On a class of Ricci-recurrent manifolds

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1 Introduction

Following Prvanivić ([P]), a semi-Riemannian manifold $(M,\ g)$ will be called conformally quasi-recurrent if its Weyl conformal curvature tensor C satisfies

$$\begin{aligned} \nabla_Z C(X,Y,V,W) = w(Z)C(X,Y,V,W) + \\ p(X)C(Z,Y,V,W) + p(Y)C(X,Z,V,W) + \\ p(V)C(X,Y,Z,W) + p(W)C(X,Y,V,Z) \end{aligned} \tag{1}$$

for some 1-forms w, p. In condition considered originally by Prvanović $w = 2p$. However, the last relation together with (1) implies that for the tensor C (and in fact for any generalized curvature tensor satisfying (1)) the second Bianchi identity must hold. The aim of this note is to give a classification of conformally quasi-recurrent manifolds in the sense of (1) which are simultaneously Ricci-recurrent, i. e. those the Ricci tensor S satisfies

$$\nabla S = b \otimes S$$

for some 1-form b.

For a generalized curvature tensor B define $\widetilde{B}$ by

$$g\left(\widetilde{B}\left(X,Y\right)V,W\right) = B\left(X,Y,V,W\right).$$

Then for a $(0,k)$ tensor field T, $k \geq 1$, and $(0,2)$ tensor field S we define the tensor fields $B \cdot T$ and $Q(S,T)$ by the formulas

$$\begin{aligned} &\left(B\cdot T\right)\left(X_1,...X_k;X,Y\right) = \\ &\quad -T\left(\widetilde{B}\left(X,Y\right)X_1,X_2,...,X_k\right) - ... - T\left(X_1,...,X_{k-1},\widetilde{B}\left(X,Y\right)X_k\right), \end{aligned}$$

$$\begin{aligned} &Q\left(S,T\right)\left(X_1,...X_k;X,Y\right) = \\ &\quad T\left(\left(X\wedge_S Y\right)X_1,X_2,...,X_k\right) + ... + T\left(X_1,...,X_{k-1},\left(X\wedge_S Y\right)X_k\right), \end{aligned}$$

where

$$\left(X\wedge_S Y\right)Z = S(Y,Z)X - S(X,Z)Y.$$

If the tensors $R{\cdot}R$ and $Q(S,R)$ are linearly dependent then the manifold is said to be Ricci-generalized pseudosymmetric one ([D-D]). It is obvious that any semisymmetric as well as any Ricci flat manifold is Ricci generalized pseudosymmetric. The manifold (M,g) is Ricci-generalized pseudosymmetric iff the relation

$$R \cdot R = LQ(S,R) \tag{2}$$

holds on the set $\{x \in M, Q(S,R)\,(x) \neq 0\}$, L being a function on M. Remark that the relation (2) with $L = 1$ is of particular importance.

All manifolds under consideration are assumed to be smooth Hausdorff connected and their metrics are not assumed to be definite.

2 Results

The first Lemma shows the difference between the 1-forms w and p :

Lemma 1 . *Suppose that at a point of the manifold M relation (1) holds. Then*

$$p_r C^r_{ijk} = 0,$$

$$w_r C^r_{ijk} = C^r{}_{ijk,r},$$

$$C_{hijk,[lm]} = \Delta w_{lm} C_{hijk} + p_{hm} C_{lijk} + p_{im} C_{hljk} + p_{jm} C_{hilk} + p_{km} C_{hijl} - \\ p_{hl} C_{mijk} - p_{il} C_{hmjk} - p_{jl} C_{himk} - p_{kl} C_{hijm},$$

where $\Delta w_{lm} = w_{l,m} - w_{m,l}$, $p_{hm} = p_{h,m} - p_h p_m$ *and comma denotes covariant differentiation.*

From the so called Patterson identity we have

Proposition 2 . *Let M be a 4-dimensional manifold with nowhere vanishing Weyl conformal curvature tensor C. If C satisfies (1), then M is conformally recurrent manifold, precisely*

$$\nabla C = (w + 2p) \otimes C.$$

In the sequel we shall assume the following hypothesis:

(H) M *is a Ricci – recurrent manifold with nowhere vanishing Weyl conformal curvature tensor and Ricci tensor. Moreover, the Weyl conformal curvature tensor satisfies* (1), p *does not vanishes on a dense subset and the Ricci tensor is not parallel.*

By hypothesis, M admits a covector field b satisfying

$$S_{ij,k} = b_k S_{ij}, \quad S_{ij,kl} = (b_{k,l} + b_k b_l) S_{ij}, \quad S_{ij,[kl]} = \Delta b_{kl} S_{ij},$$

where $\Delta b_{kl} = b_{k,l} - b_{l,k}$.

