HOMOLOGY OF COMPLETE SYMBOLS AND NON-COMMUTATIVE GEOMETRY

MOULAY-TAHAR BENAMEUR AND VICTOR NISTOR

Version: 3.0A; Revised: 6/17/2000; Run: June 22, 2000

Abstract. We identify the periodic cyclic homology of the algebra of complete symbols on a differential groupoid $\mathcal{G}$ in terms of the cohomology of $S^*(\mathcal{G})$, the cosphere bundle of $A(\mathcal{G})$, where $A(\mathcal{G})$ is the Lie algebroid of $\mathcal{G}$. We also relate the Hochschild homology of this algebra with the homogeneous Poisson homology of the space $A^*(\mathcal{G}) \smallsetminus 0 \cong S^*(\mathcal{G}) \times (0, \infty)$, the dual of $A(\mathcal{G})$ with the zero section removed. We use then these results to compute the Hochschild and cyclic homologies of the algebras of complete symbols associated with manifolds with corners, when the corresponding Lie algebroid is rationally isomorphic to the tangent bundle.

Contents

Introduction

Singular cohomology is often used in Algebraic Topology to obtaining invariants of topological spaces. In the same spirit, Hochschild and cyclic homology often provide interesting invariants of algebras. A possible important application of these algebra invariants is to the study of spaces with additional structures; these include, for instance, spaces with singularities or spaces endowed with group actions. This is one of the fundamental ideas of non-commutative geometry; see Connes' book [9] and the references therein.

Let $\mathcal{G}$ be a differentiable groupoid with units M, a manifold with corners, and Lie algebroid $A(\mathcal{G}) \to M$, (see [14] in this volume for definitions, notation, and background material). To $\mathcal{G}$ one can associate several algebras: the convolution algebras $\mathcal{C}_c^\infty(\mathcal{G})$, $L^1(\mathcal{G})$, $\Psi^\infty(\mathcal{G})$, or other variants of these algebras. These algebras have always been a favorite toy model for non-commutative geometry and have

Partially supported by the NSF Young Investigator Award DMS-9457859, NSF Grant DMS-9971951 and "collaborative research." **http://www.math.psu.edu/nistor/**.

several applications, see [5, 8, 9, 10, 22]. The algebra $\Psi^\infty(\mathcal{G})$ of pseudodifferential operators on $\mathcal{G}$, for example, is expected to play an important role in the analysis on singular spaces [14]. The ideal $\Psi^{-\infty}(\mathcal{G})$ of regularizing operators in $\Psi^\infty(\mathcal{G})$ identifies with $\mathcal{C}_c^\infty(\mathcal{G})$ and gives rise to index invariants via the boundary map in algebraic K-theory:

$$\partial : K_1^{alg}(\mathfrak{C}) \to K_0^{alg}(\Psi^{-\infty}(\mathcal{G})),$$

where $\mathfrak{C} = \mathcal{A}(M) := \Psi^\infty(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})$ or $\mathfrak{C} = \mathcal{A}_0(M) := \Psi^0(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})$, depending on the nature of the problem. Typically, much more attention has been devoted to finding cohomological invariants of $\Psi^{-\infty}(\mathcal{G})$, because its (algebraic) K_0 is so hard to compute. The computation of the K-theory of algebras associated with groupoids or their C^*-closures is far from solved and is part of the larger program involving the Baum-Connes Conjecture(s) [3].

In this paper, we concentrate on the homology of $\mathcal{A}(M)$ or variants of this algebra. We are interested in computing the Hochschild, the cyclic, and the periodic cyclic homology groups of $\mathcal{A}(M)$, denoted respectively by $\mathrm{HH}_*(\mathcal{A}(M))$, by $\mathrm{HC}_*(\mathcal{A}(M))$, or by $\mathrm{HP}_*(\mathcal{A}(M))$. Of all these, the periodic cyclic homology is the easiest to compute. The result is in terms of $A(\mathcal{G})$, the *Lie algebroid* of $\mathcal{G}$. Let $S^*(\mathcal{G})$ be the cosphere bundle of $A^*(\mathcal{G})$, that is, the set of unit vectors in the dual of the Lie algebroid of $\mathcal{G}$, and denote $\mathrm{H}^{[q]} = \oplus_{k\in\mathbb{Z}} \mathrm{H}^{q+2k}$.

Theorem 1. *Assume that the base M is σ-compact, then*

$$\mathrm{HP}_q(\mathcal{A}(M)) \cong \mathrm{H}_c^{[q]}(S^*(\mathcal{G}) \times S^1) \ \textit{and}\ \mathrm{HP}_q(\mathcal{A}_0(M)) \cong \mathrm{H}_c^{[q]}(S^*(\mathcal{G})). \tag{1}$$

The assumption that M be σ-compact can be replaced with the assumption that M be paracompact, but then we have to work with more complicated directed sets, and this is usually unnecessary in practice.

Recall that $A^*(\mathcal{G})$ has a natural Poisson structure. We do show that the natural filtration on the complex computing the Hochschild homology of the algebra $\mathcal{A}(M)$ gives rise to a spectral sequence with E^2-term identified with the homogeneous Poisson homology (Definition 2) of $A^*(\mathcal{G}) \smallsetminus 0$, the dual vector bundle of $A(\mathcal{G})$ with the zero section removed. We expect this spectral sequence to degenerate at E^2 and to be convergent to the Hochschild homology of $\mathcal{A}(M)$. The quantization of $A^*(\mathcal{G})$ with this Poisson structure was studied in [**?**].

For certain algebras associated to manifolds with corners, we identify the homogeneous Poisson homology of $A^*(\mathcal{G}) \smallsetminus 0$ in terms of a space $\mathcal{L}(S^*M)$ functorially associated to the base M. Moreover, the particular form of the resulting spectral sequence guarantees its convergence. This leads to an identification of the Hochschild homology of the Laurent complete symbols algebras $\mathcal{A}_{\mathcal{L}}(M)$.

Theorem 2. *Let $\mathcal{O}(M)$ be the ring of functions with only rational (i.e. Laurent-type) singularities at the faces of M. Assume that $\mathcal{O}(M)\Gamma(A(\mathcal{G})) \cong \mathcal{O}(M)\Gamma(TM)$ via the anchor map, then with $n = \dim(M)$*

$$\mathrm{HH}_q(\mathcal{A}_{\mathcal{L}}(M)) \cong \mathrm{H}_c^{2n-q}(\mathcal{L}(S^*M) \times S^1). \tag{2}$$

A similar result holds true in the relative case of symbols vanishing to infinite order at some subset of M, thus extending results of [20] from the case of manifolds with boundary to that of manifolds with corners.

In [19], the norm closure of the algebra of pseudodifferential operators on a manifold with corners was studied from the point of view of K-theory. However, the K-theory is sometimes too rough to identify more subtle invariants – like the η-invariant of Atiyah, Patodi, and Singer [2] – that are not homotopy invariant. This was partly remedied in [19], where the Hochschild 1-cocycle that gives the index was identified in terms of residues. This cocycle was then split into two parts that are direct analogues of the Atiyah-Singer integrand and, respectively, the η-invariant.

We now describe briefly the contents of each section. In Section 1, we introduce the class of algebras we shall work with, that is, the class of "topologically filtered algebras," (Definition 1), a class of algebras for which the multiplication is not jointly continuous, but which still has a weak continuity property for multiplication. Because of this, the usual definitions of the Hochschild and cyclic complex of a topologically filtered algebra have to be adapted to our more general framework. Namely, we have to use iterated inductive and projective limits. Then we establish some results on the spectral sequences associated to the natural filtrations of the resulting complexes. In Section 2, we establish some technical results on de Rham complexes with singularities for manifolds with corners, in the spirit of [20]. In Section 3 we identify the periodic cyclic cohomology of the algebra of complete symbols and relate the Hochschild homology of those algebras with the (homogeneous) Poisson cohomology of $A^*(\mathcal{G}) \setminus 0$. In Section 4, we compute the Hochschild homology of the algebra of complete symbols when $A(\mathcal{G})$ is rationally isomorphic to TM. For manifolds without corners (or boundary), these results are due to Wodzicki and Brylinski-Getzler [7]. Other related results were obtained by Lauter-Moroianu [13] and Moroianu [15]. We then use these results in Section 5 to study residues and to determine the cyclic homology of the algebra of complete symbols, still assuming that $A(\mathcal{G})$ is rationally isomorphic to TM. The appendix contains a short review of projective and inductive limits.

Acknowledgements. We would like to thank Alain Connes, Thierry Fack, Robert Lauter, Sergiu Moroianu, and Serge Parmentier for useful discussions.

1. Hochschild and cyclic homology of filtered algebras

We begin by recalling the definitions of Hochschild and cyclic homology groups of a topological algebra $\mathcal{A}$. A good reference is Connes' book [9]. These definitions have to be (slightly) modified when the multiplication of our algebra is only separately continuous. We thus discuss also the changes necessary to handle the class of algebras we are interested in, that of "topologically filtered algebras" (Definition 1), and then we prove some results on the homology of these algebras.

First we consider the case of a topological algebra $\mathcal{A}$. Here "topological algebra" has the usual meaning, $\mathcal{A}$ is a real or complex algebra, which is at the same time a locally convex space such that the multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is continuous. Denote by $\widehat{\otimes}$ the projective tensor product and $\mathcal{H}_n(\mathcal{A}) := \mathcal{A}^{\widehat{\otimes} n+1}$, the completion of $\mathcal{A}^{\otimes n+1}$ in the topology of the projective tensor product. Also, we denote as usual by b' and b the Hochschild differentials:

$$
\begin{aligned}
b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \\
b(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n) + (-1)^n a_n a_0 \otimes \ldots \otimes a_{n-1}.
\end{aligned} \tag{3}
$$

The *Hochschild homology groups* of the topological algebra $\mathcal{A}$, denoted $\mathrm{HH}_*(\mathcal{A})$, are then the homology groups of the complex $(\mathcal{H}_n(\mathcal{A}), b)$. By contrast, the complex

$(\mathcal{H}_n(\mathcal{A}), b')$ is often acyclic, for example when $\mathcal{A}$ has a unit. A topological algebra $\mathcal{A}$ for which $(\mathcal{H}_n(\mathcal{A}), b')$ is acyclic is called *H-unital* (or, better, *topological H-unital*), following Wodzicki [25].

We now define cyclic homology. We shall use the notation of [8]:

$$\begin{aligned} s(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= 1 \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_n, \\ t(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}, \\ B_0(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= s \sum_{k=0}^{n} t^k (a_0 \otimes a_1 \otimes \ldots \otimes a_n), \quad \text{and} \quad B = B_0(1-t). \end{aligned} \tag{4}$$

Then $[b, B]_+ := bB + Bb = B^2 = b^2 = 0$, and hence, if we define

$$\mathcal{C}(\mathcal{A})_n = \bigoplus_{k \geq 0} \mathcal{H}_{n-2k}(\mathcal{A}), \tag{5}$$

$(\mathcal{C}(\mathcal{A}), b + B)$, is a complex, called *the cyclic complex* of $\mathcal{A}$, whose homology is by definition the *cyclic homology* of $\mathcal{A}$, as introduced in [8] and [24].

Consideration of the natural periodicity morphism $\mathcal{C}_n(\mathcal{A}) \to \mathcal{C}_{n-2}(\mathcal{A})$ easily shows that cyclic and Hochschild homology are related by a long exact sequence

$$\ldots \to \mathrm{HH}_n(\mathcal{A}) \xrightarrow{I} \mathrm{HC}_n(\mathcal{A}) \xrightarrow{S} \mathrm{HC}_{n-2}(\mathcal{A}) \xrightarrow{B} \mathrm{HH}_{n-1}(\mathcal{A}) \xrightarrow{I} \ldots, \tag{6}$$

with the maps I, B, and S explicitly determined. The map S is also called the *periodicity* operator. See [8, 16]. This exact sequence exists whether or not $\mathcal{A}$ is endowed with a topology.

