TOPOLOGICAL CLASSIFICATION OF THE HYPERSPACES OF CLOSED CONVEX SUBSETS OF A BANACH SPACE

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In the talk we shall classify topologically the spaces $\mathrm{Conv}_H(X)$ of non-empty closed convex subsets of a Banach space X, endowed with Hausdorff (infinite-valued) metric

$$d_H(A,B) = \max\{\sup_{a\in A} \mathrm{dist}(a,B), \sup_{b\in B} \mathrm{dist}(b,A)\} \in [0,\infty]$$

where $\mathrm{dist}(a,B) = \inf_{b\in B} \|a-b\|$ is the distance from a point a to a subset B in X. The space $\mathrm{Conv}_H(X)$ is locally connected. The connectd component of $\mathrm{Conv}_H(X)$ containing a given closed convex set $C \subset X$ coincides with the set $\{A \in \mathrm{Conv}_H(X) : d_H(A,C) < \infty\}$.

Theorem 1. *Let X be a Banach space. Each connected component $\mathcal{H}$ of the space $\mathrm{Conv}_H(X)$ is homeomorphic to $\{0\}$, $\mathbb{R}$, $\mathbb{R}\times\mathbb{R}_+$, $Q\times\mathbb{R}_+$, or $l_2(\kappa)$ for an infinite cardinal κ. More precisely, $\mathcal{H}$ is homeomorphic to:*

1. *$\{0\}$ iff $\mathcal{H}$ contains the whole space X;*
2. *$\mathbb{R}$ iff $\mathcal{H}$ contains a half-space;*
3. *$\mathbb{R}\times\mathbb{R}_+$ iff $\mathcal{H}$ contains a linear subspace of X of codimension 1;*
4. *$Q\times\mathbb{R}_+$ iff $\mathcal{H}$ contains a linear subspace of X of finite codimension ≥ 2;*
5. *$l_2(\kappa)$ for an infinite cardinal κ iff $\mathcal{H}$ contains no half-space and no linear subspace of finite codimension.*

Here $\mathbb{R}_= = [0,\infty)$ is the half-line, $Q = [0,1]^\omega$ is the Hilbert cube and $l_2(\kappa)$ is the Hilbert space of density κ.

A closed convex subset C of a Banach space X is called

- a *half-space* if $C = f^{-1}[a,+\infty)$ for some real number a and some non-trivial linear continuous functional $f : X \to \mathbb{R}$;
- *a polyhedral set* if $C = \bigcap_{i=1}^n H_i$ is a finite intersection of half-spaces.

Theorem 2. *Let X be a finite-dimensional Banach space. For a connected component $\mathcal{H}$ of $\mathrm{Conv}_H(X)$ the following conditions are equivalent:*

1. *$\mathcal{H}$ has density $\mathrm{dens}(\mathcal{H}) < \mathfrak{c}$;*
2. *$\mathcal{H}$ is separable;*
3. *$\mathcal{H}$ contains a polyhedral convex set.*

Combining Theorems 1 and 2 we obtain the topological classification of connected components of the spaces $\mathrm{Conv}_H(\mathbb{R}^n)$.

Corollary 1. *Let X be a finite-dimensional Banach space. Each connected component $\mathcal{H}$ of the space $\mathrm{Conv}_H(X)$ is homeomorphic to $\{0\}$, $\mathbb{R}$, $\mathbb{R}\times\mathbb{R}_+$, $Q\times\mathbb{R}_+$, l_2 or l_∞. More precisely, $\mathcal{H}$ is homeomorphic to:*

1. *$\{0\}$ iff $\mathcal{H}$ contains the whole space $\mathbb{R}^n$;*
2. *$\mathbb{R}$ iff $\mathcal{H}$ contains a half-space;*

(3) $\mathbb{R} \times \mathbb{R}_+$ *iff* $\mathcal{H}$ *contains a linear subspace of* X *of codimension 1;*
(4) $Q \times \mathbb{R}_+$ *iff* $\mathcal{H}$ *contains a linear subspace of* X *of codimension* ≥ 2*;*
(5) l_2 *iff* $\mathcal{H}$ *contains a polyhedral convex set but contains no linear subspace and no half-space;*
(6) l_∞ *iff* $\mathcal{H}$ *does not contain a polyhedral convex set.*

Corollary 2. *Let* X *be a finite-dimensional Banach space. The space* $\mathrm{Conv}_H(X)$ *is homeomorphic to the topological sum:*

(1) $\{0\} \oplus \mathbb{R} \oplus \mathbb{R} \oplus (\mathbb{R} \times \mathbb{R}_+)$ *iff* $dim(X) = 1$*;*
(2) $\{0\} \oplus Q \times \mathbb{R}_+ \oplus \mathfrak{c} \times (\mathbb{R} \oplus \mathbb{R} \times \mathbb{R}_+ \oplus l_2 \oplus l_\infty)$ *iff* $\dim(X) = 2$*;*
(3) $\{0\} \oplus \mathfrak{c} \times (\mathbb{R} \oplus \mathbb{R} \times \mathbb{R}_+ \oplus Q \times \mathbb{R}_+ \oplus l_2 \oplus l_\infty)$ *iff* $dim(X) \geq 3$.

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