Proposition 3 . *Let M (dimM > 3) be a Ricci-recurrent manifold with non-parallel Ricci tensor and suppose that M admits a covector field p with properties:*
i) p does not vanish on a dense subset of M;
ii) $p_r C^r_{ijk} = 0$ *on M.*
Then the scalar curvature of M vanishes.

Lemma 4 . *Under hypothesis (H) relations*

$$p_r S^r_h = 0,$$

$$db = 0$$

$$S_{mr} C^r_{ijk} = 0$$

hold on M.

From the above Lemma it follows

Corollary 5 . *Under hypothesis* (H)

$$d(w + 2p) = 0$$

holds on M.

Making use of the Corollary 5 one can deduce

Proposition 6 . *Assume that on a manifold* $(M,\, g)$ *hypothesis* (H) *is satisfied.*
If $t = w - 2p = 0$ *on* $(M,\, g)$*, then the manifold is conformally related to a non-conformally flat conformal symmetric one* $(M,\, Exp\,(2f)\, g)$*. Conversely, if* $(M,\, g)$ *is conformally related to a non-conformally flat conformally symmetric one, then* $w - 2p = 0$.

Proposition 7 . *Suppose that on a manifold* M*,* $(dimM > 4)$*, hypothesis* (H) *is satisfied. If* $b_l \neq 0$ *at a point* $x \in M$*, then on some neighbourhood of* x *there exists null (i.e. isotropic) parallel vector field*

$$v_i = Exp[\frac{-1}{2} b]\ k_i, \qquad \partial_i b = b_i,$$

related to the Ricci tensor by

$$S_{ij} = \epsilon\ k_i k_j, \qquad |\epsilon| = 1.$$

Define the tensor

$$\begin{aligned} Q(S,C)_{hiqtpj} = {} & S_{hp}C_{jiqt} - S_{hj}C_{piqt} + S_{ip}C_{hjqt} - S_{ij}C_{hpqt} + \\ & S_{qp}C_{hijt} - S_{qj}C_{hipt} + S_{tp}C_{hiqj} - S_{tj}C_{hiqp}. \end{aligned}$$

It is well known, that if the scalar curvature vanishes and the rank of the Ricci tensor is one, then

$$Q(S,C)_{hiqtpj} = Q(S,R)_{hiqtpj}$$

Lemma 8 . *Suppose that on a manifold* M*,* $(dimM > 4)$*, hypothesis* (H) *is satisfied. Then*

$$t_r b^r \cdot Q(S,C)_{hiqtpj} = 0.$$

Lemma 9 . *Suppose that on a manifold* M*,* $(dimM > 4)$*, hypothesis* (H) *is satisfied and*

$$t_r b^r = 0.$$

Then

$$(n-4)\, b_r b^r = 0,$$

$$t_l C_{hijk} + t_j C_{hikl} + t_k C_{hilj} = 0$$

and

$$t_l \left(C_{tqlr} C^r_{jih} - C_{tqjr} C^r_{lih} \right) = 0$$

Proposition 10 . *Suppose that on a manifold M, $(dim M \geq 4)$, hypothesis (H) is satisfied. If at $x \in M$, $t_r b^r(x) \neq 0$ (hence $b_l(x) \neq 0$), then on some neighbourhood of x the Riemann-Christoffel curvature tensor has the form*

$$R_{hijk} = k_i k_j S_{hk} - k_i k_k S_{hj} + k_h k_k S_{ij} - k_h k_j S_{ik},$$

where $S_{ij} = z^r z^s R_{rijs}$, $z^r k_r = 1$.

As a consequence of Propositions 3, 7 and 10 we obtain

Corollary 11 .*Suppose that on a manifold M, $(dim M > 4)$, hypothesis (H) and and $Q(S,C)_{hiqtpj} = 0$ hold. If $b_l \neq 0$ at a point $x \in M$, then on some neighbourhood of x the metric g is of the Walker type. Moreover, M is semi-symmetric, i.e. $R \cdot R = 0$.*

On the other hand, if $Q(S,C)_{hiqtpj} \neq 0$, then Lemma 9 results in

Corollary 12 .*Let on a manifold M, $(dim M > 4)$, hypothesis (H) be satisfied. Suppose moreover that $Q(S,C)_{hiqtpj} \neq 0$ and $w - 2p \neq 0$. Then*

$$(R \cdot R)_{hiqtpj} = Q(S,R)_{hiqtpj}.$$

Thus we conclude with the following

Theorem 13 .*A conformally quasi-recurrent and Ricci-recurrent manifold of dimension $n > 4$ with nowhere vanishing Weyl conformal curvature tensor and Ricci tensor non-conformally related to a conformally symmetric one must be necessary Ricci-generalized pseudosymmetric manifold.*

References

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