For the algebras that we are interested in, however, the multiplication is usually only separately continuous, but there will exist an increasing multi-filtration $F^m_{n,l} \subset \mathcal{A}$ of $\mathcal{A}$,

$$F^m_{n,l}\mathcal{A} \subset F^{m'}_{n',l'}\mathcal{A}, \quad \text{if } n \leq n',\ l \leq l', \text{ and } m \leq m',$$

by closed subspaces satisfying the following properties:

1. $\mathcal{A} = \cup_{n,l,m} F^m_{n,l}\mathcal{A}$;
2. The union $\mathcal{A}_n := \cup_{m,l} F^m_{n,l}\mathcal{A}$ is a closed subspace such that
$$F^m_{n,l}\mathcal{A} = \mathcal{A}_n \cap \big(\cup_n F^m_{n,l}\mathcal{A} \big);$$
3. Multiplication maps $F^m_{n,l}\mathcal{A} \otimes F^m_{n',l'}\mathcal{A}$ to $F^m_{n+n',l+l'}\mathcal{A}$;
4. The maps $F^m_{n,l}\mathcal{A}/F^m_{n-1,l}\mathcal{A} \otimes F^m_{n',l'}\mathcal{A}/F^m_{n'-1,l'}\mathcal{A} \to F^m_{n+n',l+l'}\mathcal{A}/F^m_{n+n'-1,l+l'}\mathcal{A}$ induced by multiplication are continuous;
5. The quotient $F^m_{n,l}\mathcal{A}/F^m_{n-1,l}\mathcal{A}$ is a nuclear Frechet space in the induced topology;
6. The natural map
$$F^m_{n,l}\mathcal{A} \to \lim_{\leftarrow} F^m_{n,l}/F^m_{n-j,l}\mathcal{A}, \quad j \to \infty$$
is a homeomorphism; and
7. The topology on $\mathcal{A}$ is the strict inductive limit of the subspaces $F^n_{n,n}\mathcal{A}$, as $n \to \infty$ (recall that $F^n_{n,n}\mathcal{A}$ is closed in $F^{n+1}_{n+1,n+1}\mathcal{A}$).

Definition 1. An algebra $\mathcal{A}$ satisfying the conditions 1–7, above, will be called a *topologically filtered* algebra.

It follows from the definition that if $\mathcal{A}^m := \cup_{n,l} F^m_{n,l}\mathcal{A}$, then $\mathcal{A}^m$ is actually a subalgebra of $\mathcal{A}$ which is topologically filtered in its own, but with multi-filtration independent of m.

For topologically filtered algebras, the multiplication is not necessarily continuous, and the definition of the Hochschild and cyclic homologies using the projective tensor product of the algebra $\mathcal{A}$ with itself, as above, does not make much sense. For this reason, we change the definition of the space $\mathcal{H}_m(\mathcal{A})$ to be an inductive limit:

$$\mathcal{H}_q(\mathcal{A}) \cong \lim_{\rightarrow} (F^n_{n,n}\mathcal{A})^{\widehat{\otimes} q+1},$$

the tensor product being the (complete) projective tensor product. The Hochschild homology of $\mathcal{A}$ is then still the homology of the complex $(\mathcal{H}(\mathcal{A}), b)$. Since the projective tensor product is compatible with the projective limits, we also have

$$\mathcal{H}_q(\mathcal{A}) = \lim_{\rightarrow} \left(\lim_{\leftarrow} (F^n_{n,n}\mathcal{A}/F^n_{k,n}\mathcal{A})^{\widehat{\otimes} q+1} \right), \tag{7}$$

with the induced topology, where first $k \to -\infty$ (in the projective limit) and then $n \to \infty$ (in the inductive limit). The operator B extends to a well defined map $B : \mathcal{H}_q(\mathcal{A}) \to \mathcal{H}_{q+1}(\mathcal{A})$, which allows us to define the cyclic complex and the cyclic homology of the algebra $\mathcal{A}$ as the homology of the complex $(\mathcal{C}_*(\mathcal{A}), b + B)$, with $\mathcal{C}_q(\mathcal{A}) := \oplus \mathcal{H}_{q-2k}(\mathcal{A})$, as for topological algebras.

We also observe that both the Hochschild and cyclic complexes have natural filtrations given by

$$F_p\mathcal{H}_q(\mathcal{A}) := \lim_{\rightarrow} \left(\lim_{\leftarrow} \widehat{\otimes}^q_{j=0} \left(F^m_{k_j,m}\mathcal{A}/F^m_{k,m}\mathcal{A} \right) \right), \tag{8}$$

where $k_0 + \ldots + k_n \leq p$ defines the filtration. The projective and inductive limits are such that first $k, l \to -\infty$ and then $m \to \infty$.

For any topologically filtered algebra, we denote

$$Gr(\mathcal{A}) := \oplus_n \mathcal{A}_n/\mathcal{A}_{n-1}$$

the *graded algebra* associated to $\mathcal{A}$ (recall that $\mathcal{A}_n$ was defined as $\cup_{l,m} F^m_{n,l}\mathcal{A}$). Its topology is that of an inductive limit of Frechet spaces:

$$Gr(\mathcal{A}) \cong \lim_{N,m,l\to\infty} \oplus^N_{n=-N} F^m_{n,l}\mathcal{A}/F^m_{n-1,l}\mathcal{A}.$$

For the algebras like $Gr(\mathcal{A})$, we need yet a third way of topologizing its iterated tensor products. The correct definition is

$$\mathcal{H}_q(Gr(\mathcal{A})) \cong \lim_{N,m,l\to\infty} (\oplus^N_{n=-N} F^m_{n,l}\mathcal{A}/F^m_{n-1,l}\mathcal{A})^{\widehat{\otimes} q+1}.$$

The Hochschild homology of $Gr(\mathcal{A})$ is the homology of the complex $(\mathcal{H}_*(Gr(\mathcal{A})),b)$. The operator B again extends to a map $B : \mathcal{H}_q(Gr(\mathcal{A})) \to \mathcal{H}_{q+1}(Gr(\mathcal{A}))$ and we can define the cyclic homology of $Gr(\mathcal{A})$ as above.

The Hochschild and cyclic complexes of the algebra $Gr(\mathcal{A})$ decompose naturally as direct sums of complexes indexed by $p \in \mathbb{Z}$. For example, $\mathcal{H}_q(Gr(\mathcal{A}))$ is the direct sum of the subspaces $\mathcal{H}_q(Gr(\mathcal{A}))_p$, where

$$\mathcal{H}_q(Gr(\mathcal{A}))_p = \lim_{m,N,l\to\infty} \bigoplus_{k_j} \left(\widehat{\otimes}^n_{j=0} F^m_{k_j,l}\mathcal{A}/F^m_{k_j-1,l}\mathcal{A} \right),$$

$$\text{where } \ k_0 + k_1 + \ldots + k_n = p \ \text{ and } \ -N \leq k_j \leq N,$$

with the induced topology. The corresponding subcomplexes of the cyclic complex are defined similarly. We denote by $\mathrm{HH}_*(Gr(\mathcal{A}))_p$ and $\mathrm{HC}_*(Gr(\mathcal{A}))_p$ the homologies of the corresponding complexes.

Lemma 1. *Let $\mathcal{A}$ be a topologically filtered algebra. Then the natural filtrations on the Hochschild and cyclic complexes of $\mathcal{A}$ define spectral sequences $\mathrm{EH}^r_{k,h}$ and $\mathrm{EC}^r_{k,h}$ such that*

$$\mathrm{EH}^1_{k,h} \simeq \mathrm{HH}_{k+h}(Gr(\mathcal{A}))_k \quad \text{and} \quad \mathrm{EC}^1_{k,h} \simeq \mathrm{HC}_{k+h}(Gr(\mathcal{A}))_k.$$

Moreover, the periodicity morphism S induces a morphism of spectral sequences $S : \mathrm{EC}^r_{k,h} \to \mathrm{EC}^r_{k,h-2}$, which for $r = 1$ is the graded map associated to the periodicity operator $S : \mathrm{HC}_n(Gr(\mathcal{A})) \to \mathrm{HC}_{n-2}(Gr(\mathcal{A}))$.

Proof. We shall write $F_p = F_p\mathcal{H}(\mathcal{A})$, for simplicity, where the filtration is as defined in Equation (8).

The filtration of the complex computing the Hochschild homology of $\mathcal{A}$ then gives rise to a spectral sequence with $E^1_{k,h} = H_{k+h}(F_k/F_{k-1})$, by standard homological algebra. By the definition of the Hochschild complex of $Gr(\mathcal{A})$,

$$\mathrm{H}_{k+h}(F_k/F_{k-1}) \cong \mathrm{HH}_{k+h}(Gr(\mathcal{A}))_k.$$

This completes the proof for Hochschild homology. The proof for cyclic homology is similar □

Lemma 2. *Let N and M be integers. If $\mathcal{A}$ is a topologically filtered algebra with $F^m_{n,l}\mathcal{A}$ independent of m and $\mathrm{EH}^1_{k,h} = 0$ for all pairs (k, h) such that $k < N$ and $k + h \geq M$, then*

$$\mathrm{HH}_q(\mathcal{A}) \cong \oplus_{k=N}^{\infty} \mathrm{EH}^{\infty}_{k,q-k},$$

if $q \geq M$. A similar result is true for the cyclic homology spectral sequence.

The above isomorphism is not natural, in general, but comes from a filtration of $\mathrm{HH}_q(\mathcal{A})$ whose subquotients identify naturally with $\mathrm{EH}^{\infty}_{k,q-k}$, see [17, 18]. What the above lemma says, put differently, is that the spectral sequence $\mathrm{EH}^r_{k,h}$ converges to $\mathrm{HH}_{k+h}(\mathcal{A})$, for $k + h \geq M$.

Proof. We shall use

$$\mathcal{H}_q(\mathcal{A}) = \lim_{\rightarrow}\big(\lim_{\leftarrow} F_p\mathcal{H}_q(\mathcal{A})/F_{p'}\mathcal{H}_q(\mathcal{A})\big).$$

Denote $F_r = F_r\mathcal{H}(\mathcal{A})$ for simplicity. The projective limits give rise to a $\lim^1$ exact sequence

$$0 \longrightarrow \lim_{\leftarrow}{}^1 \mathrm{H}_{m+1}(F_p/F_{p'}) \longrightarrow \mathrm{H}_m(F_p) \longrightarrow \lim_{\leftarrow} \mathrm{H}_m(F_p/F_{p'}) \longrightarrow 0$$

for every fixed p (see Lemma 6 from the Appendix). From the assumption that $\mathrm{H}_q(F_{p'}/F_{p'-1}) = 0$ for $p' < N$ and $q \geq M$, we know that $\mathrm{H}_q(F_p/F_{p'})$ becomes stationary for $p' < N$ and $q \geq M$. This shows that the $\lim^1$ term above vanishes for $q \geq M$, and hence

$$\mathrm{H}_q(F_p) \simeq \lim_{\leftarrow} \mathrm{H}_q(F_p/F_{p'}) = \mathrm{H}_q(F_p/F_{N-1}),$$

if $q \geq M$. It also gives that the natural morphism $\mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{A})/F_{N-1}$ induces an isomorphism of the $E^r_{k,h}$-terms of the corresponding spectral sequences, for $r \geq 1$ and $k + h \geq M$.

Using then the fact that homology and *inductive* limits commute we obtain

$$\mathrm{HH}_q(\mathcal{A}) = \mathrm{H}_q(\lim_{p\to\infty} \lim_{\leftarrow} F_p/F_{p'}) \cong \lim_{p\to\infty} \mathrm{H}_q(\lim_{\leftarrow} F_p/F_{p'})$$
$$\cong \lim_{p\to\infty} \mathrm{H}_q(F_p/F_{N-1}) \cong \mathrm{H}_q(\mathcal{H}(\mathcal{A})/F_{N-1}) \cong \oplus_{l=N}^{\infty} E_{k,q-k}^{\infty},$$

$p' \to -\infty$, where for the last isomorphism we have used that the spectral sequence associated to $\mathcal{H}(\mathcal{A})/F_{N-1}$ is convergent because $\mathcal{H}(\mathcal{A})/F_{N-1} = \cup_q F_q/F_{N-1}$. □

Theorem 3. *Fix an integer N and M. If $\mathcal{A}$ is a topologically filtered algebra such that each of the algebras $\mathcal{A}^m := \cup_{n,l} F_{n,l}^m \mathcal{A}$ satisfies the assumptions of Lemma 2 for the given N and M, then*

$$\mathrm{HH}_q(\mathcal{A}) \cong \oplus_{k=N}^{\infty} \mathrm{EH}_{k,q-k}^{\infty}, \quad q \geq M.$$

A similar result is true for the cyclic homology spectral sequence.

