in Mathematics, 805, Springer-Verlag, Berlin-New York, 1980.

[4] R. A. Marinosci, Classification of five-dimensional generalized pointwise symmetric Riemannian spaces, *Geom. Dedicata*, **57**(1995), 11–53.

Oldřich Kowalski
Faculty of Mathematics and Physics
Charles University
Sokolovská 83
186 00 Praha
Czech Republic
e-mail: kowalski@karlin.mff.cuni.cz
URL: `http://www.karlin.mff.cuni.cz/~kowalski/`