Proof. Each of the algebras $\mathcal{A}^m = \cup_{n,l} \mathcal{A}^m$ is a topologically filtered algebra in its own if we let $F_{n',l'}^{m'} \mathcal{A}^m = F_{n',l'}^m \mathcal{A}$ (so the filtration depends only on n and l).

For the algebras $\mathcal{A}^m$, the Hochschild complex is defined as for any topologically filtered algebra, except that there is no need to take an additional direct limit with respect to m.

The Hochschild complex of $\mathcal{A}$ is then given by

$$\mathcal{H}_q(\mathcal{A}) = \lim_{\to} \mathcal{H}_q(\mathcal{A}^m), \quad m \to \infty$$

Because taking homology is compatible with inductive limits, we obtain that

$$\mathrm{HH}_q(\mathcal{A}) = \lim_{\to} \mathrm{HH}_q(\mathcal{A}^m), \quad m \to \infty.$$

Denote by $\mathrm{EH}_{k,h}^r(\mathcal{A}^m)$ the spectral sequence associated by Lemma 1 to the topologically filtered algebra $\mathcal{A}^m$. Again because homology is compatible with inductive limits, $\mathrm{EH}_{k,h}^r(\mathcal{A}) \cong \lim_{\to} \mathrm{EH}_{k,h}^r(\mathcal{A}^m)$. The result then follows from Lemma 2. □

Lemma 3. *Fix an integer N and $a \geq 1$. If $\mathcal{A}$ is a topologically filtered algebra with $F_{n,l}^m \mathcal{A}$ independent of m and $\mathrm{EH}_{k,h}^a = 0$, for all $k < N$, then the spectral sequence $\mathrm{EH}_{k,h}^r$ converges to $\mathrm{HH}_{k+h}(\mathcal{A})$. More precisely,*

$$\mathrm{HH}_q(\mathcal{A}) \cong \oplus_{k=N}^{\infty} \mathrm{EH}_{k,q-k}^{\infty}.$$

A similar result is true for the cyclic homology spectral sequence.

Proof. The assumption that $F_{n,l}^m \mathcal{A}$ is independent of m has as a consequence that we need not take inductive limits with respect to m in the definition of the Hochschild complex of $\mathcal{A}$.

Denote $F_p = F_p \mathcal{H}(\mathcal{A})$ for simplicity. The Hochschild complex of $\mathcal{A}$ is complete, in the sense that

$$\mathcal{H}(\mathcal{A}) = \lim_{\to} \big(\lim_{\leftarrow} F_p \mathcal{H}(\mathcal{A})/F_{p'} \mathcal{H}(\mathcal{A})\big), \quad p' \to -\infty \text{ and } p \to \infty.$$

The projective limits give rise to a $\lim^1$ exact sequence

$$0 \to \lim_{\leftarrow}{}^1 \mathrm{H}_{q+1}(\mathcal{H}(\mathcal{A})/F_p) \to \mathrm{H}_q(\mathcal{H}(\mathcal{A})) \to \lim_{\leftarrow} \mathrm{H}_q(\mathcal{H}(\mathcal{A})/F_p) \to 0,$$

as $p \to -\infty$, for every fixed q (see Lemma 7).

The spectral sequence $E^r_{k,h}(p)$ associated to the complex $\mathcal{H}(\mathcal{A})/F_p$ is convergent because $\mathcal{H}(\mathcal{A})/F_p = \cup_j F_j/F_p$. Consequently, the homology groups of $\mathcal{H}(\mathcal{A})/F_p$ are endowed with a filtration $F_t\,\mathrm{H}_q(\mathcal{H}(\mathcal{A})/F_p)$ such that

$$F_t\,\mathrm{H}_q(\mathcal{H}(\mathcal{A})/F_p)/F_{t-1}\,\mathrm{H}_q(\mathcal{H}(\mathcal{A})/F_p) \cong E^\infty_{t,q-t}(p) \tag{9}$$

Assume now that $p < N$, the assumption that $E^a_{k,h} = 0$ for all $k < N$ gives that $E^r_{k,h}(p) = E^r_{k,h}$ for all $k < p < N$ or $p + a < k$. Let $A_n = \mathrm{H}_q(\mathcal{H}(\mathcal{A})/F_{N-na})$, $B_n = F_{N-na+a}\,\mathrm{H}_q(\mathcal{H}(\mathcal{A})/F_{N-na})$, and $C_n = A_n/B_n$, $n \geq 2$. Then equation (9) gives that the natural map $A_{n+1} \to A_n$ gives an isomorphism $C_{n+1} \cong C_n$ and induces the zero map $B_{n+1} \to B_n$. Using Lemma 6 from the Appendix, we obtain that $\lim\limits_{\leftarrow}{}^1 A_n = 0$ and $\lim\limits_{\leftarrow} A_n = C_{n_0}$ for any fixed n_0.

Because $C_{n_0} = \oplus_{l=N}^\infty E^\infty_{l,q-l}$, the result follows. □

Theorem 4. *Fix an integer N and $a \geq 1$. If $\mathcal{A}$ is a topologically filtered algebra such that each of the algebras $\mathcal{A}^m := \cup_{n,l} F^m_{n,l}\mathcal{A}$ satisfies the assumptions of Lemma 3 for the given N and a, then the spectral sequence $\mathrm{EH}^r_{k,h}$ converges to $\mathrm{HH}_{k+h}(\mathcal{A})$. More precisely, we have*

$$\mathrm{HH}_q(\mathcal{A}) \cong \oplus_{h=N}^\infty \mathrm{EH}^\infty_{k,q-k}\,.$$

A similar result is true for the cyclic homology spectral sequence.

Proof. Each of the algebras $\mathcal{A}^m$ is a topologically filtered algebra in its own, if we let $F^{m'}_{n',l'}\mathcal{A}^m = F^m_{n',l'}\mathcal{A}$ (so the filtration really depends only on n and l). We shall write $F_{n,l}\mathcal{A}^m$ instead of $F^{m'}_{n,l}\mathcal{A}^m$, and hence $F^m_{n,l}\mathcal{A} = F_{n,l}\mathcal{A}^m$.

The Hochschild complex of $\mathcal{A}$ is then given by

$$\mathcal{H}_q(\mathcal{A}) = \lim_{\rightarrow} \mathcal{H}_q(\mathcal{A}^m), \quad m \to \infty$$

Because taking homology is compatible with inductive limits, we obtain that

$$\mathrm{HH}_q(\mathcal{A}) = \lim_{\rightarrow} \mathrm{HH}_q(\mathcal{A}^m), \quad m \to \infty.$$

Denote by $\mathrm{EH}^r_{k,h}(\mathcal{A}^m)$ the spectral sequence associated to the natural filtration of the Hochschild complex the algebra $\mathcal{A}^m$. Because homology is compatible with inductive limits, $\mathrm{EH}^r_{k,h}(\mathcal{A}) \cong \lim\limits_{\rightarrow} \mathrm{EH}^r_{k,h}(\mathcal{A}^m)$. The result then follows from Lemma 3. □

We conclude this section with a result that is in the same spirit with the above results and was independently observed by Sergiu Moroianu in a different setting.

Proposition 1. *If the graded algebra $Gr(\mathcal{A})$ of the topologically filtered algebra $\mathcal{A}$ is H-unital, then $\mathcal{A}$ is H-unital, in the sense that the complex $(\mathcal{H}_n(\mathcal{A}), b')$ is acyclic.*

Proof. This is completely analogous to the previous results, so we will be sketchy. The natural filtration on the complex $(\mathcal{H}_n(\mathcal{A}), b')$ induces a spectral sequence whose E^1 term is the b'-homology of $Gr(\mathcal{A})$. This spectral sequence is proved to be convergent as in any of the above two theorems. □

2. A rational Laurent de Rham complex

We now recall some notation and obtain some preliminary results on de Rham type complexes used in our computations. Examples of topologically filtered algebras will be discussed in the following section.

Consider a σ–compact manifold with corners M. Let $\mathcal{O}(M)$ be the ring of functions on M such that $f \in \mathcal{O}(M)$ if, and only if, on every open subset of M diffeomorphic to $[0,1)^k \times \mathbb{R}^{n-k}$, the function f is of the form $x_1^{-p_1} \ldots x_k^{-p_k} g$, with g a smooth function on M. (So $\mathcal{O}(M)$ is a quotient ring.) By abuse of terminology, we shall call $\mathcal{O}(M)$ the ring of Laurent rational functions on M. Also, we shall call $x_1, \ldots, x_k$ the *local* defining functions of the hyperfaces of M at a point x belonging to a corner of codimension k.

If there exists a boundary defining function $\rho_H \in \mathcal{C}^\infty(M)$ for each hyperface H (that is, $H = \{\rho_H = 0\}$, $\rho_H \geq 0$ on M, and $d\rho_H \neq 0$ on H), then

$$\mathcal{O}(M) = \rho^{-\mathbb{Z}} \mathcal{C}^\infty(M) = \bigcup_{j \in \mathbb{Z}} x^j \mathcal{C}^\infty(M),$$

where $\rho^{-\mathbb{Z}}$ stands for the multiplicative set consisting of all products $\rho_{H_1}^{n_1} \rho_{H_2}^{n_2} \cdots \rho_{H_m}^{n_m}$, $n_k \in \mathbb{Z}$. We then say that M has *embedded faces.* A nice discussion of how to avoid the assumption of M having embedded faces is due to Monthubert (see [**?**, **?**] and the references within). It is essential then to use groupoids.

For each k, let $\Omega^k_{c,\mathcal{L}}(M) := \mathcal{O}(M) \otimes_{\mathcal{C}^\infty(M)} \Omega^k_c(M)$ be the space of k-forms with compact support on M with only Laurent singularities at the faces. Recall then that *the Laurent–de Rham cohomology of* M is the cohomology of the Laurent–de Rham complex.

$$\cdots \longrightarrow \Omega^k_{c,\mathcal{L}}(M) \longrightarrow \Omega^{k+1}_{c,\mathcal{L}}(M) \longrightarrow \cdots \tag{10}$$

with respect to the de Rham differential. We denote by $\mathrm{H}^*_{\mathcal{L}}(M)$ these cohomology groups. (These groups were denoted $\mathrm{H}^*_b(M)$ in [20], if M is a manifold with boundary.)

We shall use also the complex

$$\cdots \longrightarrow \Omega^k_{c,\mathcal{L}}(M)_x \longrightarrow \Omega^{k+1}_{c,\mathcal{L}}(M)_x \longrightarrow \cdots . \tag{11}$$

of germs of the Laurent-de Rham complex at a point $x \in M$.

Lemma 4. *The cohomology of the complex of stalks of the Laurent-de Rham complex at the point $x \in M$ belonging to a corner of codimension k is the exterior algebra generated by $d\log x_i$, where x_i, $i = 1, \ldots, k$, are locally defining functions of the hyperfaces containing x.*

Proof. This statement is local because we are dealing with germs. So we can assume that $x = (0, 0, \ldots, 0) \in \mathbb{R}^n$ and $M = [0,\infty)^k \times \mathbb{R}^{n-k} \subset \mathbb{R}^n$. The complex whose cohomology we have to compute is then the projective tensor product of the corresponding complexes for $[0,\infty)$ and $\mathbb{R}$ an appropriate number of times. By the Poincaré Lemma, the complex corresponding to $\mathbb{R}$ has cohomology only in dimension 0. The cohomology of germs at 0 of Laurent forms on $[0,\infty)$ is seen to be generated by 1 in dimension 0 and by $d\log t$ in dimension 1 ($t > 0$). The cohomology of a tensor product of these complexes is isomorphic to the tensor product of their cohomologies as graded vector spaces (by the topological Künneth theorem). This proves the theorem. $\square$

We shall denote by $\mathcal{M}_j(M)$ the set of *codimension* j faces of M.

Theorem 5. *Let M be a manifold with corners all of whose hyperfaces H are embedded submanifolds with defining function ρ_H. Then the Laurent-de Rham cohomology spaces of M can be naturally decomposed in terms of the cohomology of its faces as*

$$\mathrm{H}^k_{\mathcal{L}}(M) = \bigoplus_{j=0}^{k} \bigoplus_{F \in \mathcal{M}_j(M)} \mathrm{H}^{k-j}(F). \tag{12}$$

Proof. Consider for each face F of M the usual de Rham complex

$$\cdots \longrightarrow \Omega^k(F) \xrightarrow{d} \Omega^{k+1}(F) \longrightarrow \cdots \tag{13}$$

whose homology is, by (a variant of) de Rham's theorem, the absolute cohomology of F. Denote by $H_1, H_2, \dots, H_m$ the hyperfaces containing F and by ρ_i their defining functions. Fix a local product structure in a neighborhood of the face F and choose a smooth cutoff function ϕ with support in that neighborhood and equal to 1 in a smaller neighborhood of F. The map

$$\Phi_F : \mathcal{C}^\infty(F; \Lambda^k) \ni \alpha \longrightarrow \alpha \wedge d(\phi \log \rho_1) \wedge \cdots \wedge d(\phi \log \rho_m) \in \rho^{-\mathbb{Z}} \mathcal{C}^\infty(M; \Lambda^{k+m}) \tag{14}$$

where the local product decomposition near F is used to lift α to a smooth form on M, is a chain map.

The de Rham complex is a resolution of the constant sheaf on a manifold with corners (no factors of x_i^{-1} are allowed). From this and the previous lemma we obtain that the cochain map $\oplus \Phi_F$, where the sum is take over all faces of M, gives an isomorphism in cohomology. This proves the proposition. $\square$

The cohomology of the above complex can be described as the cohomology of a space $\mathcal{L}(M)$ naturally associated to M and defined as follows. Consider for each face F of M the space $F \times (S^1)^k$, where k is the codimension of the face. Moreover we establish a one-to-one correspondence between the k copies of the unit circle and the faces F' of M containing F, of dimension one higher than that of F. We then identify the points of the disjoint union $\cup F \times (S^1)^k$ as follows. If $F \subset F'$ and F' corresponds to the variable $\theta_i \in S^1$ we identify $(x, \theta_1, \dots, \theta_{i-1}, 1, \theta_{i+1}, \dots, \theta_k) \in F \times (S^1)^k$ to the point $(x, \theta_1, \dots, \theta_{i-1}, \theta_{i+1}, \dots, \theta_k) \in F' \times (S^1)^{k-1}$ (same x). The resulting quotient space is by definition $\mathcal{L}(M)$. By construction there exists a continuous map $p_{\mathcal{L}} : \mathcal{L}(M) \to M$. Let $J = S^1 \cup [1, 1+\epsilon)$, for some $\epsilon > 0$, with S^1 identified with a subset of the complex plane. Then the space $\mathcal{L}(M)$ is locally modelled by $J_\epsilon^k \times \mathbb{R}^{n-k}$, above each point of M belonging to an open face of codimension k. Using the space $\mathcal{L}(M)$, one can describe also the homology of other complexes associated to M.

Let $X \subset M$ be a closed subset consisting of a closed union of faces of M. Denote by $\Omega^k_c(M, X)$ the space of compactly supported smooth forms on M that vanish to infinite order on X. We then obtain complexes

$$\cdots \longrightarrow \Omega^k_c(M, X) \xrightarrow{d} \Omega^{k+1}_c(M, X) \longrightarrow \cdots \tag{15}$$

and

$$\cdots \longrightarrow \Omega^k_{c,\mathcal{L}}(M, X) \xrightarrow{d} \Omega^{k+1}_{c,\mathcal{L}}(M, X) \longrightarrow \cdots, \tag{16}$$

where $\Omega^k_{c,\mathcal{L}}(M,X) := \mathcal{O}(M)\otimes_{\mathcal{C}^\infty(M)}\Omega^k_c(M,X)$, using the same convention as above. We denote by $\mathrm{H}^*_{\mathcal{L}}(M,X)$ the cohomology of this complex. When $X=\emptyset$, this recovers the old definitions.

Proposition 2. *The homology of the complex* (15) *is the relative (de Rham) cohomology group* $\mathrm{H}^*_c(M,X)$ *and the cohomology of the complex* (16) *is*

$$\mathrm{H}^*_{\mathcal{L}}(M,X) \cong \mathrm{H}^*_c(\mathcal{L}(M), p_{\mathcal{L}}^{-1}(X)) = \mathrm{H}^*_c(\mathcal{L}(M)\smallsetminus p_{\mathcal{L}}^{-1}(X)).$$

Moreover, the morphism on cohomology induced by the inclusion of the first complex into the second complex identifies, up to isomorphism, with the morphism $p^*_{\mathcal{L}}$.

Proof. That the cohomology of the first complex is isomorphic to $\mathrm{H}^*_c(M,X) := \mathrm{H}^*_c(M\smallsetminus X)$ is of course well known. The second isomorphism follows from the Serre-Leray spectral sequence applied to the map $p_{\mathcal{L}} : \mathcal{L}(M)\to M$ and the above proposition. □

In particular, the above theorem gives formula $\mathrm{H}^*_{\mathcal{L}}(M) \cong H^*_c(\mathcal{L}(M))$ for the Laurent–de Rham cohomology of the manifold with corners M (assumed to be σ-compact).

3. Homology of complete symbols

Let $\mathcal{G}$ be a differentiable groupoid with space of units denoted M, a manifold with corners, in general. We obtain in this section some results on the homologies of algebras of complete symbols on $\mathcal{G}$. In particular, we get a complete determination of the periodic cyclic cohomology of the algebras of complete symbols $\mathcal{A}(M,X)$, $\mathcal{A}_0(M,X)$, and $\mathcal{A}_{\mathcal{L}}(M,X)$ defined below.

Let X be a locally finite union of closed faces of the manifold with corners M. Also, let $\mathcal{I}_X\subset\mathcal{C}^\infty(M)$ be the subset of functions that vanish to infinite order on X. Then we define $\mathcal{A}(M,X) := \mathcal{I}_X(\Psi^\infty(\mathcal{G})/\Psi^{-\infty}(\mathcal{G}))$ the algebra of complete symbols on $\mathcal{G}$, supported above a compact set of M and vanishing to infinite order at X. Similarly, we let

$$\mathcal{A}_0(M,X) := \mathcal{I}_X(\Psi^0(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})),$$
$$\mathcal{A}_{\mathcal{L}}(M,X) := \mathcal{O}(M)\mathcal{I}_X(\Psi^\infty(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})).$$

The algebra $\mathcal{A}_{\mathcal{L}}(M,X)$ will be sometimes referred to as the algebra of "Laurent complete symbols vanishing rapidly at X." Note that we have $\mathcal{A}_{\mathcal{L}}(M,X) = \mathcal{A}(M)\mathcal{O}(M)\mathcal{I}_X$, too. If $X=\emptyset$, we do not include it in the notation.

Let us recall some notation from [14], in this volume. We denote by $T_{vert}(\mathcal{G}) = \cup_{x\in M}T(\mathcal{G}_x)$ the vertical tangent bundle of $\mathcal{G}$ to the fibers of the domain map $d:\mathcal{G}\to M$. Then $A(\mathcal{G})$ can be identified with the restriction of $T_{vert}(\mathcal{G})$ to M, that is, $A(\mathcal{G}) = \cup_{x\in M}T_x(\mathcal{G}_x)$. In particular, the smooth sections of $A^*(\mathcal{G})$ are canonically identified with the right $\mathcal{G}$-invariant smooth sections of $T^*_{vert}(\mathcal{G})$.

Proposition 3. *Suppose M, the space of units of $\mathcal{G}$ is σ-compact. Then the algebras $\mathcal{A}(M,X)$, $\mathcal{A}_0(M,X)$, and $\mathcal{A}_{\mathcal{L}}(M,X)$ are topologically filtered algebras.*

Proof. We shall prove this lemma for $\mathcal{A}_{\mathcal{L}}(M,X)$, the other case being completely similar, and actually even simpler. The subspaces $F^m_{n,l}\mathcal{A}$ are defined in the following way. Fix an increasing, countable exhaustion $M=\cup K_m$ by compact submanifolds. For any vector bundle $E\to M$, we denote by $\mathcal{S}^n_{rc}(E)$ the space of classical symbols supported above some compact set in the base M. We then choose a quantization map $q : \mathcal{S}^n_{rc}(A^*(\mathcal{G})) \to \Psi^n(\mathcal{G})$ as in [23], whose main property is that it induces a bijection

$$\mathcal{S}^n_{rc}(A^*(\mathcal{G}))/\mathcal{S}^{n'}_{rc}(A^*(\mathcal{G})) \cong \Psi^n(\mathcal{G})/\Psi^{n'}(\mathcal{G}),$$

for all $n' < n$ (including $n = \infty$ or $n' = -\infty$). Denote by $\pi : A^*(\mathcal{G}) \to M$ the natural projection and by $\mathcal{S}^n_{K_m}(\mathcal{A}^*(\mathcal{G}))$ the set of symbols with support in $\pi^{-1}(K_m)$, then an increasing triple filtration for $\mathcal{A}$ is defined, for $l = 0$ first, by

$$F^m_{n,0}\mathcal{A} = \big(q(\mathcal{S}^n_{K_m}(A^*(\mathcal{G}))) + \Psi^{-\infty}_{K_m}(\mathcal{G})\big)/\Psi^{-\infty}_{K_m}(\mathcal{G}).$$

Choose now a radial completion of $A^*(\mathcal{G})$, which is then a diffeomorphism of $A^*(\mathcal{G})$ onto the interior of the ball bundle $B^*(\mathcal{G}) := \{\xi \in A^*(\mathcal{G}), \|\xi\| \leq 1\}$. This identifies $S^0_K(A^*(\mathcal{G}))$ with the subset $\mathcal{C}^\infty_K(B^*(\mathcal{G})) \subset \mathcal{C}^\infty(B^*(\mathcal{G}))$ of those smooth functions on $B^*(\mathcal{G})$ with support above the compact set $K \subset M$. We use this identification to define the topology on $\mathcal{S}^0_{K_m}(A^*(\mathcal{G}))$, which in turn gives $F^m_{0,0}\mathcal{A}$ the induced topology.

To define $F^m_{n,l}\mathcal{A}$ in general, let $\mathcal{S}^{n,l}_{K_m}(A^*(\mathcal{G}))$ be the space of Laurent symbols of order $\leq n$, with support in $\pi^{-1}(K_m)$, and only rational singularities with total order $\leq l$ in each defining function of a hyperface of M outside X (we do not count negative orders). Then we define $F^m_{n,l}\mathcal{A}$ as the image of $q(\mathcal{S}^{n,l}_{K_m}(A^*(\mathcal{G}))$, with the induced topology.

This definition is such that $F^m_{n,l}\mathcal{A} = F^m_{n,0}\mathcal{A}$, if $l < 0$. The topology on this space is defined similarly. In this way, $F^m_{n,l}\mathcal{A}$ becomes a closed subset of $F^{m'}_{n',l'}\mathcal{A}$ whenever $n \leq n'$, $l \leq l'$, and $m \leq m'$. Then we endow $\mathcal{A}$ with the strict inductive limit topology. Conditions 1–7 of a topologically filtered algebra are then satisfied. $\square$

Remark. When M is compact, there is no need to consider the additional filtration with respect to m. Also, for the algebras $\mathcal{A}_0(M,X)$, there is no need to consider the variable l, as the filtration is independent of l

We now apply some of the results of the previous sections to the algebras $\mathcal{A}(M,X)$ and $\mathcal{A}_{\mathcal{L}}(M,X)$.

In this section, we first concentrate on the cyclic homology, because this will lead to a complete determination of the periodic cyclic homology of these algebras. We concentrate first on the algebra $\mathcal{A}(M) = \Psi^\infty(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})$ of complete symbols on $\mathcal{G}$, *which we shall denote by $\mathcal{A}$ throughout the rest of this section*, for simplicity. The groupoid $\mathcal{G}$ is arbitrary but fixed in what follows.

Fix a metric on $A(\mathcal{G})$ and let P be a pseudodifferential operator of order one such that $\sigma_1(P) \equiv r$ (modulo lower order symbols), where $r \in \mathcal{C}^\infty(A^*(\mathcal{G}))$ is the distance function to the origin. We know that the graded algebra $Gr(\mathcal{A})$ associated to $\mathcal{A}$ is commutative. Denote by $S^*(\mathcal{G}) = S^*(A(\mathcal{G}))$ the set of vectors of length one in $A^*(\mathcal{G})$, the dual of the Lie algebroid of $\mathcal{G}$. Then $Gr(\mathcal{A}) \simeq \mathcal{C}^\infty_c(S^*(\mathcal{G})) \otimes \mathbb{C}[r, r^{-1}]$, with grading given by the powers of r.

As noted in Section 1, the tensor products appearing in the Hochschild complex are such that $F_k\mathcal{H}_n(\mathcal{A})/F_{k-1}\mathcal{H}_n(\mathcal{A})$ is a direct sum of spaces, each of which is isomorphic to $\mathcal{C}^\infty_c(S^*(\mathcal{G}) \times S^*(\mathcal{G}) \times \cdots \times S^*(\mathcal{G}))$, such that the natural map

$$F_k\mathcal{H}_n(\mathcal{A}) \to \lim_{\leftarrow} F_k\mathcal{H}_n(\mathcal{A})/F_{k-j}\mathcal{H}_n(\mathcal{A}), \quad j \to \infty,$$

is an isomorphism. The same comments are valid for the cyclic complex.

The Hochschild, cyclic, and periodic cyclic homologies of $Gr(\mathcal{A})$ are identified using a combination of the Hochschild-Kostand-Rosenberg (HKR) isomorphism and a result of Connes, which is the analog of the HKR-isomorphism for algebras of C^∞-functions. We denote by $\Omega^l_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_d$ the set of $l-$differential forms on the manifold $A^*(\mathcal{G}) \smallsetminus 0$ that are positively d-homogeneous in the radial direction and whose support projects onto a compact subset of M. (Here $A^*(\mathcal{G}) \smallsetminus 0$ stands for the dual of $A(\mathcal{G})$ with the zero section removed.) Then, using the grading of $Gr(\mathcal{A})$, we have

$$\mathrm{HH}_l(Gr(\mathcal{A}))_d \;\cong\; \Omega^l_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_d \;\cong\; \Omega^l_c(S^*(\mathcal{G}))r^d \oplus \Omega^{l-1}_c(S^*(\mathcal{G}))r^{d-1}dr,$$

the isomorphism being obtained via the Hochschild-Kostant-Rosenberg-Connes map

$$\chi(a_0, \dots, a_l) = (1/l!)\, a_0 da_1 \wedge da_2 \wedge \dots \wedge da_l.$$

(We shall often omit the "wedge" $\wedge$ in what follows.)

We can identify $\Omega^l_c(S^*(\mathcal{G}))r^d \oplus \Omega^{l-1}(S^*(\mathcal{G}))r^{d-1}dr$ with the subspace $\Omega^l_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_d \subset \Omega^l_{rc}(A^*(\mathcal{G}) \smallsetminus 0)$ consisting of d-homogeneous forms, as above. Also, we shall sometimes complexify r and restrict it to the unit circle, so that r becomes $e^{i\theta}$ and dr becomes $ie^{i\theta}d\theta$. Let $\Omega^*_{fin}(S^1)$ be the set of polynomial forms on S^1, that is, of finite linear combinations of r^l and $r^m dr$. The complex $\Omega^*_{fin}(S^1)$ has the same cohomology as the de Rham complex of all forms on S^1, which also gives the cohomology of S^1 (with complex coefficients).

Then, we see that the complex $(\oplus_m \Omega^*_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_m, d)$, can be identified with the complex $\Omega^*_c(S^*(\mathcal{G})) \otimes \Omega^*_{fin}(S^1)$, and hence it has the same cohomology as $S^*(\mathcal{G}) \times S^1$. Next we use that

$$\Omega^{k+1}_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_m / d\Omega^k_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_m \cong \Omega^{k+1}_c(S^*(\mathcal{G})), \quad \text{ for } m \neq 0, \text{ and}$$

$$\Omega^{k+1}_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_0 / d\Omega^k_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_0 \cong$$
$$\Omega^{k+1}_c(S^*(\mathcal{G}))/d\Omega^k_c(S^*(\mathcal{G})) \oplus \Omega^k_c(S^*(\mathcal{G}))/d\Omega^{k-1}_c(S^*(\mathcal{G})).$$

The first isomorphism above is given by

$$(\alpha, \beta) \in \Omega^{k+1}_c(S^*\mathcal{G})r^m \oplus \Omega^k_c(S^*\mathcal{G})r^{m-1}dr \mapsto m^{-1}\alpha - d\beta.$$

Lemma 5. *The E^1-term of the spectral sequence associated by Lemma 1 to the topologically filtered algebra $\mathcal{A}$ is given by $\mathrm{EC}^1_{k,h} \cong \Omega^{k+h}_c(S^*(\mathcal{G}))$ if $k \neq 0$ and*

$$\mathrm{EC}^1_{0,h} \cong \Omega^h_c(S^*(\mathcal{G}))/d\Omega^{h-1}_c(S^*(\mathcal{G})) \oplus \Omega^{h-1}_c(S^*(\mathcal{G}))/d\Omega^{h-2}_c(S^*(\mathcal{G})) \oplus$$
$$\oplus_{j>0}\, \mathrm{H}^{h-2j}_c(S^*(\mathcal{G}) \times S^1).$$

Moreover, the periodicity morphism $S : \mathrm{EC}^1_{k,h} \to \mathrm{EC}^1_{k,h-2}$ vanishes if $k \neq 0$ and is the natural projection if $k = 0$.

Proof. We know from Lemma 1 that $\mathrm{EC}^1_{k,h} \simeq \mathrm{HC}_{k+h}(Gr(\mathcal{A}))_k$. Using again the HKR-isomorphism, we obtain that $\mathrm{HC}_m(Gr(\mathcal{A}))_d$ is isomorphic to

$$\Omega^m_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_d / d\Omega^{m-1}_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_d \oplus \oplus_{j>0}\, \mathrm{H}^{m-2j}(A^*(\mathcal{G}) \smallsetminus 0)_d,$$

and the operator $S : \mathrm{EC}^1_{k,h} \to \mathrm{EC}^1_{k,h-2}$ is the projection which sends $\Omega^{k+h}_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_k / d\Omega^{k+h-1}_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_k$ to 0, is the inclusion

$$\mathrm{H}^{k+h-2}(A^*(\mathcal{G}) \smallsetminus 0)_k \hookrightarrow \Omega^{k+h-2}_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_k / d\Omega^{k+h-3}_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_k$$

and the identity on the other factors. The above computation of $\Omega^m_{rc}(A^*(\mathcal{G}) \smallsetminus 0)/d\Omega^{m-1}_{rc}(A^*(\mathcal{G}) \smallsetminus 0)$ then gives the result. □

From the above lemma we obtain the following immediate consequence.

Corollary 1. *We have* $\mathrm{EC}^1_{k,h} = 0$ *if* $k < 0$ *and* $k + h > \dim S^*(\mathcal{G})$.

The following proposition is essential in determining the periodic cyclic homology of $\mathcal{A}$.

Proposition 4. *If* $q > \dim S^*(\mathcal{G})$, *then* $\mathrm{HC}_q(\mathcal{A}) \cong \oplus_{k\in\mathbb{Z}} \mathrm{H}^{q-2k}_c(S^*(\mathcal{G}) \times S^1)$ *with* $S : \mathrm{HC}_{q+2}(\mathcal{A}) \to \mathrm{HC}_q(\mathcal{A})$ *also an isomorphism.*

Proof. We shall use Theorem 3 and Lemma 5. For $q > \dim(S^*\mathcal{G}) + 1$, Lemma 5 gives

$$\mathrm{EC}^1_{k,q-k} = 0 \text{ for all } k \neq 0, \text{ and } \mathrm{EC}^1_{0,q} = \oplus_{j\in\mathbb{Z}} \mathrm{H}^{q-2j}_c(S^*\mathcal{G} \times S^1).$$

Moreover, the assumptions of Theorem 3 are satisfied, by Corollary 1, and this gives

$$\mathrm{HC}_q(\mathcal{A}) \simeq \oplus_k \mathrm{EC}^1_{k,q-k} \cong \oplus_{j>0} \mathrm{H}^{q-2j}_c(S^*(\mathcal{G}) \times S^1).$$

This gives the result. □

From this we obtain

Theorem 6. *The periodic cyclic homology groups of* $\mathcal{A} = \Psi^\infty(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})$, *the algebra of complete symbols on* $\mathcal{G}$, *are given by* $\mathrm{HP}_q(\mathcal{A}) \cong \mathrm{H}^{[q]}_c(S^*(\mathcal{G}) \times S^1)$.

Proof. Whenever the periodicity operator S of the exact sequence (6) is surjective, we have $\mathrm{HP}_q(\mathcal{A}) \cong \lim\limits_{\leftarrow} \mathrm{HC}_{q+2j}$, the projective limit being taken with respect to the operator S. The conclusion then follows from Proposition 4. □

Similarly, we have the following determination of the periodic cyclic homology groups of the algebra $\mathcal{A}_0 = \Psi^0(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})$.

Theorem 7. $\mathrm{HP}_q(\mathcal{A}_0) \cong \mathrm{H}^{[q]}_c(S^*(\mathcal{G}))$.

Proof. The proof is essentially the same as for the corresponding result for the algebra $\mathcal{A}$, so we will be brief. The filtration on $\mathcal{A}_0$ is induced from the filtration on $\mathcal{A}$. This and the specific form of the EC^1-terms then give

$$\mathrm{EC}^1_{k,h}(\mathcal{A}_0) \simeq \mathrm{EC}^1_{k,h}(\mathcal{A}) \text{ if } k < 0, \text{ and } \mathrm{EC}^1_{k,h}(\mathcal{A}_0) \simeq \{0\} \text{ if } k > 0,$$
$$\text{and } \mathrm{EC}^1_{0,h}(\mathcal{A}_0) \cong \oplus_{j\in\mathbb{Z}} \mathrm{H}^{h-2j}_c(S^*(\mathcal{G})) \text{ if } h \geq \dim S^*(\mathcal{G}).$$

Moreover, the differential d^1 is the same as that for $\mathcal{A}$ if $k \leq 0$, but is trivial for $k > 0$. Thus, if $q > \dim(S^*\mathcal{G})$, we get as above that

$$\forall k \in \mathbb{Z} \smallsetminus \{0\},\ \mathrm{EC}^1_{k,q-k}(\mathcal{A}_0) \simeq 0.$$

Consequently,

$$\mathrm{EC}^1_{0,q}(\mathcal{A}_0) \simeq \oplus_{j\in\mathbb{Z}} \mathrm{H}^{q-2j}_c(S^*(\mathcal{G})) =: \mathrm{H}^{[q]}(S^*(\mathcal{G})).$$

□

A similar approach can be used to treat variants of the algebra $\mathcal{A}$ when the Schwarz symbols of our operators vanish to infinite order at certain hyperfaces of M.

Let X be a closed union of faces of the manifold with corners M. Let $\mathcal{I}_X \subset \mathcal{C}^\infty(M)$ be the ideal of functions that vanish to infinite order on X. We now study the algebras $\mathcal{A}_{\mathcal{L}}(M,X)$ of "Laurent complete symbols vanishing rapidly at X," supported above a compact subset of M, which, we recall, are given by

$$\mathcal{A}_{\mathcal{L}}(M,X) := \mathcal{O}(M)\mathcal{I}_X\mathcal{A}(\mathcal{G}) = \mathcal{O}(M)\mathcal{I}_X(\Psi^{\infty}(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})),$$

and $\mathcal{A}_{\mathcal{L}}(\mathcal{G}) = \mathcal{A}_{\mathcal{L}}(\mathcal{G},\emptyset)$.

We shall usually denote by $\pi_Y : Y \to M$ the projection associated with a typical fibration Y over M. As before, we denote by $\Omega^t_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0, \pi^{-1}_{A^*(\mathcal{G})\smallsetminus 0}(X))$ the space of t-differential forms on $A^*(\mathcal{G}) \smallsetminus 0$ that vanish to infinite order in the base variable on $\pi^{-1}_{S^*(\mathcal{G})\setminus 0}(X)$ and are supported above a compact set in M. The above results on the Hochschild and cyclic homology of the algebras $\mathcal{A}$ and $\mathcal{A}_0$ extend then almost right away to the algebras $\mathcal{A}_{\mathcal{L}}(M,X)$. Recall that $p_{\mathcal{L}} : \mathcal{L}(Y) \to Y$ is the projection defined in section 2 for a manifold with corners Y. We summarize results in the following two propositions. To simplify notation, we shall denote also by $p_{\mathcal{L}} : \mathcal{L}(S^*(\mathcal{G})) \times S^1 \to M$ the induced projection.

Proposition 5. *For $q > \dim(S^*(\mathcal{G}))$, we have*

$$\mathrm{HC}_q(\mathcal{A}_{\mathcal{L}}) \cong \oplus_{k\geq 0}\, \mathrm{H}^{q-2k}_c(\mathcal{L}(S^*(\mathcal{G})) \times S^1 \smallsetminus p^{-1}_{\mathcal{L}}(X)).$$

Thus, $\mathrm{HP}_q(\mathcal{A}_{\mathcal{L}}) \cong \mathrm{H}^{[q]}_c(\mathcal{L}(S^*(\mathcal{G})) \times S^1 \smallsetminus p^{-1}_{\mathcal{L}}(X))$.

Proof. We first prove the analogue of Lemma 5 in our new settings: namely, the E^1-term of the cyclic spectral sequence associated to the topologically filtered algebra $\mathcal{A}_{\mathcal{L}}(M,X)$ by Lemma 1 is given by

$$\mathrm{EC}^1_{k,h} \cong \Omega^{k+h}_{c,\mathcal{L}}(S^*(\mathcal{G}), \pi^{-1}_{S^*(\mathcal{G})}(X)),$$

if $k \neq 0$, and otherwise by

$$\begin{aligned}\mathrm{EC}^1_{0,h} \cong \Omega^{h}_{c,\mathcal{L}}(S^*(\mathcal{G}), \pi^{-1}_{S^*(\mathcal{G})}(X))/d\Omega^{h-1}_{c,\mathcal{L}}(S^*(\mathcal{G}), \pi^{-1}_{S^*(\mathcal{G})}(X)) \\ \oplus\, \Omega^{h-1}_{c,\mathcal{L}}(S^*(\mathcal{G}), \pi^{-1}_{S^*(\mathcal{G})}(X))/d\Omega^{h-2}_{c,\mathcal{L}}(S^*(\mathcal{G}), \pi^{-1}_{S^*(\mathcal{G})}(X)) \oplus \\ \oplus_{j>0}\, \mathrm{H}^{h-2j}_c(\mathcal{L}(S^*(\mathcal{G})) \times S^1, p^{-1}_{\mathcal{L}}(X)).\end{aligned}$$

Moreover, the periodicity morphism $S : \mathrm{EC}^1_{k,h} \to \mathrm{EC}^1_{k,h-2}$ vanishes if $k \neq 0$ and is the natural projection if $k = 0$.

Indeed, Lemma 1(i) together with a relative version of the HKR isomorphism give that

$$\begin{aligned}\mathrm{EC}^1_{k,h} \cong \frac{\Omega^{k+h}_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0, \pi^{-1}_{A^*(\mathcal{G})\smallsetminus 0}(X))_k}{d\Omega^{k+h-1}_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0, \pi^{-1}_{A^*(\mathcal{G})\smallsetminus 0}(X))_k} \oplus \\ \oplus_{j>0}\, \mathrm{H}^{k+h-2j}_{rc}(A^*(\mathcal{G}) \smallsetminus 0, \pi^{-1}_{A^*(\mathcal{G})\smallsetminus 0}(X))_k.\end{aligned}$$

Moreover, it follows that the operator S is the projection just above. Now we can compute each term and use the homotopy invariance of the relative de Rham cohomology to see that the k-homogeneous relative cohomology groups $H^{h-2j}_{\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0, \pi^{-1}_{A^*(\mathcal{G})\smallsetminus 0}(X))_k$ vanish for $k \neq 0$ and that for $k = 0$ we have

$$H^{h-2j}_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0, \pi^{-1}_{A^*(\mathcal{G})\smallsetminus 0}(X))_0 \cong H^{h-2j}_{c,\mathcal{L}}(S^*(\mathcal{G}) \times S^1, \pi^{-1}_{S^*(\mathcal{G})\times S^1}(X)).$$

On the other hand, we have

$$\frac{\Omega^{k+h}_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0, \pi^{-1}_{A^*(\mathcal{G})\smallsetminus 0}(X))_k}{d\Omega^{k+h-1}_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0, \pi^{-1}_{A^*(\mathcal{G})\smallsetminus 0}(X))_k} \cong \Omega^{k+h}_{c,\mathcal{L}}(S^*(\mathcal{G}), \pi^{-1}_{S^*(\mathcal{G})}(X)),$$

if $k \neq 0$, and, for $k = 0$, we have

$$\frac{\Omega^{k+h}_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0, \pi^{-1}_{A^*(\mathcal{G})\smallsetminus 0}(X))_0}{d\Omega^{k+h-1}_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0, \pi^{-1}_{A^*(\mathcal{G})\smallsetminus 0}(X))_0}$$
$$\cong \frac{\Omega^{h}_{c,\mathcal{L}}(S^*(\mathcal{G}), \pi^{-1}_{S^*(\mathcal{G})}(X))}{d\Omega^{h-1}_{c,\mathcal{L}}(S^*(\mathcal{G}), \pi^{-1}_{S^*(\mathcal{G})}(X))} \oplus \frac{\Omega^{h-1}_{c,\mathcal{L}}(S^*(\mathcal{G}), \pi^{-1}_{S^*(\mathcal{G})}(X))}{d\Omega^{h-2}_{c,\mathcal{L}}(S^*(\mathcal{G}), \pi^{-1}_{S^*(\mathcal{G})}(X))}.$$

The conclusion follows.

The first assertion is a direct consequence of the above discussion and Theorem 4. The computation of HP_q then follows as in the proof of Theorem 6. □

Proposition 5 is formulated in such a way that it remains true if we drop the subscript $\mathcal{L}$ (and if we remove $p_{\mathcal{L}}$). If we proceed then as indicated, we obtain the periodic cyclic homology of $\mathcal{A}(M, X)$.

We now take a closer look at the Hochschild homology of $\mathcal{A} = \Psi^{\infty}(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})$ and of the other related algebras.

We shall use below the Poisson structure of $A^*(\mathcal{G})$, which we now recall for the benefit of the reader. The natural regular Poisson structure of $T^*_{vert}(\mathcal{G})$ induces a Poisson structure on $A^*(\mathcal{G})$. (This is recalled in [23], Lemma 7, for example). If $r : \mathcal{G} \to M$ is the range map, then the image of the differential of r restricted to $A(\mathcal{G})$ determines a possibly singular foliation S on M. On the other hand, the kernel of r_* is a family of Lie algebras whose fiber at $x \in M$ is the Lie algebra of the Lie group $\mathcal{G}^x_x$. When the groups $\mathcal{G}^x_x$ are 0-dimensional, this foliation has no singularities, and $A(\mathcal{G})$ becomes the tangent bundle to the leaves of S. In this case, the Poisson structure of $A^*(\mathcal{G}) \cong T^*S$ is induced from the symplectic structures of the leaves. In general, the Poisson structure on $A^*(\mathcal{G})$ is defined by a two tensor

$$G \in \mathcal{C}^{\infty}(A^*(\mathcal{G}), \Lambda^2(T(A^*(\mathcal{G}))))$$

so that $\{f, g\} = i_G(df \wedge dg)$.

Let i_G be the contraction by G. Then we obtain as in [4] a differential

$$\delta := i_G \circ d - d \circ i_G : \Omega^k(A^*(\mathcal{G})) \to \Omega^{k-1}(A^*(\mathcal{G})), \tag{17}$$

Explicitly, for any open subset $V \subset A^*(\mathcal{G})$ of the form $V \cong [0,1)^l \times \mathbb{R}^{n-l}$ and any $(f_0, ..., f_k) \in \mathcal{C}^{\infty}(V)^{k+1}$, the differential δ is given locally by the formula

$$\delta(f_0 df_1 \wedge df_2 \wedge \ldots \wedge df_k) = \sum_{1 \le j \le k} (-1)^{j+1} \{f_0, f_j\} df_1 \wedge \ldots \wedge \widehat{df_j} \wedge \ldots \wedge df_k$$
$$+ \sum_{1 \le i < j \le k} (-1)^{i+j} f_0 d\{f_i, f_j\} \wedge df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge \widehat{df_j} \wedge \ldots \wedge df_k.$$

This formula is valid also when M has corners. It is easy to check that the differential δ is homogeneous of degree -1 with respect to the action of $\mathbb{R}^*_+$ on $A^*(\mathcal{G}) \smallsetminus 0$ and hence maps

$$\delta : \Omega^p(A^*(\mathcal{G}) \smallsetminus 0)_d \to \Omega^{p-1}(A^*(\mathcal{G}) \smallsetminus 0)_{d-1}.$$

Moreover, δ preserves the support, so it maps the space $\Omega^p_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_d$ of d-homogeneous forms supported above a compact set of M to $\Omega^{p-1}_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_{d-1}$.

Definition 2. The d-homogeneous Poisson p-homology space $\mathrm{H}^\delta_p(A^*(\mathcal{G}) \smallsetminus 0)_d$ of the conic Poisson manifold with corners $A^*(\mathcal{G}) \smallsetminus 0$ is defined by

$$\mathrm{H}^\delta_p(A^*(\mathcal{G}) \smallsetminus 0)_d := \frac{\ker(\delta : \Omega^p_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_d \to \Omega^{p-1}_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_{d-1})}{\delta(\Omega^{p+1}_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_{d+1}}.$$

Let $\Omega^p_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0)_d$ be the set of Laurent differential p-forms which are d-homogeneous under the radial action of $\mathbb{R}^*_+$ and such that their support projects onto a compact subset of M. Then it can be checked that δ is well defined on $\Omega^*_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0)_d$ and is given by the same formula. In fact the covector G is here tangent to the faces because the groupoid $\mathcal{G}$ is differentiable. We thus have the following Laurent complexes indexed by $p \in \mathbb{Z}$

$$0 \to \mathcal{P}^{p,n-p} \xrightarrow{\delta} \mathcal{P}^{p,n-p-1} \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{P}^{p,-p} \to 0,$$

where $\mathcal{P}^{p,d} = \Omega^{p+d}_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0)_d$. The homology of the resulting complex is, by definition, the Laurent-Poisson homogeneous homology of $A^*(\mathcal{G})\smallsetminus 0$. This homology will be denoted by

$$\mathrm{H}^\delta_{\mathcal{L},p+d}(A^*(\mathcal{G}) \smallsetminus 0)_d := \frac{\ker(\delta : \mathcal{P}^{p,d} \to \mathcal{P}^{p,d-1})}{\delta(\mathcal{P}^{p,d+1})}.$$

We begin again with a result on the algebra $\mathcal{A} = \Psi^\infty(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})$.

Proposition 6. *The algebra $\mathcal{A}$ is H-unital. Let*

$$\chi : \mathrm{HH}_l(Gr(\mathcal{A}))_d \to \Omega^l_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_d$$

be the HKR-isomorphism, and let $d_1 : \mathrm{EH}^1_{k,h} \to \mathrm{EH}^1_{k-1,h}$ be the first differential of the spectral sequence associated to $\mathcal{A}$ by Lemma 1. Then $\chi \circ d_1 \circ \chi^{-1} = -\sqrt{-1}\delta$, and hence $\mathrm{EH}^2_{k,h} \simeq \mathrm{H}^\delta_{k+h}(A^*(\mathcal{G}) \smallsetminus 0)_k$.

Proof. The H-unitality follows from Propositions 1 and 3. For the second part, we proceed essentially as in [7], Theorem 1. Choose an anti-symmetric tensor in the last m-variables

$$\eta = \sum \mathrm{sign}(\sigma) f_0 \otimes f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(m)},$$

with $f_j \in S^\infty_c(A^*(\mathcal{G}))$. We denote by

$$q(\eta) = \sum \mathrm{sign}(\sigma) q(f_0) \otimes q(f_{\sigma(1)}) \otimes \ldots \otimes q(f_{\sigma(m)})$$

the *quantization* of η. Let $k = \deg f_0 + \ldots + \deg f_m$ be the total degree. Because

$$[q(a), q(b)] = -\sqrt{-1} q(\{a, b\}) + \ldots ,$$

where the dots represent terms of order at most $\deg a + \deg b - 2$, the quantity $b \circ q(\eta)$ is of total order at most $k - 1$ and hence, modulo terms of order $k - 2$, $\chi \circ b \circ q(\eta))$ is easily checked to be exactly $\delta(\eta)$. ☐

We think that the above spectral sequence is always convergent and that it actually degenerates at E^2. We prove this, for instance, for differentiable groupoids associated with manifolds with corners, see in Section 4. This spectral sequence is also studied in [6] for foliations. In [11] it is proved that the groupoid algebras $\Psi^{-\infty}(\mathcal{G})$ are H-unital.

In the same way we can generalize Proposition 6 and state:

Proposition 7. *The algebra $\mathcal{A}_{\mathcal{L}}(M,X)$ is H-unital. The* EH^2 *term of the spectral sequence associated by Lemma 1 to $\mathcal{A}_{\mathcal{L}}(M,X)$ is given by*

$$\mathrm{EH}^2_{k,h} \simeq \mathrm{H}^{\delta}_{\mathcal{L},k+h}((A^*(\mathcal{G}) \smallsetminus 0) \smallsetminus \pi^{-1}_{A^*(\mathcal{G})\smallsetminus 0}(X))_k.$$

Proof. The H-unitality follows from Propositions 1 and 3, as above.

To prove the rest of this proposition, we can either use the same method as the one used to prove Proposition 6, or we can argue that this proposition actually follows from Proposition 6. ☐

4. Hochschild homology for manifolds with corners

We now restrict our study to groupoids $\mathcal{G}$ in a particular class. Thus, we shall assume throughout the rest of this paper that the anchor map $\varrho : A(\mathcal{G}) \to TM$ is such that the induced map $\varrho_* : \Gamma(A(\mathcal{G})) \to \Gamma(TM)$ of $\mathcal{C}^\infty(M)$-modules becomes an isomorphism after tensoring with $\mathcal{O}(M)$. That is, we assume that

$$\varrho_* : \mathcal{O}(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(A(\mathcal{G})) \cong \mathcal{O}(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(TM). \tag{18}$$

We then say that $A(\mathcal{G})$ is *rationally isomorphic to TM*. If that is the case, it also follows that ϱ induces an isomorphism of $A(\mathcal{G})$ to TM on the interior M_0 of M:

$$A(\mathcal{G})|_{M_0} \cong TM_0.$$

Let $\mathcal{O}(M)$ be the ring of smooth functions on the interior M with only Laurent singularities at the corners, as in Section 2. Also, we assume that $A(\mathcal{G})$ is such that where TM is the tangent bundle to M. We proceed in this section to the computation of the Hochschild homology groups of $\mathcal{A} := \Psi^\infty(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})$, the algebra of complete symbols on $\mathcal{G}$ and of various other related algebras.

Throughout this and the following sections, n denotes the dimension of M, the space of units of $\mathcal{G}$.

Let $X \subset M$ be a closed union of faces of M and denote by $\mathcal{I}_X \subset \mathcal{C}^\infty(M)$ the ideal of functions that vanish to infinite order on X. Recall that we have defined the algebra $\mathcal{A}_{\mathcal{L}}(M,X)$ of "Laurent complete symbols vanishing rapidly at X" by

$$\mathcal{A}_{\mathcal{L}}(M,X) := \mathcal{O}(M)\mathcal{I}_X\mathcal{A} = \mathcal{O}(M)\mathcal{I}_X(\Psi^\infty(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})),$$

$(\mathcal{A}_{\mathcal{L}}(M) = \mathcal{A}_{\mathcal{L}}(M,\emptyset))$.

We keep the notations of Section 3 so that $\pi_{S^*M\times S^1} : S^*M \times S^1 \to M$ is the natural projection. Also, note that $S^*(\mathcal{G}) \cong S^*M$, but not canonically. The homotopy class of this homeomorphism is canonical though. Also, $p_{\mathcal{L}}$ denotes the structural map $\mathcal{L}(S^*(\mathcal{G})) \times S^1 \to M$.

Theorem 8. *Let $X \subset M$ be a union of hyperfaces of M and assume that $A(\mathcal{G})$ is rationally isomorphic to TM (that is, it satisfies the condition* (18) *above). Then we have*

$$\mathrm{HH}_*(\mathcal{A}_{\mathcal{L}}(M,X)) \simeq \mathrm{H}^{2n-*}_c(\mathcal{L}(S^*M) \times S^1 \smallsetminus p_{\mathcal{L}}^{-1}(X))$$

Proof. The computation of $\mathrm{HH}_*(\mathcal{A}_{\mathcal{L}}(M,X))$ proceeds exactly as in the case of manifolds with boundary. From Proposition 7 we know that the E^2 term of the spectral sequence associated with the order filtration on the Hochschild complex is given by

$$\mathrm{EH}^2_{k,h} \cong \mathrm{H}^{\delta}_{\mathcal{L},k+h}(A^*(\mathcal{G}) \smallsetminus 0, \pi^{-1}_{A^*(\mathcal{G})\smallsetminus 0}(X))_k.$$

In our case,

$$\Omega^{k+h}_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0, \pi^{-1}_{A^*(\mathcal{G})\smallsetminus 0}(X))_k \cong \Omega^{k+h}_{rc,\mathcal{L}}(T^*M \smallsetminus 0, \pi^{-1}_{T^*M\smallsetminus 0}(X))_k,$$

which is the main reason why this result is true, and explains the thinking behind the condition (18). Thus the Poisson structure on $A^*(\mathcal{G}) \smallsetminus 0$ can be related to the natural Poisson structure on the cotangent bundle $T^*M \smallsetminus 0$. In particular, the restriction of the given Poisson structure to the interior of $A^*(\mathcal{G}) \smallsetminus 0$ is a symplectic structure.

We now recall the definition of the symplectic Hodge operator $*_G$. Let ω be the two form on the interior of M that defines the symplectic form. The form ω has only Laurent singularities at the hyperfaces due to our assumptions on $A^*(\mathcal{G})$. This then defines the $*_G$-operator by the formula [4]

$$*_G : \Omega^t_{rc,\mathcal{L}}(T^*M) \to \Omega^{2n-t}_{rc,\mathcal{L}}(T^*M), \quad \alpha \wedge *_G(\alpha') = \wedge^n(G)(\alpha, \alpha')\omega^n/n!.$$

Then $*_G^2 = 1$ and $*_G \circ \delta \circ *_G = (-1)^k d$ on the interior of M, by [4], and hence also on $\Omega^k_{rc,\mathcal{L}}(T^*M \smallsetminus 0)$ because the forms in the latter space are determined by their restriction to the interior of M.

Now, keeping in mind that by construction the operator $*_G$ maps k-homogeneous q-forms into $k+n-q$-homogeneous $(2n-q)$-forms, we get

$$\begin{aligned}\mathrm{EH}^2_{k,h} &\cong H^{\delta}_{\mathcal{L},k+h}(A^*(\mathcal{G}) \smallsetminus 0, \pi^{-1}_{A^*(\mathcal{G})\smallsetminus 0}(X))_k \\ &\cong H^{2n-k-h}_{\mathcal{L}}(T^*M \smallsetminus 0, \pi^{-1}_{T^*M\smallsetminus 0}(X))_{n-h}.\end{aligned}$$

Using Proposition 2 and the relative homotopy invariance of the relative De Rham cohomology, we see that the only non-zero homogeneous component corresponds to $h = n$, and then that it is given by

$$\mathrm{EH}^2_{k,n} \cong \mathrm{H}^{n-k}_{\mathcal{L}}(S^*M \times S^1, \pi^{-1}_{S^*M\times S^1}(X)).$$

From the fact that $\mathrm{EH}^2_{k,h} = 0$, if $h \neq n$, we see that $d_r = 0$, for all $r \geq 2$, and hence we deduce the result. The convergence of the spectral sequence to $\mathrm{HH}_*(\mathcal{A}_{\mathcal{L}}(M, X))$ follows from Theorem 4.

By replacing k with $k-n$ we obtain the stated formula for $\mathrm{HH}_k(\mathcal{A}_{\mathcal{L}}(M, X))$. $\square$

Let us look now in more detail at one of the constructions used in the above proof in a particular case. Suppose then that the sections of the vector bundle $A(\mathcal{G})$ at the hyperface $H = \{x = 0\}$ are generated by ∂_y and $x^{c_H}\partial_x$. Here c_H are integers ≥ 1 associated to each face and the y's are coordinates on H, using a tubular neighborhood of H. (This definition depends on a choice of defining functions for the hyperfaces of M.) Let us call this calculus the c_n-calculus.

Then the 2-covector is a covector on the manifold with corners $A^*(\mathcal{G}) \smallsetminus 0$ which is given in any open subset of M diffeomorphic to one of the form $[0,1)^k \times \mathbb{R}^{n-k}$ (with defining functions $(x_1, \dots, x_k)$) by

$$G = \sum_{k+1 \leq j \leq n} \partial_{\xi_j} \wedge \partial_{y_j} + \sum_{1 \leq j \leq k} x_j^{c_j} \partial_{\xi_j} \wedge \partial_{x_j}.$$

Here the convention of sign is that of [4]. The symplectic form is then given by

$$\omega = \sum_{k+1 \leq j \leq n} dy_j \wedge d\xi_j + \sum_{1 \leq j \leq k} x_j^{-c_j} dx_j \wedge d\xi_j.$$

These two formulæ can be checked in the case $[0,1) \times \mathbb{R}$ and globalized just like in the case of the usual cusp calculus. We have for instance in the case of a manifold with boundary given by $\{x = 0\}$ that $*_G(f) = \pm(f/x^c)dx \wedge d\xi$ while $*_G(f dx \wedge d\xi) = \pm f x^c$.

5. Applications

We keep the assumptions of the previous section. The same methods as the ones in the previous section can be used also to determine the homology of some quotient algebras. Let X be a closed union of faces of M. First we prove a result that allows us to reduce the computation of the homologies of the relative algebras $\mathcal{A}_{\mathcal{L}}(M, X)$ to the computation of those of an "absolute algebra."

More precisely, denote by $\mathcal{A}_{\mathcal{L}}(M \smallsetminus X)$ the algebra of Laurent complete symbols on $\mathcal{G}|_{M \smallsetminus X}$, the groupoid obtained by restricting $\mathcal{G}$ to the invariant subset $M \smallsetminus X$. Also, let $\mathcal{L}(S^*M)$ be the space introduced at the end of Section 2 (with M in place of S^*M), and denote by $p_{\mathcal{L}} : \mathcal{L}(S^*M) \times S^1 \to M$ the resulting natural map. Then we have the following excision result.

Proposition 8. *The inclusion $\mathcal{A}_{\mathcal{L}}(M \smallsetminus X) \to \mathcal{A}_{\mathcal{L}}(M, X)$ induces isomorphisms in Hochschild, cyclic, and periodic cyclic homologies.*

Proof. By a standard argument using the 'SBI'-exact sequence, it is enough to prove this statement for Hochschild homology.

Consider then the spectral sequences associated to the two algebras and the order filtration on their Hochschild complexes by Lemma 1. The induced map is then an isomorphism at the E^2-term, by Theorem 8, because

$$\mathcal{L}(S^*M) \times S^1 \smallsetminus p_{\mathcal{L}}^{-1}(X) = \mathcal{L}(S^*(M \smallsetminus X)) \times S^1.$$

□

Proposition 9. *The quotient $\mathcal{A} := \mathcal{A}_{\mathcal{L}}(M)/\mathcal{A}_{\mathcal{L}}(M, X)$ is a topologically filtered algebra with*

$$\mathrm{HH}_q(\mathcal{A}) \cong \mathrm{H}_c^{2n-q}(p_{\mathcal{L}}^{-1}(X)).$$

Moreover, the exact sequence in Hochschild homology associated to the exact sequence $0 \to \mathcal{A}_{\mathcal{L}}(M, X) \to \mathcal{A}_{\mathcal{L}}(M) \to \mathcal{A} \to 0$ is naturally isomorphic to the long exact sequence in cohomology associated to the pair $(\mathcal{L}(S^*M) \times S^1, p_{\mathcal{L}}^{-1}(X))$.

Proof. Only the last statement is not similar to some other proofs in the previous sections. Choose an open neighborhood U of X in M such that X is a deformation retract of U. Also, let ϕ be a smooth function supported in U which is one in some smaller neighborhood of X. This data gives rise to a lifting of any smooth function on $\pi^{-1}(X)$ to a function on S^*M with support in U, and hence also to a linear lift $\psi : \mathcal{H}_*(\mathcal{A}) \to \mathcal{H}_*(\mathcal{A}_{\mathcal{L}}(M))$. The computation of $[b, \psi]$ identifies the boundary map in the Hochschild cohomology spectral sequence and gives the result. □

It is interesting to notice that as a consequence of our computations we obtain that the traces live only on the minimal faces. Obviously, the faces of M that are manifolds without corners are exactly the ones that are minimal with respect to inclusion in the set of faces of M. Let then Y be the union of all faces of M that have no corners (and no boundary).

Proposition 10. *Let $\tau \in \mathrm{HH}^0(\mathcal{A}_{\mathcal{L}}(M))$ be a trace. Then it is induced from a trace on $\mathcal{A}_{\mathcal{L}}(M)/\mathcal{A}_{\mathcal{L}}(M,Y)$.*

Proof. Let n be the dimension of M, as before. The dual of Proposition 9 holds true for Hochschild cohomology. This implies that the inclusion

$$\mathrm{HH}_0(\mathcal{A}_{\mathcal{L}}(M)/\mathcal{A}_{\mathcal{L}}(M,Y)) \to \mathrm{HH}_0(\mathcal{A}_{\mathcal{L}}(M))$$

is isomorphic (in the sense of the above Proposition) to the dual of the map $\mathrm{H}_c^{2n}(\mathcal{L}(M)) \to \mathrm{H}_c^{2n}(p_{\mathcal{L}}^{-1}(Y))$. But this last map is checked to be an isomorphism. □

The above proof gives as a corollary then the "number" of residue traces on M.

Corollary 2. *The dimension of the space of traces of $\mathcal{A}_{\mathcal{L}}(M)$ is the number of minimal faces of M.*

The above two results can be proved directly, without using singular cohomology groups.

We continue to assume that $A(\mathcal{G})$ is rationally isomorphic to TM (that is, it satisfies Equation (18)), as in the previous section. For the algebras that we have considered, the cyclic homology can be computed directly in terms of the Hochschild homology as we shall see below. Let $X \subset M$ be a union of closed faces of M.

Consider Connes' 'SBI'-exact sequence associated to the algebra $\mathcal{A}_{\mathcal{L}}(M,X)$ (see Equation (6)).

Theorem 9. *We have $B=0$ in the 'SBI'-exact sequence of the algebra $\mathcal{A}_{\mathcal{L}}(M,X)$, and hence*

$$\mathrm{HC}_m(\mathcal{A}_{\mathcal{L}}(M,X)) = \oplus_{k\geq 0}\, \mathrm{HH}_{m-2k}(\mathcal{A}_{\mathcal{L}}(M,X)). \tag{19}$$

Proof. We will proceed by induction on m to show that the morphism

$$B : \mathrm{HC}_{m-1}(\mathcal{A}_{\mathcal{L}}(M,X)) \to \mathrm{HH}_m(\mathcal{A}_{\mathcal{L}}(M,X))$$

is zero for any m.

Let n denote the dimension of the manifold M, as above. The statement of the theorem is obviously true for $m=0$. The proof of Theorem 8, show that the groups $\mathrm{HH}_q(\mathcal{A}_{\mathcal{L}}(M,X))$ are generated by elements of order $-n+q$ with respect to the degree filtration, and that all cycles of order less that $-n+q$ are boundaries. This is a direct consequence of the computation of the E^1 terms of the spectral sequences associated to the degree filtration.

Assuming now the statement to be true for all values less than m, we obtain from the 'SBI' exact sequence that the groups $\mathrm{HC}_{m-1}(\mathcal{A}_{\mathcal{L}}(M,X))$ are isomorphic to $\mathrm{HC}_{m-1}(\mathcal{A}_{\mathcal{L}}(M,X)) \simeq \oplus_{k\geq 0}\, \mathrm{HH}_{m-2k-1}(\mathcal{A}_{\mathcal{L}}(M,X))$. This shows that the groups $\mathrm{HC}_{m-1}(\mathcal{A}_{\mathcal{L}}(M,X))$ are generated by elements of order at most $-n+m-1$ in the degree filtration. It follows that they map to elements of order less than $-n+m$ in $\mathrm{HH}_m(\mathcal{A}_{\mathcal{L}}(M,X))$, and hence they vanish by the above remark.

The last statement follows directly from the vanishing of B in the Connes' exact sequence. □

Appendix A. Projective limits

In this appendix we recall some well known facts about projective (or inverse) limits and the homology of projective limits, using this also as an opportunity to fix notation. An important role is played by $\lim^1 = \varprojlim^1$, the first (and only) derived functor of the projective limit functor $\varprojlim$.

Let $\phi_n : V_{n+1} \to V_n$, $n \in \mathbb{N}$, be an inverse system of vector spaces. Define $F : \prod V_n \to \prod V_n$ by $F(v_k) = (v_k - \phi_k(v_{k+1}))$. Then $\varprojlim V_n$ is the kernel of F and $\varprojlim^1$ is the cokernel of F, by definition.

Suppose now that V_n are complexes of vector spaces and the maps ϕ_n are surjective. By writing the homology long exact sequence associated to the short exact sequence of complexes

$$0 \to \varprojlim V_n \to \prod V_n \xrightarrow{F} \prod V_n \to 0,$$

we obtain the following well known exact sequence:

Lemma 6. *If the maps $\phi_n : V_{n+1} \to V_n$ are surjective, then the homology of the inverse limit satisfies*

$$0 \to \varprojlim{}^1 \mathrm{H}_{q+1}(V_n) \to \mathrm{H}_q(\varprojlim V_n) \to \varprojlim \mathrm{H}_q(V_n) \to 0.$$

If $B_n \subset A_n$ is a subspace preserved by $\phi_n : A_{n+1} \to A_n$ and $C_n = A_n/B_n$, then we have an exact sequence

$$0 \to \prod B_n \to \prod A_n \to \prod C_n \to 0$$

and F is an endomorphism of this exact sequence. The ker-coker lemma (see [1], for example) then gives the following exact sequence:

Lemma 7. *If A_n, B_n, and C_n are as above, then we have the exact sequence*

$$0 \to \varprojlim B_n \to \varprojlim A_n \to \varprojlim C_n \to \varprojlim{}^1 B_n \to \varprojlim{}^1 A_n \to \varprojlim{}^1 C_n \to 0.$$

References

[1] M. F. Atiyah and MacDonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, Mass.-London, 1969.

[2] M.F. Atiyah, Patodi and I. Singer, *Spectral asymmetry and Riemannian geometry, I*, Math. Proc. Camb. Phil. Soc. **77** (1975), 43–69.

[3] P. Baum, A. Connes, and N. Higson, *Classifying spaces for proper actions and K-theory of group C^*-algebras*, C^*-algebras: 1943–1993 (San Antonio, TX, 1993), 240–291, Contemp. Math. **167**, AMS, Providence, RI, 1994.

[4] J.-L. Brylinski, *A differential complex for Poisson manifolds*, J.Diff. Geom. **28** (1988), 93–114.

[5] J.-L. Brylinski and V. Nistor, *Cyclic cohomology of etale groupoids*, K-Theory **8** (1994), 341–365.

[6] M. Benameur and V. Nistor, *Residues and an index theorem for foliations*, work in progress.

[7] J.-L. Brylinski and E. Getzler, *The homology of Algebras of Pseudo-differential Symbols and the Noncommutative Residue*, K-Theory **1** (1987), 385–403.

[8] A. Connes, *Noncommutative differential geometry*, Publ. Math. IHES **62** (1985), 41–144.

[9] A. Connes. *Noncommutative Geometry*, Academic Press, New York - London, 1994.

[10] A. Connes and J. Cuntz, *Quasihomomorphisms, cohomology cyclique et positivite. Communications in Mathematical Physics*, 114:515–526, 1988.

[11] M. Crainic and I. Moerdijk, *Foliation groupoids and their cyclic homology*, Preprint 2000.

[12] G. Hochschild, B. Kostant, and A. Rosenberg, *Differential forms on regular affine algebras*, Trans. AMS **102** (1962), 383–408.

[13] R. Lauter and S. Moroianu, *Homology of pseudo-differential operators on manifolds with fibered boundaries*, Mainz University preprint 2000.

[14] R. Lauter and V. Nistor, *Analysis of geometric operators on open manifolds: a groupoid approach*, these proceedings (based on a Muenster University preprint, 2000).

[15] S. Moroianu, *Residue functionals on the algebra of adiabatic pseudo-differential operators*, MIT thesis, 1999.

[16] N. P. Landsman and B. Ramazan, *Quantization of Poisson algebras associated to Lie algebroids*, math-ph/001005.

[17] J.-L. Loday and D. Quillen, *Cyclic homology and the Lie homology of matrices*, Comment. Math. Helv. **59** (1984), 565–591.

[18] S. Mac Lane, *Homology*, Springer-Verlag, Berlin-Heidelberg-New York, 1995.

[19] S. Mac Lane and I. Moerdijk, *Sheaves in geometry and logic. A first introduction to topos theory*, Universitext, Springer-Verlag, Berlin-Heidelberg-New York, 1994.

[20] R. Melrose and V. Nistor, *K-Theory of C^*-algebras of b-pseudodifferential operators*, GAFA **8**, (1998), 88-122.

[21] R. Melrose and V. Nistor, *Homology of pseudodifferential operators I. Manifolds with boundary*, accepted to appear in Amer. J. Math.

[22] B. Monthubert, *Pseudodifferential calculus on manifolds with corners and groupoids*, Proc. Amer. Math. Soc. **12** (1999), 2871–2881.

[23] B. Monthubert and P.-Y. Le Gall, *K-theory of the indicial algebra of manifolds with corners*, Preprint 2000.

[24] V. Nistor, *Higher Index Theorems and the boundary map in cyclic cohomology*, Documenta Mathematica (electronic) (1997), 263–296.

[25] V. Nistor, A. Weinstein and Ping Xu, *Pseudodifferential operators on differential groupoids*, Pacific J. Math. **189** (1999), 117-152.

[26] B. L. Tsygan. Homology of matrix Lie algebras over rings and Hochschild homology. *Uspekhi Math. Nauk.*, 38:217–218, 1983.

[27] M. Wodzicki. Excision in cyclic homology and in rational algebraic K-theory. *Annals of Mathematics*, 129:591–640, 1989.

Pennsylvania State University and Inst. Desargues, Lyon, France
E-mail address: `benameur@desargues.univ-lyon1.fr`

Inst. Desargues and Pennsylvania State University, University Park, PA 16802
E-mail address: `nistor@math.psu.edu`